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A THEOREM ON ALTERNATIVES FOR PAIRS OF MATRICES

H. A. ANTOSIEWICZ

The theory of linear inequalities has come into prominence anew in recent years because of its importance in the solution of linear programming problems. In this note we present a simple algebraic proof of an interesting theorem on alternatives for pairs of matrices. This problem was suggested by A. W. Tucker.

Let A and B be matrices, n by m and n by p, respectively, and let x, y, u be column vectors of dimensions m, p, n, respectively.

STATEMENT I. *Either* $A'u>0$, $B'u\geqq 0$ *for some* u *or* $Ax+By=0$ *for some* $x\geq 0$, $y\geqq 0$.[1]

STATEMENT II. *Either* $A'u\geq 0$, $B'u\geqq 0$ *for some* u *or* $Ax+By=0$ *for some* $x>0$, $y\geqq 0$. [7].

We shall prove the following theorem.

THEOREM. *Statement* I *implies, and is implied by, Statement* II.

Note that for the special case when $A=-a$ (column vector) Statement I (or II) reduces to a result of Farkas [2]. If $B=0$, then Statements I and II are two theorems of Stiemke [6]. More importantly, if the matrix $[B, C, -C]$ is substituted for B, where C is a n by q matrix, and y is replaced by the vector $\begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix}$, then Statement I gives the well-known transposition theorem of Motzkin [4, 5]. We refer to [4] for several proofs and further references.

Before proving our theorem, let us make the following preliminary observations. Define the matrix $M=[A, B]$ and the column vector $z=\begin{bmatrix} x \\ y \end{bmatrix}$, and consider the system of equations $Mz=0$. Assume that the vectors $s_1, s_2, \cdots, s_k$ span the linear manifold $\mathcal{S}$ of solutions of this system. Then every solution z can be written in the form $z=S'c$ where $S'=[s_1, s_2, \cdots, s_k]$ and c is a k-dimensional (column) vector. Observe that the rows of the matrix M span the orthogonal complement $\mathcal{S}^*$ of $\mathcal{S}$, that is, every solution of the system $Sz^*=0$ can be represented as $z^*=M'd$ where d is a n-dimensional (column) vector.

It will be convenient to write $S=[S_1, S_2]$ where S_1 and S_2 are the k by m and k by p matrices, respectively, into which S can be parti-

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[1] Throughout, transposition is indicated by a dash; also, $x\geq 0$ means $x\geqq 0$ with $x=0$ excluded.

tioned; accordingly, we introduce two column vectors v, w with m and p components, respectively, and write $z^* = \begin{bmatrix} v \\ w \end{bmatrix}$.

Clearly, the alternatives in each Statement are mutually exclusive as can be seen by multiplying $Ax+By=0$ on the left by u'. To prove the theorem suppose, at first, that $A'u \geq 0$, $B'u \geqq 0$ for no u and $Ax+By=0$ has *no* solution $x>0$, $y \geqq 0$. Then there exists no c such that

$$S_1'c > 0 , \qquad S_2'c \geqq 0 .$$

Hence, by Statement I, the system $S_1 v + S_2 w = 0$ must be satisfied for some $v \geq 0$, $w \geqq 0$. Since every solution of

$$Sz^* \equiv S_1 v + S_2 w = 0$$

is of the form $z^* = M'd$, there must exist a vector d such that $A'd \geq 0$, $B'd \geqq 0$, which is a contradiction. Thus Statement I implies Statement II. Conversely, if $A'u > 0$, $B'u \geqq 0$ for no u and $Ax+By=0$ has *no* solution $x \geq 0$, $y \geqq 0$, then there exists no c such that $S_1'c \geq 0$, $S_2'c \geqq 0$. Hence, by Statement II, the system $S_1 v + S_2 w = 0$ must be satisfied for some $v > 0$, $w \geqq 0$, that is, there must exist a vector d such that $A'd > 0$, $B'd \geqq 0$; but this is a contradiction. Thus Statement II implies Statement I.

For applications to linear programming Statements I and II are modified by adjoining in them the inequality $u \geqq 0$ to $B'u \geqq 0$, that is, by replacing the matrix B by $[B, I]$; in this form they can be used to prove the duality theorem, [1, 3].

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THE AMERICAN UNIVERSITY

ON SECOND-ORDER NON-LINEAR OSCILLATIONS

F. V. ATKINSON

1. **In this paper** we establish criteria regarding the behaviour, oscillatory or otherwise, near $x=\infty$ of the solutions of

$$y''+fy^{2n-1}=0\ , \tag{1.1}$$

where $f=f(x)$ is positive and continuous for $x\geqslant 0$ and n is an integer greater than 1. A solution, not identically zero, will be said to be *oscillatory* if it has infinitely many zeros for $x\geqslant 0$.

The three possibilities to be distinguished are that the solutions of (1.1) might be (i) all oscillatory, (ii) some oscillatory and some not, and (iii) all nonoscillatory. We give here a necessary and sufficient condition for (i) to hold, and a sufficient condition for (iii).

In the linear case, $n=1$, a number of criteria have been found for cases (i) and (iii); in the linear case (ii) is impossible. A very sensitive procedure is afforded by the chain of logarithmic tests studied by J. C. P. Miller [3], P. Hartman [1], and W. Leighton [2]; some further developments in this field have been given recently by Ruth L. Potter [4], who has in particular a result [Theorem 5.1] bearing on the limitations of this procedure. There does not, however, seem to have been found any simple necessary and sufficient condition for (i) to hold in the linear case, so it is noteworthy that such a criterion exists in the nonlinear case.

2. **The result** in question is:

THEOREM 1. *Let $f=f(x)$ be positive and continuous for $x\geqslant 0$, and let n be an integer greater than unity. Then a necessary and sufficient condition for all solutions of* (1.1) *to be oscillatory is*

$$\int_0^\infty xf\,dx=\infty\ . \tag{2.1}$$

We remark that in the linear case the criterion is necessary but not sufficient.

It should be mentioned that no solution of (1.1) becomes infinite for any finite positive x-value; this is ensured by the positiveness of $f(x)$.

We prove first that if (1.1) has a nonoscillatory solution, then (2.1) cannot hold; this will prove the sufficiency of the criterion.

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Let then y denote a nonoscillatory solution of (1.1); y will then be ultimately of one sign, which we may without loss of generality take to be positive. It follows from (1.1) that y'' will be ultimately negative, so that y' will tend either to a positive limit, or to zero, or to a negative limit, or to $-\infty$. The last two cases can be excluded since they would imply that y is ultimately negative. Thus y must be ultimately monotonic increasing, and y' must tend to a finite nonnegative limit.

