Pacific Journal of Mathematics

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Vol. 5, No. 1 September 1955

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Introduction. The number $N_k(D)$ of solutions (u, w) of the congruence

$$u^k + D \equiv w^2 \pmod p \tag{1}$$

can be expressed in terms of the Gaussian cyclotomic numbers (i, j) of order $\mathrm{LCM}(k, 2)$ as has been done by Vandiver [7], or in terms of the character sums introduced by Jacobsthal [4] and studied in special cases by von Schrutka [6], Chowla [1], and Whiteman [8]. In the special cases $k = 3, 4, 5, 6$, and 8, the answer can be expressed in terms of certain quadratic partitions of p, but unless D is a kth power residue there remained an ambiguity in sign, which we will be able to eliminate in some cases in the present paper. Theorems 2 and 4 were first conjectured from the numerical evidence provided by the SWAC and later proved by the use of cyclotomy. They improve Jacobsthal's results for all p for which 2 is not a quartic residue. Similarly Theorem 6 improves von Schrutka's and Chowla's results for those p's which do not have 2 for a cubic residue. Only in case $k = 2$ and in the cases where k is oddly even and D is a $(k/2)$th but not a kth power residue is $N_k(D)$ a function of p alone and is in fact $p - 1$. This result appears in Theorem 1. In case $k = 4$, Vandiver [7a] gives an unambiguous solution, which requires the determination of a primitive root.

1. Character sums. It is clear that the number of solutions $N_k(D)$ of (1) can be written

$$N_k(D) = \sum_{u=0}^{p-1} \left[1 + \left(\frac{u^k + D}{p}\right)\right] = p + \sum_{u=0}^{p-1} \left(\frac{u^k + D}{p}\right),$$

or

$$N_k(D) = p + \left(\frac{D}{p}\right) + \psi_k(D), \tag{2}$$

Received July 10, 1953.
Pacific J. Math. 5 (1955), 103-118

where the function

$$\psi_k(D) = \sum_{u=1}^{p-1} \left(\frac{u^k + D}{p}\right) \tag{3}$$

is connected with the Jacobsthal sum

$$\phi_k(D) = \sum_{u=1}^{p-1} \left(\frac{u}{p}\right)\left(\frac{u^k + D}{p}\right) \tag{4}$$

by the relations

$$\psi_k(D) = \left(\frac{D}{p}\right) \phi_k(\bar{D}), \quad k \text{ odd and } D\bar{D} \equiv 1 \pmod p, \tag{5}$$

and

$$\psi_{2k}(D) = \psi_k(D) + \phi_k(D). \tag{6}$$

Other pertinent relations are

$$\begin{cases} \phi_k(m^k D) = \left(\frac{m}{p}\right)^{k+1} \phi_k(D) \\ \psi_k(m^k D) = \left(\frac{m}{p}\right)^{k} \psi_k(D) \end{cases} \qquad (m \not\equiv 0 \pmod p) \tag{7}$$

and

$$\begin{cases} \phi_k(\bar{D}) = -\left(\frac{D}{p}\right) \phi_k(D) \\ \psi_k(\bar{D}) = \left(\frac{D}{p}\right) \psi_k(D). \end{cases} \qquad (k \text{ even}) \tag{8}$$

Also, for k odd and ρ a primitive root,

$$\sum_{\nu=0}^{k-1} \phi_k(\rho^\nu) = -k. \tag{9}$$

These relations are either well known or are paraphrases of known relations and are all easily derivable from the definitions. If k is odd, it follows from (5) and (6) that

$$(10) \qquad \psi_{2k}(D) = \phi_k(D) + \left(\frac{D}{p}\right) \phi_k(\overline{D}).$$

If D is a kth power residue, then so is $\overline{D}$ and hence by (7) for k odd $\phi_k(D) = \phi_k(\overline{D}) = \phi_k(1)$, and we have

$$(11) \qquad \psi_{2k}(D) = \phi_k(D)\left[1 + \left(\frac{D}{p}\right)\right] = \begin{cases} 2\phi_k(D) & \text{if } \left(\frac{D}{p}\right) = +1 \\ 0 & \text{if } \left(\frac{D}{p}\right) = -1. \end{cases}$$

Hence from (2) we obtain:

THEOREM 1. *If k is odd and if $D = m^k$, where m is a nonresidue of $p = 2kh + 1$, then the number $N_{2k}(m^k)$ of solutions (u, w) of*

$$u^{2k} + m^k \equiv w^2 \pmod p$$

is exactly $p - 1$.

