

Pacific Journal of Mathematics

ON THE DIVISIBILITY OF THE CLASS NUMBER OF QUADRATIC FIELDS

NESMITH CORNETT ANKENY AND S. CHOWLA

ON THE DIVISIBILITY OF THE CLASS NUMBER OF QUADRATIC FIELDS

N. C. ANKENY AND S. CHOWLA

1. Introduction. It is well known that there exist infinitely many quadratic extensions of the rationals each with class number divisible by 2. In fact, if the discriminant of the field contains more than two prime factors, then 2 divides the class number. Max Gut [1] generalized this result to show that there exist infinitely many quadratic imaginary fields each with class number divisible by 3. In this present paper we prove that there exist infinitely many quadratic imaginary fields each with class number divisible by g where g is any given rational integer.

The method extends to yield certain results about quadratic real fields, but these are not as sharp as on quadratic imaginary fields.

2. Theorem. In the following we may assume without loss of generality that g is positive, sufficiently large, and even.

LEMMA 1. Denote by N the number of square-free integers of the form

$$3^g - x^2, \text{ where } 2 \mid x, 0 < x < (2 \cdot 3^{g-1})^{1/2}.$$

Then, for g sufficiently large,

$$N \geq \frac{1}{25} 3^{g/2}.$$

Proof. Denote by d the expression

$$(1) \quad d = 3^g - x^2,$$

where

$$(2) \quad 2 \mid x, \quad 0 < x < (2 \cdot 3^{g-1})^{1/2}.$$

Received September 28, 1953.

Pacific J. Math. 5 (1955), 321-324

The number of such d is

$$\frac{1}{2} (2 \cdot 3^{g-1})^{1/2} + O(1).$$

As $2 \mid x$, none of the d 's are divisible by 2. The number of d divisible by 3, and, hence, by 9, is less than

$$\frac{1}{6} (2 \cdot 3^{g-1})^{1/2} + O(1).$$

For p an odd prime greater than 3, the number of d divisible by p^2 is less than

$$\frac{1}{2p^2} (2 \cdot 3^{g-1})^{1/2} + 2.$$

Hence the number of square-free d is

$$\begin{aligned} N &\geq \frac{1}{2} (2 \cdot 3^{g-1})^{1/2} - \frac{1}{6} (2 \cdot 3^{g-1})^{1/2} + O(1) - \sum_{\substack{p \geq 5 \\ p^2 < 3^g}} \frac{1}{2p^2} (2 \cdot 3^{g-1})^{1/2} + 2 \\ &\geq \frac{1}{2} (2 \cdot 3^{g-1})^{1/2} \left(1 - \frac{1}{3} - \sum_{p \geq 5} \frac{1}{p^2} \right) - 2 \sum_{p^2 < 3^g} 1 + O(1) \\ &\geq \frac{1}{2} (2 \cdot 3^{g-1})^{1/2} \left(1 - \frac{1}{3} - \sum_{n=5} \frac{1}{n^2} \right) + O\left(\frac{1}{g} 3^{g/2}\right), \end{aligned}$$

by the prime-number theorem. Hence

$$N \geq \frac{1}{2} (2 \cdot 3^{g-1})^{1/2} \left(1 - \frac{1}{3} - \frac{1}{4} \right) + O\left(\frac{1}{g} 3^{g/2}\right) \geq \frac{1}{25} 3^{g/2}.$$

THEOREM 1. *For the square-free integers d which satisfy (1) and (2) we have $g \mid h$, where h denotes the class number of the field $R(\sqrt{-d})$.*

Proof. Consider the quadratic extension of the rationals $R(\sqrt{-d})$. Since

$$3^g = x^2 + d,$$

where x is prime to 3 as d is square free, we see that

$$x^2 + d \equiv O \pmod{3}.$$

Hence, by the well-known criterion for the splitting of rational primes in quadratic extensions, $(3) = P_1 P_2$ where (3) denotes the principal ideal generated by 3 in $R(\sqrt{-d})$, and P_1, P_2 are two distinct conjugate prime ideals in $R(\sqrt{-d})$.

Let m be the least positive integer such that P_1^m is a principal ideal in $R(\sqrt{-d})$. If possible let $m < g$, and $P_1^m = (\alpha)$ for some integer $\alpha \in R(\sqrt{-d})$. Since $2 \mid g$, we have $2 \mid x$, and, by (1), $d \equiv 1 \pmod{4}$. Then

$$\alpha = u + v\sqrt{-d}$$

for rational integers u and v .

Then

$$(3^m) = P_1^m P_2^m = (u + v\sqrt{-d})(u - v\sqrt{-d}) = (u^2 + v^2d),$$

or

$$(3) \quad 3^m = u^2 + v^2d.$$

By (1) and (2), we have $d > 3^{g-1}$; but if $m < g$, (3) implies

$$3^{g-1} \geq u^2 + v^2d,$$

so $v = 0$. But then

$$P_1^m = (u), P_2^m = (u), \quad \text{or} \quad P_1^m = P_2^m, P_1 = P_2,$$

which is false as P_1, P_2 are two distinct prime ideals in $R(\sqrt{-d})$.