We next integrate (1.1) over $(0, x)$, getting

$$y'(x)-y'(0)+\int_0^x fy^{2n-1}dt=0 . \tag{2.2}$$

Since $y'(x)$ tends to a limit as $x\to\infty$, this implies that the integral on the left of (2.2) converges as $x\to\infty$; we may therefore integrate (1.1) over (x, ∞), getting now

$$y'(\infty)-y'(x)+\int_x^\infty fy^{2n-1}dt=0 ,$$

whence, since $y'(\infty)\geqslant 0$,

$$y'(x)\geqslant\int_x^\infty fy^{2n-1}dt \tag{2.3}$$

Still with the assumption that y is ultimately positive, let a be an x-value such that $y(x)>0$ for $x\geqslant a$. We integrate (2.3) over (a, x), where $x>a$, and get

$$y(x)-y(a)\geqslant\int_a^x du\int_u^\infty fy^{2n-1}dt=\int_a^x(t-a)fy^{2n-1}dt+(x-a)\int_x^\infty fy^{2n-1}dt ,$$

and hence, for $x>a$,

$$y(x)\geqslant\int_a^x(t-a)fy^{2n-1}dt ,$$

which we re-write in the form

$$(x-a)fy^{2n-1}\left\{\int_a^x(t-a)fy^{2n-1}dt\right\}^{1-2n}\geqslant(x-a)f . \tag{2.4}$$

We now take any x_1, x_2 such that $a<x_1<x_2$, and integrate (2.4) over (x_1, x_2). This gives

$$(2-2n)^{-1}\left[\left(\int_a^x(t-a)fy^{2n-1}dt\right)^{2-2n}\right]_{x_1}^{x_2}\geqslant\int_{x_1}^{x_2}(x-a)fdx .$$

If now we make $x_2\to\infty$, the left side remains finite; this proves that

$$\int_{x_1}^\infty(x-a)fdx<\infty ,$$

which is equivalent to

$$\int_0^\infty xf\,dx < \infty \ , \tag{2.5}$$

in contradiction of (2.1). Thus the sufficiency of the criterion is proved.

As to the necessity, we shall show that if (2.5) is the case then for any prescribed value of $y(\infty)$, for example 1, there exists a solution of (1.1) such that

$$y(\infty)=1\ , \quad y'(\infty)=0\ , \tag{2.6}$$

which is obviously nonoscillatory.

It is easily verified that if the integral equation

$$y(x)=1-\int_x^\infty (t-x)f(t)\{y(t)\}^{2n-1}dt \tag{2.7}$$

has a solution y which is continuous and uniformly bounded as $x\to\infty$, then it is also a solution of (1.1) with the supplementary conditions (2.6). The existence of a bounded continuous solution of (2.7) may be established by the Picard method of successive approximation. We define a sequence of functions

$$y_m(x) \qquad\qquad (m=0,\,1,\,\cdots)\ , \quad x\geqslant 0\ ,$$

by

$$y_0(x)\equiv 0\ ,$$

$$y_{m+1}(x)=1-\int_x^\infty (t-x)f(t)\{y_m(t)\}^{2n-1}dt \qquad (m=0,\,1,\,\cdots)\ .$$

The remainder of the argument need only be sketched. We can prove by induction that if x is so large that

$$\int_x^\infty (t-x)f(t)dt<1\ ,$$

assuming now (2.5), then $0\leqslant y_m(x)\leqslant 1$. We have also

$$y_{m+2}(x)-y_{m+1}(x)=\int_x^\infty (t-x)f(t)\{(y_m(t))^{2n-1}-(y_{m+1}(t))^{2n-1}\}dt\ ,$$

whence, for sufficiently large x,

$$|y_{m+2}(x)-y_{m+1}(x)|\leqslant(2n-1)\max_{t\geqslant x}|y_m(t)-y_{m+1}(t)|\int_x^\infty (t-x)f(t)dt\ .$$

From this we deduce the convergence of the sequence $y_m(x)$ $(m=0,1,\cdots)$, for x so large that

$$(2n-1)\int_x^\infty (t-x)f(t)dt<1 \ ;$$

the continuity of the limiting function is easily established. This proves the existence of a nonoscillatory solution of (1.1) for sufficiently large x, which is enough for our purpose.

This completes the proof of Theorem 1.

3. **We conclude** with a simple sufficient criterion for nonoscillatory solutions which happens also to be true in the linear case [4, Lemma 1.2].

THEOREM 2. *Let $f(x)$ be positive and continuously differentiable for $x\geqslant 0$, and let $f'\leqslant 0$. Let also*

$$\int_0^\infty x^{2n-1} f\, dx<\infty \ . \tag{3.1}$$

Then (1.1) *has no oscillatory solutions.*

We observe first of all that the result

$$\frac{d}{dx}\left\{\frac{1}{2}y'^2+\frac{1}{2n}fy^{2n}\right\}=\frac{1}{2n}f'y^{2n}\leqslant 0$$

implies that, for any solution, y' remains bounded as $x\to\infty$.

Supposing if possible that (1.1) had an oscillatory solution, let x_0, x_1, $\cdots$ be its successive zeros. Let x_m be for convenience a zero for which $y'(x_m)>0$, and let x'_m be the unique zero of y' in (x_m, x_{m+1}). Integrating (1.1) over (x_m, x'_m), we have

$$y'(x'_m)-y'(x_m)+\int_{x_m}^{x_m'} fy^{2n-1}dx=0 \ ,$$

or

$$y'(x_m)=\int_{x_m}^{x_m'} fy^{2n-1}dx \ . \tag{3.2}$$

Now y' is positive and decreasing in (x_m, x'_m), and $y(x_m)=0$; hence for $x_m\leqslant x\leqslant x'_m$ we have

$$0\leqslant y\leqslant y'(x_m)(x-x_m) \ .$$

Thus from (3.2) we derive

$$y'(x_m)\leqslant\{y'(x_m)\}^{2n-1}\int_{x_m}^{x_m'} f(x)(x-x_m)^{2n-1}dx \ ,$$

and so

$$1 \leqslant \{y'(x_m)\}^{2n-2} \int_{x_m}^{\infty} f x^{2n-1} dx .$$

This however becomes impossible as x_m becomes large, since $y'(x_m)$ has been proved to remain bounded as $x_m \to \infty$, while by (3.1) we have

$$\int_{x_m}^{\infty} f x^{2n-1} dx \to 0 .$$

Since we have obtained a contradiction it follows that (1.1) has under these assumptions no oscillatory solutions. This proves the theorem.

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UNIVERSITY COLLEGE,
IBADAN, NIGERIA

FOURIER ANALYSIS AND DIFFERENTIATION OVER REAL SEPARABLE HILBERT SPACE

F. H. BROWNELL

1. Introduction. Let l_2 denote as usual the space of square summable real sequences, the prototype of real separable Hilbert space. It is well known that l_2 possesses no non-trivial, translation invariant Borel measures. However, l_2 does have infinitely many subspaces X, locally compact in the l_2 norm relative topology, which we may call translation spaces and for which such measures φ exist [2]. Here the spaces X are not groups under l_2 vector addition, so the notion of translation invariance must be appropriately modified. For any such X we may of course use the corresponding φ to define over $z \in l_2$ a Fourier transform F of $f \in L_1(X, \mathscr{B}, \varphi)$ by

$$F(z) = \int_X f(x) e^{i(z,x)} d\varphi(x) \ .$$

However, in order to get the expected inverse formula, it seems necessary to be able to make X into a group—roughly speaking to define a vector in X corresponding to $x+y$ when this l_2 vector sum $\notin X$. This is a severe restriction on our translation spaces X, and the only natural ones still available seem to be essentially modifications of Jessen's infinite torus [9]. With orthogonal coordinates this is the space X_0 defined below, a modified Hilbert cube.