Since $\phi_1(D) = -1$, it follows from (11) that $\psi_2(D) = -2$, if D is a residue, and zero otherwise. Hence by (2), $N_2(D) = p - 1$ for all D. This is a well known result in quadratic congruences. We will next discuss the case $k = 4$, which is connected with Jacobsthal's theorem.

Jacobsthal [4] proved that if D is a residue and if $p = x^2 + 4y^2$, then

$$(12) \qquad \phi_2(D) = -2x\left(\frac{\sqrt{D}}{p}\right), \qquad x \equiv 1 \pmod 4;$$

but if D is a nonresidue then he was able to prove only that

$$(13) \qquad \phi_2(D) = \pm 4y.$$

Hence for D a residue, it follows from the fact that $\psi_2(D) = -2$, using (6) and (2), that

$$(14) \qquad N_4(D) = p - 1 - 2x\left(\frac{\sqrt{D}}{p}\right), \quad x \equiv 1 \pmod 4.$$

However, the corresponding result for D nonresidue would read

$$N_4(D) = p - 1 \pm 4y. \tag{15}$$

In order to eliminate this ambiguity in sign at least for some cases we now turn to the cyclotomic approach.

2. Cyclotomy. If we define as usual the cyclotomic number $(i, j)_k$ as the number of solutions (ν, μ) of the congruence

$$g^{k\nu+i} + 1 \equiv g^{k\mu+j} \pmod{p} \tag{16}$$

then if D belongs to class s with respect to some primitive root g (that is, if $\operatorname{ind}_g D \equiv s \pmod{k}$), we can write the number of nonzero solutions of (1) for k even as follows:

$$N_k^*(D) = 2k \sum_{\nu=1}^{k/2} (k-s, 2\nu - s)_k. \tag{17}$$

We now assume that 2 is a nonresidue and choose g so that 2 belongs to the first class, or $s = 1$; then

$$N_4(2) = N_4^*(2) = 8[(3,1)_4 + (3,3)_4]. \tag{18}$$

These cyclotomic constants have been calculated by Gauss [3] in terms of x and y in the quadratic partition $p = x^2 + 4y^2$ and are for $p = 8n + 5$

$$16(3,3)_4 = p - 2x - 3, \quad 16(3,1)_4 = p + 2x - 8y + 1. \tag{19}$$

Substituting this into (18) we obtain

$$N_4(2) = p - 1 - 4y, \qquad \left(\frac{2}{p}\right) = -1. \tag{20}$$

To determine the sign of y we recall a lemma of our previous paper [5] which states that $(0, s)$ is odd or even according as 2 belongs to class s or not. Hence in our case $(0, 0)$ is even, while $(0, 1)$ is odd. These numbers have been given by Gauss as follows,

$$16(0,0)_4 = p + 2x - 7, \quad 16(0,1)_4 = p + 2x + 8y + 1. \tag{21}$$

Hence

$$p + 2x - 7 \equiv 0 \pmod{32} \text{ and } p + 2x + 8y + 1 \equiv 16 \pmod{32}.$$

Subtracting the first congruence from the second we have, dividing by 8,

$$y \equiv 1 \pmod 4. \tag{22}$$

This makes (20) unambiguous, and returning to (2) we find by (6), since $\psi_2(2) = 0$, that for $(2/p) = -1$

$$\psi_4(2) = \phi_2(2) = -4y, \qquad y \equiv 1 \pmod 4. \tag{23}$$

Hence by (7)

$$\phi_2(2m^2) = -4y\left(\frac{m}{p}\right), \qquad \left(\frac{2}{p}\right) = -1. \tag{24}$$