Thus we have shown that $m \geq g$; but as $3^g = x^2 + d$, $m = g$. Hence, there exists in $R(\sqrt{-d})$ a prime ideal P_1 whose g th power but none lower is a principal ideal. This immediately implies $g \mid h$.

3. Application. To show that there exist infinitely many fields each with class number divisible by g , we proceed as follows. Theorem 1 shows that there are at least $(1/25) 3^{g/2}$ with class number divisible by g . Let $g^t = g_1$ be such that the class number of none of these fields is divisible by g_1 . Then, as before, we find at least $(1/25) 3^{g_1/2}$ fields with class number divisible by g_1 . These fields must be distinct from the previous fields. Repeating this method we see there exist infinitely many quadratic fields with class number divisible

by g .

4. A further result. We shall prove:

THEOREM 2. *If d is square free number of the form $d = n^{2g} + 1$, where $n > 4$, then $g \mid h$, where h is the class number of the field $R(\sqrt{d})$.*

Proof. We need only outline the proof of Theorem 2, as in most aspects it is very similar to the proof of Theorem 1. We first show that

$$(n) = \mathfrak{X}\mathfrak{X}' \text{ in } R(\sqrt{d}),$$

where \mathfrak{X} , \mathfrak{X}' are two relatively prime conjugate ideals. We then show that $u^2 - dv^2$ (u, v integers) represents no integer other than 0 and 1 whose absolute value is less than \sqrt{d} . This follows from the fact that d is of the form $d = w^2 + 1$. Hence the least power of \mathfrak{X} which is a principal ideal is the g th power. This immediately implies $g \mid h$.

The interest of Theorem 2 is somewhat lessened by the fact that it is unknown at present if there exists an infinite number of square-free numbers of the form $n^{2g} + 1$. Hence we are unable to prove a theorem similar to Theorem 1 with regard to quadratic real extensions of the rationals.

REFERENCE

1. Max Gut, *Kubische Klassenkörper über quadratischimaginären Grundkörpern*, Nieuw Arch. Wiskunde (2) **23** (1951), 185-189.

JOHNS HOPKINS UNIVERSITY,
UNIVERSITY OF COLORADO

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H.L. ROYDEN

Stanford University
Stanford, California

E. HEWITT

University of Washington
Seattle 5, Washington

R. P. DILWORTH

California Institute of Technology
Pasadena 4, California

* Alfred Horn

University of California
Los Angeles 24, California

ASSOCIATE EDITORS

H. BUSEMANN

HERBERT FEDERER

MARSHALL HALL

P.R. HALMOS

HEINZ HOPF

ALFRED HORN

R.D. JAMES

BØRGE JESSEN

PAUL LÉVY

GEORGE PÓLYA

J.J. STOKER

KOSAKU YOSIDA

SPONSORS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA, BERKELEY
UNIVERSITY OF CALIFORNIA, DAVIS
UNIVERSITY OF CALIFORNIA, LOS ANGELES
UNIVERSITY OF CALIFORNIA, SANTA BARBARA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
OREGON STATE COLLEGE
UNIVERSITY OF OREGON

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD RESEARCH INSTITUTE
STANFORD UNIVERSITY
UNIVERSITY OF UTAH
WASHINGTON STATE COLLEGE
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
HUGHES AIRCRAFT COMPANY
SHELL DEVELOPMENT COMPANY

UNIVERSITY OF SOUTHERN CALIFORNIA

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors. Manuscripts intended for the outgoing editors should be sent to their successors. All other communications to the editors should be addressed to the managing editor, Alfred Horn, at the University of California Los Angeles 24, California.

50 reprints of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50; back numbers (Volumes 1, 2, 3) are available at \$2.50 per copy. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to the publishers, University of California Press, Berkeley 4, California.

Printed at Ann Arbor, Michigan. Entered as second class matter at the Post Office, Berkeley, California.

* During the absence of E.G. Straus.

UNIVERSITY OF CALIFORNIA PRESS • BERKELEY AND LOS ANGELES

COPYRIGHT 1955 BY PACIFIC JOURNAL OF MATHEMATICS

Pacific Journal of Mathematics

Vol. 5, No. 3

November, 1955

Nesmith Cornett Ankeny and S. Chowla, <i>On the divisibility of the class number of quadratic fields</i>	321
Cecil Edmund Burgess, <i>Collections and sequences of continua in the plane</i>	325
Jane Smiley Cronin Scanlon, <i>The Dirichlet problem for nonlinear elliptic equations</i>	335
Arieh Dvoretzky, <i>A converse of Helly's theorem on convex sets</i>	345
Branko Grünbaum, <i>On a theorem of L. A. Santaló</i>	351
Moshe Shimrat, <i>Simple proof of a theorem of P. Kirchberger</i>	361
Michael Oser Rabin, <i>A note on Helly's theorem</i>	363
Robert E. Edwards, <i>On factor functions</i>	367
Robert E. Edwards, <i>On certain algebras of measures</i>	379
Harley M. Flanders, <i>Methods in affine connection theory</i>	391
Alfred Huber, <i>The reflection principle for polyharmonic functions</i>	433
Geoffrey Stuart Stephen Ludford, <i>Generalised Riemann invariants</i>	441
Ralph Gordon Selfridge, <i>Generalized Walsh transforms</i>	451