Since X_0 is a locally compact abelian topological group, Fourier analysis upon it becomes standard procedure. We are able to extend some standard one-variable theorems (see [1]), relating Fourier transforms and the operation of differentiation, to the situation here, which seems new. In a summary at the end we discuss the significance of these results as related to the work in functional analysis of Fréchet, Gâteaux, Lévy, Hille, Zorn, Cameron and Martin, and Friedrichs.

2. Fourier integrals on X_0. Let

$$X_0 = \{x \in l_2 \mid -h_n < x_n \leq h_n \text{ for integer } n \geq 1\}$$

where the fixed sequence of extended real h_n, $0 < h_n \leq +\infty$, has

$$\sum_{n=N+1}^{\infty} h_n^2 < +\infty$$

for some fixed integer $N \geq 0$. For simplicity we assume $h_n = +\infty$ for

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$1 \leq n \leq N$ if $N \geq 1$. Define $+'$ addition as l_2 vector addition modulo the subgroup $I_0 = \{x \in l_2 \mid x_n = 0 \text{ for } n \leq N,\ x_n/2h_n = m_n, \text{ an integer, for } n \geq N+1\}$. Define $P(x)$ for $x \in l_2$ as the unique element of X_0 in the coset $x + I_0$; thus clearly $x +' y = P(x+y) \in X_0$ for x and $y \in X_0$. After defining the inverse $-'x = P(-x)$ for $x \in X_0$, we see that X_0 becomes a group under $+'$ and $-'$. However, the operation $+'$ is not continuous under the metric $\|x-y\|$ defined by the l_2 norm

$$\|x\| = \left[\sum_{n=1}^{\infty} x_n^2\right]^{\frac{1}{2}} .$$

Thus, following Gelfand [5], we introduce the modified norm $|||x||| = \|P(x)\|$ for $x \in l_2$. That $+'$ and $-'$ are continuous under the resulting metric $|||x-y|||$ is clear from the easily verified statements

$$\begin{aligned} |||(\tilde{x} +' \tilde{y}) - (x +' y)||| &= \|P(\tilde{x} - x + \tilde{y} - y)\| \leq \|P(\tilde{x} - x) + P(\tilde{y} - y)\| \\ &\leq |||\tilde{x} - x||| + |||\tilde{y} - y||| \quad \text{and} \quad |||(-'y) - (-'x)||| = |||y - x||| . \end{aligned}$$

Thus X_0 is a topological group under the metric topology of the modified norm. Note that $P(x)$ is continuous from l_2 onto X_0 under the appropriate l_2 and modified norm metrics, since

$$|||P(x) - P(y)||| = \|P(x-y)\| \leq \|x - y\| .$$

We can easily verify that the as yet unused condition

$$\sum_{n=N+1}^{\infty} h_n^2 < +\infty$$

is necessary and sufficient for X_0 to be locally compact under either the l_2 norm or modified norm metric topologies. Thus X_0, under the latter topology, possesses a regular Haar measure φ defined over $\mathscr{B}$, the Borel subsets of X_0; and φ is unique up to constant factors. Hence φ is non-trivial and invariant under $+'$, though, as we remarked above, this φ could be constructed for $+$ alone without making X_0 into a group, (see [2]). To fix φ, let

$$V_1 = \{x \in X_0 \mid |x_n| < \tfrac{1}{2} \text{ for } n \leq N\} ;$$

thus V_1, being non-void and open with compact closure, must satisfy $0 < \varphi(V_1) < +\infty$. We specify φ uniquely by requiring $\varphi(V_1) = 1$.

In order to get Fourier analysis on X_0 following Godement [6] or Weil [11], we need to determine the continuous characters on X_0, that is all continuous complex valued functions $\psi(x)$ on $x \in X_0$ with $|\psi(x)| = 1$ and $\psi(x +' y) = \psi(x)\psi(y)$. Here let

$$Z_0 = \left\{ z \in l_2 \mid z_n = \frac{\pi p_n}{h_n} \text{ with } p_n \text{ an integer for } n \geq N+1 \right\} .$$

Note that since $\sum_{n=N+1}^{\infty} h_n^2 < +\infty$ and $z \in l_2$ make $h_n \to 0$ and $z_n \to 0$ as $n \to \infty$, each $z \in Z_0$ must have $p_n \equiv 0$ and thus $z_n \equiv 0$ for sufficiently large n. Let

$$(x, y) = \sum_{n=1}^{\infty} x_n y_n$$

denote the l_2 inner product.

LEMMA 1. *The group of characters $\tilde{X}_0$ is isomorphic with Z_0, each character having the form $\psi(x) = e^{i(z,x)}$ with $z \in Z_0$.*

Proof. Let $\exp[i\Phi(x)] = \psi(P(x))$ for any $\psi \in \tilde{X}_0$, with $\Phi(0) = 0$ and $\Phi(x)$ defined uniquely by requiring continuity. Thus $\Phi(x)$ is a continuous linear functional over l_2, so $\Phi(x) = (x, z) = (z, x)$ for some unique $z \in l_2$. For $h_n < +\infty$, taking $x_j = 2h_n$ if $j = n$ and $x_j = 0$ if not, we see that $P(x) = 0$. Hence $2\pi p_n = \Phi(x) = (z, x)$ makes $z_n = \pi p_n / h_n$, so $z \in Z_0$.

Let $Z_0 \subseteq l_2$ be topologized relatively from l_2. Clearly this topology is equivalent to the product of the euclidean E_N topology with the discrete topology on the part $n > N$, where $z_n = \pi p_n / h_n$ and $h_n \to 0$. Z_0 so topologized forms a locally compact abelian topological group under l_2 vector addition, η denoting its Haar measure. Clearly this topology on Z_0 is equivalent to the Hausdorff space topology with neighborhoods as finite intersections of sets of the form

$$N_{\rho,F}(z_0) = \{z \in Z_0 \mid |(z - z_0, x)| < \rho \text{ for } x \in F\},$$

$\rho > 0$ and F a norm bounded subset of X_0. Equivalently on $\tilde{X}_0$ this topology is given by

$$\tilde{N}_{\delta,F}(\psi_0) = \{\psi \in X_0 \mid |\psi(x) - \psi_0(x)| < \delta \text{ for } x \in F\}.$$

Now $(X, \mathscr{B}, \varphi)$ is a σ-finite measure space, so $L_\infty(X_0, \mathscr{B}, \varphi)$ is the conjugate space of $L_1(X_0, \mathscr{B}, \varphi)$. Thus the argument of Godement, [6, p. 87], is valid and Z_0 is homeomorphic to $\tilde{X}_0 \subseteq L_\infty(X_0, \mathscr{B}, \varphi)$ under the weak topology defined by $L_1(X_0, \mathscr{B}, \varphi)$.