This gives a slight strengthening of Jacobsthal's theorem, namely:

THEOREM 2. *If 2 is a nonresidue of* $p = x^2 + 4y^2$, *where* $x \equiv y \equiv 1 \pmod 4$, *then*

$$\phi_2(D) = \begin{cases} -2x\left(\dfrac{m}{p}\right), & \text{if } D \equiv m^2 \pmod p \\[2ex] -4y\left(\dfrac{m}{p}\right), & \text{if } D \equiv 2m^2 \pmod p. \end{cases}$$

Hence by (2) we have:

THEOREM 3. *If 2 is a nonresidue of* $p = x^2 + 4y^2$, $x \equiv y \equiv 1 \pmod 4$ *then the number of solutions of* $u^4 + D \equiv w^2 \pmod p$ *is given by*

$$N_4(D) = \begin{cases} p - 1 - 2x\left(\dfrac{m}{p}\right), & \text{if } D \equiv m^2 \pmod p \\[2ex] p - 1 - 4y\left(\dfrac{m}{p}\right), & \text{if } D \equiv 2m^2 \pmod p. \end{cases}$$

We now suppose that 2 is a quadratic residue but a quartic nonresidue, hence we may choose g such that $\sqrt{2}$ belongs to class 1 and calculate $N(\sqrt{2})$ by (18). The cyclotomic constants of order 4 for $p = 8n + 1$ are

$$16(3,1)_4 = p - 2x + 1, \quad 16(3,3)_4 = p + 2x + 8y - 3. \tag{25}$$

Hence by (18)

$$N_4(\sqrt{2}) = p - 1 + 4y; \tag{26}$$

but in this case y turns out to be even, so that it is not sufficient to determine y modulo 4 and it is necessary to introduce the cyclotomic numbers of order 8 to determine the sign of y. It also becomes necessary to distinguish the cases $p = 16n + 1$ and $16n + 9$.

Case 1. $p = 16n + 1 = x^2 + 4y^2 = a^2 + 2b^2$, $x \equiv a \equiv 1 \pmod 4$.

Since $\sqrt{2}$ belongs to class 1, 2 belongs to class 2 and by our lemma $(0,0)_8$ is even, while $(0,2)_8$ is odd. Dickson [2] gives

$$64(0,0)_8 = p - 23 + 6x. \tag{27}$$

Since $(0,0)_8$ is even, we have

$$6x \equiv -p + 23 \pmod{128}. \tag{28}$$

In order to complete our discussion it was necessary to calculate $(0,2)_8$ and $(1,2)_8$ by solving 15 linear equations involving the constants $(i,j)_8$ given by Dickson, which we list in the Appendix. We obtained

$$64(0,2)_8 = p - 7 - 2x - 16y - 8a,\quad 64(1,2)_8 = p + 1 - 6x + 4a. \tag{29}$$

Substituting $p - 23$ for $-6x$ from (28) into $64(1,2)_8$ we obtain

$$2a \equiv 11 - p \pmod{32}. \tag{30}$$

Since $(0,2)_8$ is odd we have, multiplying (29) by 3,

$$\begin{aligned}3p - 21 - 6x - 48y - 24a &\equiv 3p - 21 + (p - 23) - 48y - 12(11 - p)\\ &\equiv 64 \pmod{128};\end{aligned} \tag{31}$$

or, dividing out a 16 and solving for y, we get

$$y \equiv 3(p + 1) \equiv -2 \pmod 8. \tag{32}$$

Case 2. $p = 16n + 9$. In this case Dickson gives

$$64(0,4)_8 = p + 1 + 6x + 24a, \tag{33}$$

while we have calculated [see Appendix]

$$(34)\qquad 64(0,2)_8 = p + 1 - 2x + 16y\ ,$$

$$(35)\qquad 64(2,0)_8 = p - 7 + 6x\ ,$$

$$(36)\qquad 64(1,2)_8 = p + 1 + 2x - 4a\ .$$

From (35)

