

Pacific Journal of Mathematics

A THEOREM ON ALTERNATIVES FOR PAIRS OF MATRICES

HENRY A. ANTOSIEWICZ

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The theory of linear inequalities has come into prominence anew in recent years because of its importance in the solution of linear programming problems. In this note we present a simple algebraic proof of an interesting theorem on alternatives for pairs of matrices. This problem was suggested by A. W. Tucker.

Let A and B be matrices, n by m and n by p , respectively, and let x , y , u be column vectors of dimensions m , p , n , respectively.

STATEMENT I. *Either $A'u > 0$, $B'u \geq 0$ for some u or $Ax + By = 0$ for some $x \geq 0$, $y \geq 0$.¹*

STATEMENT II. *Either $A'u \geq 0$, $B'u \geq 0$ for some u or $Ax + By = 0$ for some $x > 0$, $y \geq 0$. [7].*

We shall prove the following theorem.

THEOREM. *Statement I implies, and is implied by, Statement II.*

Note that for the special case when $A = -a$ (column vector) Statement I (or II) reduces to a result of Farkas [2]. If $B = 0$, then Statements I and II are two theorems of Stiemke [6]. More importantly, if the matrix $[B, C, -C]$ is substituted for B , where C is a n by q matrix, and y is replaced by the vector $\begin{bmatrix} y \\ y_1 \\ y_2 \end{bmatrix}$, then Statement I gives the well-known transposition theorem of Motzkin [4, 5]. We refer to [4] for several proofs and further references.

Before proving our theorem, let us make the following preliminary observations. Define the matrix $M = [A, B]$ and the column vector $z = \begin{bmatrix} x \\ y \end{bmatrix}$, and consider the system of equations $Mz = 0$. Assume that the vectors s_1, s_2, \dots, s_k span the linear manifold \mathcal{S} of solutions of this system. Then every solution z can be written in the form $z = S'c$ where $S' = [s_1, s_2, \dots, s_k]$ and c is a k -dimensional (column) vector. Observe that the rows of the matrix M span the orthogonal complement \mathcal{S}^* of \mathcal{S} , that is, every solution of the system $Sz^* = 0$ can be represented as $z^* = M'd$ where d is a n -dimensional (column) vector.

It will be convenient to write $S = [S_1, S_2]$ where S_1 and S_2 are the k by m and k by p matrices, respectively, into which S can be parti-

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¹ Throughout, transposition is indicated by a dash; also, $x \geq 0$ means $x \geq 0$ with $x = 0$ excluded.

tioned; accordingly, we introduce two column vectors v, w with m and p components, respectively, and write $z^* = \begin{bmatrix} v \\ w \end{bmatrix}$.

Clearly, the alternatives in each Statement are mutually exclusive as can be seen by multiplying $Ax + By = 0$ on the left by u' . To prove the theorem suppose, at first, that $A'u \geq 0$, $B'u \geq 0$ for no u and $Ax + By = 0$ has no solution $x > 0$, $y \geq 0$. Then there exists no c such that

$$S'_1 c > 0, \quad S'_2 c \geq 0.$$

Hence, by Statement I, the system $S_1 v + S_2 w = 0$ must be satisfied for some $v \geq 0$, $w \geq 0$. Since every solution of

$$Sz^* \equiv S_1 v + S_2 w = 0$$

is of the form $z^* = M'd$, there must exist a vector d such that $A'd \geq 0$, $B'd \geq 0$, which is a contradiction. Thus Statement I implies Statement II. Conversely, if $A'u > 0$, $B'u \geq 0$ for no u and $Ax + By = 0$ has no solution $x > 0$, $y \geq 0$, then there exists no c such that $S'_1 c \geq 0$, $S'_2 c \geq 0$. Hence, by Statement II, the system $S_1 v + S_2 w = 0$ must be satisfied for some $v > 0$, $w \geq 0$, that is, there must exist a vector d such that $A'd > 0$, $B'd \geq 0$; but this is a contradiction. Thus Statement II implies Statement I.

For applications to linear programming Statements I and II are modified by adjoining in them the inequality $u \geq 0$ to $B'u \geq 0$, that is, by replacing the matrix B by $[B, I]$; in this form they can be used to prove the duality theorem, [1, 3].

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