We may normalize η uniquely by requiring the Fourier inversion formula (2.2), which must hold as stated in Lemmas 2 and 3 following. The formulae are:

$$F(z) = \int_{X_0} e^{i(z,x)} f(x) d\varphi(x). \tag{2.1}$$

$$(2.2) \qquad f(x)=\int_{Z_0} e^{-i(z,x)}F(z)d\eta(z) .$$

Here we note that any $f\in L_1(X_0, \mathscr{B}, \varphi)$ has its Fourier transform $F(z)$ defined and continuous on Z_0 by (2.1); and if such $F\in L_1(Z_0, \mathscr{B}', \eta)$, $\mathscr{B}'$ being the Borel subsets of Z_0, then the right side of (2.2) also exists and is continuous. For Lemmas 2 and 3 let $\mathscr{M}$ be the class of all convolutions

$$[u*v](x)=\int_{X_0} u(x-'y)v(y)d\varphi(y)$$

of continuous functions $u(x)$ and $v(x)$ vanishing outside compact subsets of X_0. (For proof of these following well-known lemmas see [6, p. 90-94]. The density of $\mathscr{M}$ in Lemma 2 follows from the regularity of φ.)

LEMMA 2. *$\mathscr{M}$ is dense in $L_1(X_0, \mathscr{B}, \varphi)$ and $L_2(X_0, \mathscr{B}, \varphi)$, and each $f\in\mathscr{M}$ has its Fourier transform $F\in L_1(Z_0, \mathscr{B}', \eta)$ with* (2.2) *holding at each $x\in X_0$ for the inverse transformation.*

LEMMA 3. *If $f\in L_2(X_0, \mathscr{B}, \varphi)$, then there exists a unique Plancherel transform $F\in L_2(Z_0, \mathscr{B}', \eta)$ such that every sequence $\{f_k\}\subseteq\mathscr{M}$ with the L_2 norm $\|f-f_k\|_2\to 0$ also has $\|F-F_k\|_2\to 0$. Moreover, every sequence $\{f_k\}\subseteq\mathscr{M}$ with $\|F-F_k\|_2\to 0$ also has $\|f-f_k\|_2\to 0$. This Plancherel transformation takes $L_2(X_0, \mathscr{B}, \varphi)$ onto $L_2(Z_0, \mathscr{B}', \eta)$ as a Hilbert space isomorphism,*

$$(2.3) \qquad \int_{X_0} f(x)\overline{g(x)}d\varphi(x)=\int_{Z_0} F(z)\overline{G(z)}d\eta(z) , \qquad f, g\in L_2 .$$

In order to determine η explicitly, let S be the set of all integer valued sequences $\zeta=\{p_n\}$ over $n>N$ such that $p_n=0$ for large enough n for each sequence; thus S is countable. Let $z=(\omega;\zeta)$ be defined for $\omega\in E_N$, $\zeta\in S$ by $z_n=\omega_n$ for $n\le N$ and $z_n=\pi p_n/h_n$ for $n>N$. Letting $\chi_A(z)$ be the characteristic function of any $A\in\mathscr{B}'$, with μ_N Lebesgue measure on E_N,

$$(2.4) \quad \eta(A)=\left(\frac{1}{2\pi}\right)^N \sum_{\zeta\in S}\left\{\int_{E_N}\chi_A(\omega;\zeta)\,d\mu_N(\omega)\right\}=\int_{E_N}\left\{\sum_{\zeta\in S}\chi_A(\omega;\zeta)\right\}\frac{d\mu_N(\omega)}{(2\pi)^N}$$

follows, by applying Lemma 3 to the Gaussian

$$f(x)=\exp\left(-\frac{1}{2}\sum_{n=1}^{N}x_n^2\right)$$

to determine the normalization.

3. Fourier transforms and X_0 differentiation. Here let X_n denote X_0 with the nth coordinate omitted, φ_n the corresponding measure over the σ-algebra $\mathscr{B}_n$ of Borel subsets of X_n, and $\mathscr{G}_1$ the Borel σ-algebra of E_1 if $n \leq N$, of $(-h_n, h_n)$ if $n > N$. Then [7, p. 222], we see that $\mathscr{B} = \mathscr{B}_n \times \mathscr{G}_1$ as the uncompleted product; also, using the uniqueness of Haar measure, $\varphi = \varphi_n \times \mu_1$ or $= \varphi_n \times (\mu_1/2h_n)$ according as $n \leq N$ or $> N$. Now consider $f \in L_1(X_0, \mathscr{B}, \varphi)$, let $\tilde{x}$ denote x with the nth coordinate omitted, and define $K_n(t, x_n) = 1$ if $-h_n < t \leq x_n$, $K(t, x_n) = 0$ if not. Clearly $K_n(t, x_n) f(x_1, \cdots, x_{n-1}, t, x_{n+1}, \cdots)$ is measurable $(\mathscr{B}_n \times \mathscr{G}_1 \times \mathscr{G}_1 = (\mathscr{B} \times \mathscr{G}_1)$ over $(\tilde{x}, x_n, t) \in X_n \times E_1 \times E_1$ if $n \leq N$, or $X_n \times (-h_n, h_n] \times (-h_n, h_n]$ if $n > N$. Thus if we define

$$\int f(x)\, dx_n = \int_{-\infty}^{\infty} K_n(t, x_n) f(x, t) dt,$$

then the Fubini theorem makes $\int f(x)\, dx_n \in L_1(X_n \times I, \mathscr{B}, \varphi)$ for any finite x_n interval I.

For the following theorems we will say that $f(x)$ is x_n absolutely continuous if for all $\tilde{x} \in X_n - A$, where A is some set $\in \mathscr{B}_n$ having $\varphi_n(A) = 0$, we have $f(P(\tilde{x}, x_n))$ absolutely continuous as a function of x_n over every finite interval of E_1.