$$(37)\qquad 6x \equiv 7 - p \pmod{64}.$$

Substituting this into (36) we find

$$(38)\qquad 12a \equiv 2p + 10 \pmod{64}.$$

Since $(0,4)_8$ is even we obtain, using (38),

$$(39)\qquad p + 1 + 6x + 24a \equiv p + 1 + 6x + 4p + 20 \equiv 0 \pmod{128}.$$

This gives an improvement of (37), namely,

$$(40)\qquad 6x \equiv -(5p + 21) \pmod{128}.$$

Finally substituting all this into $(0,2)_8$ which is odd, we have, after multiplying (34) by 3,

$$3p + 3 - 6x + 48y \equiv 3p + 3 + 5p + 21 + 48y \equiv 8p + 24 + 48y \equiv 64 \pmod{128},$$

or dividing out an 8 and noting that $p \equiv 9 \pmod{16}$ we obtain

$$y \equiv +2 \pmod 8.$$

Hence the sign of y in (26) is now determined as follows if $(\sqrt{2}/p) = -1$:

$$(41)\qquad N_4(\sqrt{2}) = p - 1 + 4y,\ \text{ where } y/2 \equiv -(-1)^{(p-1)/8} \pmod 4.$$

From this we have as before by (2) and (6) for $(\sqrt{2}/p) = -1$:

$$(42)\qquad \psi_4(\sqrt{2}) = \phi_2(\sqrt{2}) = -4y,\ \text{ where } y/2 \equiv (-1)^{(p-1)/8} \pmod 4,$$

and we can write a slight improvement of Jacobsthal's theorem in the case in which 2 is a quadratic but not a quartic residue of p:

THEOREM 4. *If 2 is a quadratic residue, but a quartic nonresidue of* $p = x^2 + 4y^2 = 8n + 1$, *then*

$$\varphi_2(D) = \begin{cases} -2x\left(\dfrac{m}{p}\right) & \text{if } D \equiv m^2 \pmod p \\ -4y\left(\dfrac{m}{p}\right) & \text{if } D \equiv \sqrt{2}m^2 \pmod p, \end{cases}$$

where $x \equiv 1 \pmod 4$ *and* $y/2 \equiv (-1)^n \pmod 4$.

THEOREM 5. *If 2 is a quadratic residue, but a quartic nonresidue of* $p = x^2 + 4y^2 = 8n + 1$, *then the number of solutions* (u, w) *of* $u^4 + D \equiv w^2 \pmod p$ *is given by*

$$N_4(D) = \begin{cases} p - 1 - 2x\left(\dfrac{m}{p}\right) & \text{if } D \equiv m^2 \pmod p \\ p - 1 - 4y\left(\dfrac{m}{p}\right) & \text{if } D \equiv \sqrt{2}m^2 \pmod p, \end{cases}$$

where $x \equiv 1 \pmod 4$ *and* $y/2 \equiv (-1)^n \pmod 4$.

In order to obtain an improvement on Jacobsthal's theorem in the case in which 2 is a quartic residue, or to improve the results for ϕ_4 and ψ_4 in order to obtain N_8, it appears necessary to examine the cyclotomic constants of order 16, or to go through a determination of a specified primitive root as in Vandiver [7a]. The known results for ϕ_4 and ψ_4 are as follows:

$$\phi_4(D) = \begin{cases} -4a\left(\dfrac{m}{p}\right) & \text{if } D \equiv m^4 \pmod p \\ 0 & \text{if } D \equiv m^2 \not\equiv m_1^4 \pmod p \\ \pm 4b & \text{otherwise}, \end{cases}$$

and

$$\psi_4(D) = \begin{cases} -2x\left(\dfrac{m}{p}\right) - 2 & \text{if } D \equiv m^2 \pmod p \\ \pm 4y & \text{otherwise}. \end{cases}$$

It follows from this that

$$(43)\qquad N_8(D) = \begin{cases} p - 1 - 2x - 4a\left(\dfrac{m}{p}\right) & \text{if } D \equiv m^4 \pmod p \\ p - 1 + 2x\left(\dfrac{m}{p}\right) & \text{if } D \equiv m^2 \not\equiv m_1^4 \pmod p \\ p - 1 \pm 4b \pm 4y & \text{otherwise.} \end{cases}$$

3. Case $k = 3$. The known results for the case $k = 3$ can be stated as follows:

$$(44)\qquad \phi_3(D) = \begin{cases} -2A - 1 & \text{if } D \text{ is a cubic residue} \\ A \pm 3B - 1 & \text{if } D \text{ is a cubic nonresidue,} \end{cases}$$

where $p = A^2 + 3B^2 = 6n + 1$, $A \equiv 1 \pmod 3$.

This can be obtained either by summing the appropriate cyclotomic constants of order 6, or by using the results of Schrutka or Chowla, as was done in Whiteman [8]. From this it follows by (2) and (5) that

$$(45)\qquad N_3(D) = \begin{cases} p - \left(\dfrac{D}{p}\right) 2A & \text{if } D \text{ is a cubic residue} \\ p + \left(\dfrac{D}{p}\right)(A \pm 3B) & \text{if } D \text{ is a cubic nonresidue.} \end{cases}$$

We are again faced with an ambiguity in sign in case D is a cubic nonresidue, which can be resolved in case 2 is a cubic nonresidue. For in this case by (9)

$$(46)\qquad \phi_3(1) + \phi_3(2) + \phi_3(4) = -3.$$

By (44), $\phi_3(1) = -2A - 1$, while Chowla proved that $\phi_3(4) = L - 1$, where $4p = L^2 + 27M^2$, $L \equiv 1 \pmod 3$. Hence by (46)

$$(47)\qquad \phi_3(2) = 2A - L - 1 \quad (2 \text{ a cubic nonresidue}).$$

Hence by (7) we can write a slight generalization of Chowla's or Schrutka's theorem:

THEOREM 6. *If 2 is a cubic nonresidue of* $p = A^2 + 3B^2$, *and if* $4p = L^2 + 27M^2$, $A \equiv L \equiv 1 \pmod 3$, *then*

$$\phi_3(D) = \begin{cases} -(2A+1) & \text{if } D \equiv m^3 \pmod p \\ 2A - L - 1 & \text{if } D \equiv 2m^3 \pmod p \\ L - 1 & \text{if } D \equiv 4m^3 \pmod p. \end{cases}$$

Using (5) and (2) we obtain the corresponding theorem for $N_3(D)$:

THEOREM 7. *If 2 is a cubic nonresidue of* $p = A^2 + 3B^2$, *and if* $4p = L^2 + 27M^2$, $A \equiv L \equiv 1 \pmod 3$, *then*

$$N_3(D) = \begin{cases} p - \left(\frac{D}{p}\right) 2A & \text{if } D \equiv m^3 \pmod p \\ p + \left(\frac{D}{p}\right) L & \text{if } D \equiv 2m^3 \pmod p \\ p + \left(\frac{D}{p}\right)(2A - L) & \text{if } D \equiv 4m^3 \pmod p. \end{cases}$$

For $k = 6$, it follows from (10) by substituting the values for $\phi_3(D)$ from (44) (remembering that D and $\bar{D}$ are either both cubic residues, or both non-residues), that:

$$(48) \qquad \psi_6(D) = \begin{cases} -(2A+1)\left[1 + \left(\frac{D}{p}\right)\right] & \text{if } D \text{ is a cubic residue} \\ (A-1)\left[1 + \left(\frac{D}{p}\right)\right] \pm 3B\left[1 - \left(\frac{D}{p}\right)\right] & \text{otherwise}. \end{cases}$$

Substituting this into (2) we have

$$(49) \qquad N_6(D) = \begin{cases} p - 2A\left[1 + \left(\frac{D}{p}\right)\right] - 1 & \text{if } D \text{ is a cubic residue} \\ p + A\left[1 + \left(\frac{D}{p}\right)\right] \pm 3B\left[1 - \left(\frac{D}{p}\right)\right] - 1 & \text{otherwise}. \end{cases}$$