THEOREM 4. *If $f \in L_1(X_0, \mathscr{B}, \varphi)$, if f is x_n absolutely continuous, and if f'_n, the resulting x_n first partial, is $\in L_1(X_0, \mathscr{B}, \varphi)$ also, then the* (2.1) *defined Fourier transforms F_n and F of f'_n and f have $F_n(z) = -iz_n F(z)$ over $z \in Z_0$.*

Proof. Consider first $h_n < +\infty$, so we know almost everywhere (φ) on X_0 that

$$f(x) = \int f'_n(x)\, dx_n + f(P(\tilde{x}, -h_n)) = \int f'_n(x) dx_n + f(\tilde{x}, h_n).$$

Now

$$\int_{-h_n}^{h_n} e^{iz_n t} dt = 0 \qquad \text{for } z_n \neq 0,$$

so

$$F(z) = \int_{X_n} \int_{-h_n}^{h_n} e^{i(z,x)} \left\{ \int f'_n(x_n)\, dx_n \right\} \frac{dx_n}{2h_n} d\varphi_n(\tilde{x}).$$

But

$$\int_{-h_n}^{h_n} e^{iz_n s} \left\{ \int_{-h_n}^{s} f'_n(\tilde{x}, t)\, dt \right\} ds$$

$$= \frac{e^{iz_n h_n}}{iz_n} \int_{-h_n}^{h_n} f'_n(\tilde{x}, t)\, dt - \frac{1}{iz_n} \int_{-h_n}^{h_n} e^{iz_n s} f'_n(\tilde{x}, s)\, ds$$

by integrating by parts, and

$$\int_{-h_n}^{h_n} f'_n(\tilde{x},\ t)dt = f(P(\tilde{x},\ h_n)) - f(P(\tilde{x},\ -h_n)) = 0\ .$$

Thus $F(z) = -(1/iz_n)F_n(z)$ for $z_n \neq 0$. If $z_n = 0$, then

$$\int_{-h_n}^{h_n} f'_n(\tilde{x},\ t)dt = 0$$

makes $F_n(z) = 0$, so $F_n(z) = -iz_nF(z)$ for all $z \in Z_0$.

Secondly if $h_n = +\infty$, we know

$$f(x) = \int f'_n(x)dx_n + C(\tilde{x})$$

almost everywhere (φ_n) over $\tilde{x} \in X_n$. Thus $f(\tilde{x},\ x_n) \to C(\tilde{x})$ as $x_n \to -\infty$, so $f(\tilde{x},\ x_n) \in L_1(E_1)$ in x_n almost everywhere (φ_n) requires $C(\tilde{x}) = 0$, $f(x) = \int f'_n(x)dx_n$, and similarly $\int_{-\infty}^{\infty} f'_n(\tilde{x},\ t)dt = 0$ almost everywhere (φ_n). Thus

$$\int_{-\infty}^{\infty} e^{iz_ns} f(\tilde{x},\ s)ds = \int_{-\infty}^{\infty} e^{iz_ns} \left\{ \int_{-\infty}^{s} f'_n(\tilde{x},\ t)\,dt \right\} ds$$

$$= \lim_{a,b\to\infty} \left[\frac{e^{iz_ns}}{iz_n} \int_{-\infty}^{s} f'_n(\tilde{x},\ t)\,dt \right]_{-a}^{b} - \frac{1}{iz_n} \int_{-\infty}^{\infty} e^{iz_ns} f'_n(\tilde{x},\ s)ds$$

$$= \frac{-1}{iz_n} \int_{-\infty}^{\infty} e^{iz_ns} f'_n(\tilde{x},\ s)ds, \quad \text{so} \quad F(z) = -\frac{1}{iz_n} F_n(z) \qquad \text{for } z_n \neq 0.$$

If $z_n = 0$, then $\int_{-\infty}^{\infty} f_n'(\tilde{x},\ t)dt = 0$ makes $F_n(z) = 0$, so $F_n(z) = -iz_nF(z)$ for all $z \in Z_0$.

For the next lemma we need to remark that $T(x;\ y) = (x;\ y -' x)$ is a homeomorphism of $X_0 \times X_0$ into itself, and hence leaves unchanged the Borel class $\mathscr{B} \times \mathscr{B}$, [7, p. 257]. Thus $A \in \mathscr{B}$ has $T(X_0 \times A) \in \mathscr{B} \times \mathscr{B}$, so clearly any $f(x)$ measurable $(\mathscr{B})$ has $f(x +' y)$ measurable $(\mathscr{B} \times \mathscr{B})$. Let ${}_ne \in l_2$ be defined by ${}_ne_k = \delta_{n,k}$, and we then easily see, using

$$\{(x\,;\ y) \in X_0 \times X_0 \mid y_k = 0 \text{ for } k \neq n\} \in \mathscr{B} \times \mathscr{B}\,,$$

that such f also have $f(x +' t_ne)$ measurable $(\mathscr{B} \times \mathscr{G}_1)$ over $x \in X_0$ and t real.

LEMMA 5. *If $f \in L_r(X_0,\ \mathscr{B},\ \varphi)$ with real $r \geq 1$, if f is x_n absolutely continuous, and if the resulting $f'_n \in L_r(X_0,\ \mathscr{B},\ \varphi)$, then defining*

$$ {}_nf_h(x)=\frac{1}{h}\{f(x+'h\,{}_ne)-f(x)\} $$

over real $h\to 0$ yields

$$ \lim_{h\to 0}\|{}_nf_h-f'_n\|_r=0\ . $$

Proof. Since $x+'h\,{}_ne=P(x+h\,{}_ne)$, we know that

$$ {}_nf_h(x)-f'_n(x)=\frac{1}{h}\int_0^h\{f'_n(x+'t\,{}_ne)-f'_n(x)\}\,dt $$

almost everywhere (φ_n) over $\tilde{x}\in X_n$. With $1/r'=1-1/r$ if $r>1$, $1/r'$ replaced by 0 if $r=1$. The Schwarz-Hölder inequality thus yields

$$ |{}_nf_h(x)-f'_n(x)|\leq|h|^{1/r'-1}\left|\int_0^h|f'_n(x+'t\,{}_ne)-f'_n(x)|^r\,dt\right|^{1/r}. $$

Then by the Fubini theorem

$$ \begin{aligned} \|{}_nf_h-f'_n\|_r^r&\leq\frac{1}{|h|}\left|\int_0^h\left\{\int_{X_0}|f'_n(x+'t\,{}_ne)-f'_n(x)|^r\,d\varphi(x)\right\}dt\right| \\ &\leq\sup_{|t|\leq|h|}\|g_t-g\|_r^r \end{aligned} $$

where $g(x)=f'_n(x)\in L_r$ and $g_t(x)=g(x+'t\,{}_ne)$. The functions $u(x)$, continuous on X_0 under the modified norm topology and vanishing outside compact subsets of X_0, are L_r norm dense in $L_r(X_0,\mathscr{B},\varphi)$ by the regularity of φ; and such u have $\|u_t-u\|_r\to 0$ as $t\to 0$ by their uniform continuity. Also $\|g_t-u_t\|_r=\|g-u\|_r$ by φ invariance, so

$$ \|{}_nf_h-f'_n\|_r\leq 2\|g-u\|_r+\sup_{|t|\leq|h|}\|u_t-u\|_r $$

and hence $\|{}_nf_h-f'_n\|_r\to 0$ as $h\to 0$.

We also have the following converse for $r=2$.

LEMMA 6. *If f and $g\in L_2(X_0,\mathscr{B},\varphi)$ and if $\lim_{h\to 0}\|{}_nf_h-g\|_2=0$, then $f(x)=\tilde{f}(x)$ almost everywhere (φ) for some $\tilde{f}(x)$ measurable $(\mathscr{B})$ which is x_n absolutely continuous and has its derivative $\tilde{f}'_n(x)=g(x)$ almost everywhere (φ).*

Proof.

$$ \|{}_nf_h-g\|_2^2=K\int_{X_n}\left\{\int_{-h_n}^{h_n}|{}_nf_h(x)-g(x)|^2dx_n\right\}d\varphi_n(\tilde{x}) $$

by the Fubini theorem, so using a Riesz-Fischer subsequence $h=t_k\to 0$

we have

$$\lim_{t_k\to 0}\int_{-h_n}^{h_n} |{}_nf_{t_k}(x)-g(x)|^2 dx_n=0$$

for almost (φ_n) all $\tilde{x}\in X_n$. This reduces our statement to the one real variable analogue, where the result is well known (see for example Bochner, [1, p. 131], if $h_n=+\infty$). Since we may take

$$\tilde{f}(x)=\int_0^{x_n} g(\tilde{x}, t)\,dt+\tilde{f}(\tilde{x}, 0)$$

almost everywhere (φ_n) with

$$\tilde{f}(\tilde{x}, 0)=\frac{1}{a}\int_0^a \left\{f(\tilde{x}, s)-\int_0^s g(\tilde{x}, t)\,dt\right\}ds$$

for $0<a<h_n$, clearly $\tilde{f}(x)$ may be taken measurable ($\mathscr{B}$).

The L_2 counterpart of Theorem 4 now follows.

THEOREM 7. *If $f\in L_2(X_0, \mathscr{B}, \varphi)$, if f is x_n absolutely continuous, and if the resulting $f'_n\in L_2(X_0, \mathscr{B}, \varphi)$ too, then the Plancherel transforms F and F_n of f and f'_n satisfy $F_n(z)=-iz_nF(z)$ almost everywhere (η).*

Proof. Using the Fubini theorem in (2.1) and the translation invariance of φ, we have

$${}_nF_h(z)=\frac{1}{h}(e^{-iz_nh}-1)F(z)$$

for the transform of ${}_nf_h$ in case $f\in L_1\cap L_2$, and hence for all $f\in L_2$ by the Plancherel Lemma 3 with $L_1\cap L_2$ dense in L_2. Since

$$\lim_{h\to 0}\frac{1}{h}(e^{-iz_nh}-1)=-iz_n$$

and since $\|{}_nF_h-F_n\|_2\to 0$ as $h\to 0$ by Lemma 5 and (2.3), the Riesz-Fischer theorem yields $F_n(z)=-iz_nF(z)$ as desired.

It is easy to get an extended converse of Theorem 7.

THEOREM 8. *If f and $g\in L_2(X_0, \mathscr{B}, \varphi)$ and have transforms F and G satisfying $G(z)=(-iz_n)^kF(z)$ for integer $k>0$, then $f(x)=\tilde{f}(x)$ almost everywhere (φ) for some $\tilde{f}(x)$ measurable ($\mathscr{B}$) such that $\tilde{f}(x)$ possesses everywhere up to $(k-1)$st order x_n partials which are $\in L_2(X_0, \mathscr{B}, \varphi)$,* *the* $(k-1)$*st* $\tilde{f}^{(k-1)}_{n,n\cdots,n}(x)$ *is* x_n *absolutely continuous, and*

$$\tilde{f}^{(k)}_{n,n,\cdots,n}(x)=g(x)$$

almost everywhere (φ).

Proof. From $(-iz_n)^k F(z)$ and $F(z)\in L_2(Z_0, \mathscr{B}', \eta)$ clearly $(-iz_n)^p F(z)$ $\in L_2$ also for $p=0, 1, \cdots, k-1$, and by taking inverse Plancherel transforms we get ${}_pg\in L_2(X_0, \mathscr{B}, \varphi)$ transforming into $(-iz_n)^p F(z)$. As we have seen before the difference quotient ${}_pg_h$ of ${}_pg$ will have the transform

$$\frac{1}{h}(e^{-iz_nh}-1)(-iz_n)^p F(z)=\left\{\int_0^1 e^{-iz_nht}dt\right\}(-iz_n)^{p+1}F(z)\ .$$

Since $|\{\quad\}|\leq 1$ and $\{\quad\}\to 1$, this transform $\to(-iz_n)^{p+1}F(z)$ in L_2 norm as $h\to 0$. Hence $\|{}_pg_h-{}_{p+1}g\|_2\to 0$ as $h\to 0$ by the Plancherel lemma, and so Lemma 6 with ${}_0g=f$ and ${}_kg=g$ gives the result.

The following converse of Theorem 8 is considerably deeper than Theorem 7. We remark that if f and $g\in L_1(X_0, \mathscr{B}, \varphi)$, then $f*g\in L_1$ also and has the Fourier transform $F(z)G(z)$, where

$$[f*g](x)=\int_{X_0} f(x-'y)g(y)d\varphi(y)$$

exists almost everywhere (φ). More important for us, if f and $g\in L_2(X_0, \mathscr{B}, \varphi)$, then $f*g$ is the inverse Fourier transform of $F(z)G(z)\in L_1(Z_0, \mathscr{B}, \eta)$, defined pointwise by (2.2), and hence also the inverse Plancherel transform if $FG\in L_2$. This follows by noting that $e^{i(z,x)}\overline{F(z)}$ is the transform of $\overline{f(x-'y)}$ as a function of y and by using (2.3).

THEOREM 9. *If* $f\in L_2(X_0, \mathscr{B}, \varphi)$ *and possesses everywhere up to* $(k-1)$*st order* x_n *partials, if* $f^{(k-1)}_{n,\cdots,n}(x)$ *is* x_n *absolutely continuous, and if* $f^{(k)}_{n,\cdots,n}(x)\in L_2(X_0, \mathscr{B}, \varphi)$, *then also* $f^{(p)}_{n,\cdots,n}(x)\in L_2(X_0, \mathscr{B}, \varphi)$ *for* $p=1, 2, \cdots, k$, *and such* $f^{(p)}_{n,\cdots,n}$ *have the transforms* $(-iz_n)^p F(z)$.

Proof. First we construct rather arbitrarily a smoothing transform

$$G(z)=\exp\left(-\frac{1}{2}\sum_{j=1}^{N}\omega_j^2-\frac{1}{2}\gamma_n z_n^2\right)\rho(\zeta)$$

for $z=(\omega\,;\,\zeta)$ of $\omega\in E_N$ and $\zeta\in S$ using the notation of (2.4), where $\gamma_n=0$ if $n\leq N$ and $\gamma_n=1$ if $n>N$. S being countable we may set $S=\{{}_k\zeta\}$ and define $\rho(\zeta)$ on S by setting $\rho({}_k\zeta)=e^{-k}$. We see clearly from (2.4) for each integer $p\geq 0$ that $(-iz_n)^p G(z)\in L_1\cap L_2\cap L_\infty$ for the measure space $(Z_0, \mathscr{B}', \eta)$, since

$$|z_n|^p e^{-\frac{1}{2}z_n^2}$$

is bounded and $O(e^{-|z_n|})$ as $|z_n| \to \infty$. Also $G(z) > 0$ everywhere on Z_0, these two conditions being all we really need. Take g as the unique element of $L_2(X_0, \mathscr{B}, \varphi)$ transforming into G, and by Theorem 8 we may take $g(x)$ to possess all order derivatives in x_n with $g^{(p)}_{n,\cdots,n} \in L_2$ transforming into $(-iz_n)^p G(z)$.