In case 2 is a cubic nonresidue, however, we can substitute more exact values for $\phi_3(D)$ from Theorem 6 into (10) to obtain:

THEOREM 7. *If 2 is a cubic nonresidue of* $p = A^2 + 3B^2$ *and if* $4p = L^2 + 27M^2$. $A \equiv L \equiv 1 \pmod 3$, *then*

$$\psi_6(D) = \begin{cases} -(2A+1)\left[1+\left(\frac{D}{p}\right)\right] & \text{if } D \equiv m^3 \pmod{p} \\ 2A + L\left[\left(\frac{D}{p}\right)-1\right] - \left[1+\left(\frac{D}{p}\right)\right] & \text{if } D \equiv 2m^3 \pmod{p} \\ \left(\frac{D}{p}\right) 2A - L\left[\left(\frac{D}{p}\right)-1\right] - \left[1+\left(\frac{D}{p}\right)\right] & \text{if } D \equiv 4m^3 \pmod{p}. \end{cases}$$

Substituting these values intc (2) we obtain:

THEOREM 8. *If 2 is a cubic nonresidue of* $p = A^2 + 3B^2$ *and if* $4p = L^2 + 27M^2$, $A \equiv L \equiv 1 \pmod 3$, *then the number of solutions of* $u^6 + D \equiv v^2 \pmod{p}$ *is given by*

$$N_6(D) = \begin{cases} p-1-2A\left[1+\left(\frac{D}{p}\right)\right] & \text{if } D \equiv m^3 \pmod{p} \\ p-1+2A+L\left[\left(\frac{D}{p}\right)-1\right] & \text{if } D \equiv 2m^3 \pmod{p} \\ p-1+\left(\frac{D}{p}\right) 2A - L\left[\left(\frac{D}{p}\right)-1\right] & \text{if } D \equiv 4m^3 \pmod{p}. \end{cases}$$

4. Congruences in three variables. In conclusion we can apply our results to the number of solutions of congruences in three variables. We have:

THEOREM 9. *The number* $N_{k,k}(D)$ *of solutions* (u, v, w) *of*

$$u^k + Dv^k \equiv w^2 \pmod{p} \tag{50}$$

is

$$N_{k,k}(D) = \begin{cases} p^2 & \text{if } k \text{ is odd} \\ p^2 + (p-1)\left[1+\left(\frac{D}{p}\right)+\psi_k(D)\right] & \text{if } k \text{ is even.} \end{cases}$$

Proof. Replacing D by $D\nu^k$ in (2) and summing over $\nu = 1, 2, \cdots, p-1$, we obtain

$$\sum_{\nu=1}^{p-1} N_k(D\nu^k) = p(p-1) + \left(\frac{D}{p}\right)\sum_{\nu=1}^{p-1}\left(\frac{\nu}{p}\right)^k + \sum_{\nu=1}^{p-1}\psi_k(\nu^k D).$$

By (7) this becomes

$$\sum_{\nu=1}^{p-1} N_k(D\nu^k) = p(p-1) + \left(\frac{D}{p}\right) \sum_{\nu=1}^{p-1} \left(\frac{\nu}{p}\right)^k + \psi_k(D) \sum_{\nu=1}^{p-1} \left(\frac{\nu}{p}\right)^k.$$

But

$$\sum_{\nu=1}^{p-1} \left(\frac{\nu}{p}\right)^k = \begin{cases} 0 & k \text{ odd} \\ p-1 & k \text{ even}, \end{cases}$$

while the number of solutions with $\nu = 0$ is p for k odd and $2p-1$ for k even. Hence

$$N_{k,k}(D) = \begin{cases} p(p-1) + p = p^2 & \text{for } k \text{ odd} \\ p(p-1) + (p-1)\left[\left(\frac{D}{p}\right) + \psi_k(D)\right] + 2p - 1, & k \text{ even}. \end{cases}$$

Hence the theorem.