Now for $h_n < +\infty$ and $0 \leq p \leq k$, by integrating by parts we see that

$$\int_{-h_n}^{h_n} g^{(p)}_{n,\cdots,n}(x-'y) f(y)\, dy_n = \int_{-h_n}^{h_n} g(x-'y) f^{(p)}_{n,\cdots,n}(y) dy_n$$

existent finite for almost (φ_n) all $\tilde{y} \in X_n$ for each $x \in X_0$, using the periodicity of $g(P(x-y))$ and $f(P(y))$ at the endpoints $y_n = \pm h_n$. If $h_n = +\infty$ we still get the same result by a slightly different argument. Here we know $f^{(k)}_{n,\cdots,n}(x) \in L_2(-\infty, \infty)$ over x_n for almost (φ_n) all $\tilde{x} \in X_n$, so by the Schwarz inequality follows

$$f^{(k-1)}_{n,\cdots,n}(x) = O(|x_n|)$$

as $|x_n| \to \infty$ for such $\tilde{x}$. Thus by further integration

$$f^{(p)}_{n,\cdots,n}(x) = O(|x_n|^k)$$

as $|x_n| \to \infty$ for such $\tilde{x}$, $0 \leq p \leq k-1$. Now clearly

$$g(x) = e^{-\frac{1}{2}x_n^2} g_1(\tilde{x}) \ ,$$

so

$$g^{(p)}_{n,\cdots,n}(x) = O(e^{-|x_n|})$$

as $|x_n| \to \infty$. These two estimates are enough to make the endpoint terms vanish in integration by parts, so

$$\int_{-\infty}^{\infty} g^{(p)}_{n,\cdots,n}(x-'y) f(y)\, dy_n = \int_{-\infty}^{\infty} g(x-'y) f^{(p)}_{n,\cdots,n}(y) dy_n.$$

Thus with $K=1$ or $1/2h_n$ we have

$$[g^{(p)}_{n,\cdots,n} * f](x) = K \int_{X_n} \int_{-h_n}^{h_n} g(x-'y) f^{(p)}_{n,\cdots,n}(y) dy_n d\varphi_n(\tilde{y})$$

existent finite in the order written for $0 \leq p \leq k$ and all $x \in X_0$.

Now for $p=k$ we are given $f^{(k)}_{n,\cdots,n} \in L_2$, so the Schwarz inequality shows $g(x-'y) f^{(k)}_{n,\cdots,n}(y)$ to be $\in L_1$. Thus by the Fubini theorem

$$[g_{n,\dots,n}^{(k)} * f](x) = [g * f_{n,\dots,n}^{(k)}](x)$$

at all $x \in X_0$. By our remarks preceding this theorem, since $(-iz_n)^k G(z)$ and $G(z) \in L_\infty$ make $(-iz_n)^k G(z)F(z)$ and $G(z)F_k(z) \in L_2$, for the Plancherel transforms we have $[(-iz_n)^k G(z)]F(z) = G(z)F_k(z)$. Thus since $G(z) > 0$ everywhere, $F_k(z) = (-iz_n)^k F(z)$ with $F_k \in L_2$ the transform of $f_{n,\dots,n}^{(k)} \in L_2$. Thus Theorem 8 gives the result.

THEOREM 10. *If f and $g \in L_2(X_0, \mathscr{B}, \varphi)$ and if their transforms F and G satisfy*

$$G(z) = -\left(\sum_{n=1}^{\infty} z_n^2\right) F(z),$$

then there exists a sequence of functions ${}_nf(x)$ measurable ($\mathscr{B}$) such that ${}_nf(x) = f(x)$ almost everywhere (φ), ${}_nf(x)$ is x_n absolutely continuous as well as its everywhere existent first x_n derivative ${}_nf'_n(x)$, ${}_nf'_n$ and ${}_nf''_{nn} \in L_2(X_0, \mathscr{B}, \varphi)$, and $\sum_{n=1}^{M} {}_nf''_{nn}$ converges in L_2 norm to g as $M \to \infty$.

Proof. Let $g_n \in L_2(X_0, \mathscr{B}, \varphi)$ be defined uniquely by requiring $G_n(z) = -z_n^2 F(z)$, since $|z_n^2 F(z)| \leq |G(z)|$ makes $z_n^2 F(z) \in L_2(Z_0, \mathscr{B}', \eta)$. Now $\sum_{n=1}^{\infty} z_n^2$ is actually a finite sum for each $z \in Z_0$, and also

$$\left|\sum_{n=1}^{M} z_n^2\right| |F(z)| \leq |G(z)| \in L_2,$$

so by dominated convergence $\sum_{n=1}^{M} G_n(z) \to G(z)$ in L_2 norm as $M \to \infty$, and hence also $\sum_{n=1}^{M} g_n \to g$ in L_2 norm. Finally Theorem 8 for each n gives the desired ${}_nf(x) = f(x)$ almost everywhere (φ), ${}_nf'_n$ and ${}_nf''_{nn} \in L_2$, and ${}_nf''_{nn}(x) = g_n(x)$ almost everywhere (φ) as desired.

THEOREM 11. *Let f and $g \in L_2(X_0, \mathscr{B}, \varphi)$ and let a sequence of functions ${}_nf(x)$ measurable ($\mathscr{B}$) satisfy the conditions: ${}_nf(x) = f(x)$ almost everywhere (φ); ${}_nf$ everywhere possesses a first x_n partial ${}_nf'_n$ which is x_n absolutely continuous; ${}_nf''_{nn} \in L_2(X_0, \mathscr{B}, \varphi)$; and $\sum_{n=1}^{M} {}_nf''_{nn} \to g$ in $L_2(X_0, \mathscr{B}, \varphi)$ norm as $M \to \infty$. Then the transforms F and G satisfy*

$$G(z) = -\left(\sum_{n=1}^{\infty} z_n^2\right) F(z)$$

almost everywhere (η).