Using the expressions derived for special values of k earlier we can write down the following special cases:

$$N_{2,2}(D) = p^2.$$

By (14),

$$N_{4,4}(D) = p^2 - 2x\left(\frac{\sqrt{D}}{p}\right)(p-1) \quad \text{if} \left(\frac{D}{p}\right) = +1, \ x \equiv 1 \pmod 4.$$

By (24),

$$N_{4,4}(2m^2) = p^2 - 4y(p-1) \qquad \text{if} \left(\frac{2}{p}\right) = -1 \text{ and } y \equiv 1 \pmod 4.$$

By (42),

$$N_{4,4}(\sqrt{2}m^2) = p^2 - 4y(p-1) \quad \text{if } \frac{\sqrt{2}}{p} = -1 \text{ and } y/2 \equiv (-1)^{(p-1)/8} \pmod 4.$$

By (48),

$$N_{6,6}(m^3) = p^2 - 2A\left[1 + \left(\frac{m}{p}\right)\right](p-1).$$

By Theorem 7,

$$\left.\begin{aligned} N_{6,6}(2m^3) &= p^2 + \left\{2A + L\left[\left(\frac{m}{p}\right) - 1\right]\right\}(p-1) \\ N_{6,6}(4m^3) &= p^2 + \left\{\left(\frac{m}{p}\right)2A - L\left[\left(\frac{m}{p}\right) - 1\right]\right\}(p-1) \end{aligned}\right\} \text{ if 2 is a cubic nonresidue.}$$

By (43),

$$N_{8,8}(m^4) = p^2 - \left[2x + 4a\left(\frac{m}{p}\right)\right](p-1).$$

We note that $N_{6,6}(m^3) = p^2$ if m is a nonresidue. It can be readily seen that this is a special case of a general theorem, namely:

THEOREM 10. *If k is oddly even and D is a $k/2$th power residue, but not a kth power residue, then*

$$N_{k,k}(D) = p^2.$$

This follows from Theorem 9 and the fact that the corresponding $\psi_k(D)$ is zero in this case by (11).

We hope to take up the cases $k = 5$ and $k = 10$ in a future paper.

APPENDIX: **Cyclotomic constants of order 8.**

The 64 constants $(i, j)_8$ have at most 15 different values for a given p. These values are expressible in terms of p, x, y, a and b in

$$p = x^2 + 4y^2 = a^2 + 2b^2, \quad (x \equiv a \equiv 1 \pmod 4)).$$

There are two cases.

Case I. $p = 16n + 1$.

Table of $(i, j)_8$

$j \backslash i$	0	1	2	3	4	5	6	7
0	(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
1	(0,1)	(0,7)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,2)
2	(0,2)	(1,2)	(0,6)	(1,6)	(2,4)	(2,5)	(2,4)	(1,3)
3	(0,3)	(1,3)	(1,6)	(0,5)	(1,5)	(2,5)	(2,5)	(1,4)
4	(0,4)	(1,4)	(2,4)	(1,5)	(0,4)	(1,4)	(2,4)	(1,5)
5	(0,5)	(1,5)	(2,5)	(2,5)	(1,4)	(0,3)	(1,3)	(1,6)
6	(0,6)	(1,6)	(2,4)	(2,5)	(2,4)	(1,3)	(0,2)	(1,2)
7	(0,7)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,2)	(0,1)

These 15 fundamental constants $(0,0), \cdots, (2,5)$ are given by the relations contained in the following table.

	If 2 is a quartic residue	If 2 is not a quartic residue
$64(0,0)$	$p - 23 - 18x - 24a$	$p - 23 + 6x$
$64(0,1)$	$p - 7 + 2x + 4a + 16y + 16b$	$p - 7 + 2x + 4a$
$64(0,2)$	$p - 7 + 6x + 16y$	$p - 7 - 2x - 8a - 16y$
$64(0,3)$	$p - 7 + 2x + 4a - 16y + 16b$	$p - 7 + 2x + 4a$
$64(0,4)$	$p - 7 - 2x + 8a$	$p - 7 - 10x$
$64(0,5)$	$p - 7 + 2x + 4a + 16y - 16b$	$p - 7 + 2x + 4a$
$64(0,6)$	$p - 7 + 6x - 16y$	$p - 7 - 2x - 8a + 16y$
$64(0,7)$	$p - 7 + 2x + 4a - 16y - 16b$	$p - 7 + 2x + 4a$
$64(1,2)$	$p + 1 + 2x - 4a$	$p + 1 - 6x + 4a$
$64(1,3)$	$p + 1 - 6x + 4a$	$p + 1 + 2x - 4a - 16b$
$64(1,4)$	$p + 1 + 2x - 4a$	$p + 1 + 2x - 4a + 16y$
$64(1,5)$	$p + 1 + 2x - 4a$	$p + 1 + 2x - 4a - 16y$
$64(1,6)$	$p + 1 - 6x + 4a$	$p + 1 + 2x - 4a + 16b$
$64(2,4)$	$p + 1 - 2x$	$p + 1 + 6x + 8a$
$64(2,5)$	$p + 1 + 2x - 4a$	$p + 1 - 6x + 4a$

Case II. $p = 16n + 9$.

Table of $(i, j)_8$

$j \backslash i$	0	1	2	3	4	5	6	7
0	(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
1	(1,0)	(1,1)	(1,2)	(1,3)	(0,5)	(1,3)	(0,3)	(1,7)
2	(2,0)	(2,1)	(2,0)	(1,7)	(0,6)	(1,3)	(0,2)	(1,2)
3	(1,1)	(2,1)	(2,1)	(1,0)	(0,7)	(1,7)	(1,2)	(0,1)
4	(0,0)	(1,0)	(2,0)	(1,1)	(0,0)	(1,0)	(2,0)	(1,1)
5	(1,0)	(0,7)	(1,7)	(1,2)	(0,1)	(1,1)	(2,1)	(2,1)
6	(2,0)	(1,7)	(0,6)	(1,3)	(0,2)	(1,2)	(2,0)	(2,1)
7	(1,1)	(1,2)	(1,3)	(0,5)	(0,3)	(1,6)	(1,3)	(1,0)

where

	If 2 is a quartic residue	If 2 is not a quartic residue
$64(0,0)$	$p - 15 - 2x$	$p - 15 - 10x - 8a$
$64(0,1)$	$p + 1 + 2x - 4a + 16y$	$p + 1 + 2x - 4a - 16b$
$64(0,2)$	$p + 1 + 6x + 8a - 16y$	$p + 1 - 2x + 16y$
$64(0,3)$	$p + 1 + 2x - 4a - 16y$	$p + 1 + 2x - 4a - 16b$
$64(0,4)$	$p + 1 - 18x$	$p + 1 + 6x + 24a$
$64(0,5)$	$p + 1 + 2x - 4a + 16y$	$p + 1 + 2x - 4a + 16b$
$64(0,6)$	$p + 1 + 6x + 8a + 16y$	$p + 1 - 2x - 16y$
$64(0,7)$	$p + 1 + 2x - 4a - 16y$	$p + 1 + 2x - 4a + 16b$
$64(1,0)$	$p - 7 + 2x + 4a$	$p - 7 + 2x + 4a + 16y$
$64(1,1)$	$p - 7 + 2x + 4a$	$p - 7 + 2x + 4a - 16y$
$64(1,2)$	$p + 1 - 6x + 4a + 16b$	$p + 1 + 2x - 4a$
$64(1,3)$	$p + 1 + 2x - 4a$	$p + 1 - 6x + 4a$
$64(1,7)$	$p + 1 - 6x + 4a - 16b$	$p + 1 + 2x - 4a$
$64(2,0)$	$p - 7 - 2x - 8a$	$p - 7 + 6x$
$64(2,1)$	$p + 1 + 2x - 4a$	$p + 1 - 6x + 4a$

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