Proof. By Theorem 9 we also have ${}_nf'_n \in L_2$ and ${}_nf''_{nn}$ has the transform $G_n(z) = -z_n^2 F(z)$. From $\sum_{n=1}^{M} {}_nf''_{nn} \to g$ in L_2 we thus know

$\sum_{n=1}^{M} G_n \to G$ in L_2 norm as $M \to \infty$, where

$$\sum_{n=1}^{M} G_n(z) = -\Big(\sum_{n=1}^{M} z_n^2\Big) F(z) .$$

Since $\sum_{n=1}^{\infty} z_n^2$ is actually a finite sum at each $z \in Z_0$, Riesz-Fischer subsequences yield

$$G(z) = -\Big(\sum_{n=1}^{\infty} z_n^2\Big) F(z)$$

as desired.

4. Significance of results. The main results of this paper are Theorems 7 through 11 relating Fourier transforms over X_0, a modification of the Hilbert cube, to the operations of differentiation in an L_2 sense. It is clear that Theorems 10 and 11 allow one to use Fourier transforms to define a generalized Laplace differential operator for scalar functions on X_0. This definition is in a global L_2 sense, which gives a pointwise definition only by using Riesz-Fischer subsequences. The ideas of pointwise infinite dimensional derivatives seem to go back to Fréchet and Gâteaux. Hille [8, pp. 71–90], Zorn [12], and others have developed a notion of analyticity from similar complex differentiability on complex Banach spaces.

To be precise, for real l_2 consider a real valued function $f(x)$ over $x \in l_2$ and define the gradient $\nabla f(x) = y$ at each x such that there exists $y \in l_2$ having over $u \in l_2$

$$\lim_{\|u\| \to 0} \|u\|^{-1} |f(x+u) - f(x) - (u, y)| = 0 , \tag{4.1}$$

such y being clearly unique. This is a Fréchet type definition. If we let $\{w_n\}$ be a complete orthonormal system in l_2, we have where $\nabla f(x)$ exists that

$$(w_n, \nabla f(x)) = \Big[\frac{d}{d\lambda} f(x + \lambda w_n)\Big]_{\lambda=0} . \tag{4.2}$$

This equation could also serve as a Gâteaux type definition of $\nabla f(x)$, possibly depending on $\{w_n\}$, wherever the squares of the right hand terms are summable. For the divergence, if $T(x) \in l_2$ for each $x \in l_2$, we may formulae the Gâteaux type definition

$$(\nabla, T(x)) = \sum_{n=1}^{\infty} (w_n, \nabla \psi_n(x)) \text{ for } \psi_n(x) = (T(x), w_n) , \tag{4.3}$$

which is independent of the choice of base $\{w_n\}$ if

$$(4.4) \qquad \sum_{n=1}^{\infty} \|\nabla\psi_n(x)\| < +\infty \quad \text{and}$$

$$0 = \lim_{\|u\|\to 0} \|u\|^{-1}\left[\sum_{n=1}^{\infty} |\psi_n(x+u) - \psi_n(x) - (u, \nabla\psi_n(x))|^2\right]^{\frac{1}{2}}.$$

Finally we can define the Laplacian $\nabla^2 f(x) = (\nabla, \nabla f(x))$, so that

$$(\nabla, \nabla f(x)) = \sum_{n=1}^{\infty}\left[\frac{d^2}{d\lambda^2} f(x+\lambda w_n)\right]_{\lambda=0}$$

shows this definition to agree pointwise with the expression in Theorems 10 and 11, $\sum_{n=1}^{\infty} f''_{nn}(x)$.

Lévy has also constructed a vector analysis for Hilbert space, though he is led to define

$$\lim_{M\to\infty} \frac{1}{M}\left\{\sum_{n=1}^{M} f''_{nn}(x)\right\}$$

as the Laplacian, [5, p. 248]. He differs more seriously from our approach by using a development of mean values of functions instead of integration with respect to a non-trivial, translation invariant measure. Thus he has no need to reduce l_2 to X_0, though naturally his theory of mean values must pay for this by certain anomalous features. Cameron and Martin have also done a great deal of functional analysis in terms of Wiener measure on the continuous functions ([3] and others), but since this is not translation invariant, it has little contact with our work.

It seems that our results relating Fourier transforms and differentiation over real Hilbert space may be useful in a mathematical formulation of quantum radiation theory, just as finite dimensional differential operators are very conveniently defined self-adjointly in terms of Fourier transforms. Friedrichs has discussed such problems and is led to still a different method of integration over Hilbert space, [4, p. 60]. However, the set functions induced by his method are not σ-additive and apparently not translation invariant either.

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UNIVERSITY OF WASHINGTON

REMARK ON THE AVERAGES OF REAL FUNCTIONS

R. E. CHAMBERLIN

1. Introduction. Let $f(x)$ be a continuous function defined on the closed interval $[a, b]$. It is known that if for each x in the open interval (a, b) there is a positive number t such that

$$[x-t,\ x+t]\subset(a,\ b) \quad \text{and} \quad f(x)=\frac{f(x-t)+f(x+t)}{2}$$

then $f(x)$ is linear (see [2, p. 253]). The same method of proof shows that if there is such a t for each $x\in(a,\ b)$ with

$$f(x)=\frac{1}{2t}\int_{x-t}^{x+t} f(s)\, ds$$

then $f(x)$ is linear. Suppose $f(x)$ is such that for each $x\in(a,\ b)$ there exists a t with $[x-t,\ x+t]\subset(a,\ b)$ and

$$(1) \qquad \frac{f(x+t)+f(x-t)}{2}=\frac{1}{2t}\int_{x-t}^{x+t} f(s)ds\,.$$

Is $f(x)$ necessarily linear? On page 231 of [1] it is shown that if (1) holds for each x and all t such that $[x-t,\ x+t]\subset(a,\ b)$ then $f(x)$ is linear. The question arises whether or not one can relax the requirement that (1) holds for all t in the above intervals and still conclude that $f(x)$ is linear.

In this note a continuously differentiable non-linear function $f(x)$ is given which satisfies (1) for every $x\in(a,\ b)$ and an infinity of t's. The values of t depend on x but they may be chosen arbitrarily small for each x. Conditions which together with (1) make $f(x)$ linear are given and the note is concluded with some remarks on the approximation to a function by its averages

$$f(x,\ t)=\frac{1}{2t}\int_{x-t}^{x+t} f(s)\, ds\,.$$

DEFINITION. A continuous function $f(x)$ on $[a,\ b]$ will be said to have property (1) if for each $x\in(a,\ b)$ there are arbitrarily small values of t for which (1) is true.

2. An example. We give an example of a continuously differentiable function having property (1) which is not linear. Let

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$$f(x)=\sum_{n=1}^{\infty}\frac{\cos 10^{2n}\pi x}{n^2\cdot 10^{2n}} \tag{2}$$

It is clear that $f(x)$ is not linear and is continuously differentiable. To show that $f(x)$ has property (1) we begin with the following

LEMMA. *For every* x,

$$\varlimsup_{n\to\infty}|\cos 10^{2n}\pi x|\geq 10^{-3}.$$

Since the functions $\cos 10^{2n}\pi x$ $(n\geq 1)$ all have 1 as a period it is clear we need only consider $x\in[0, 1]$ in the proof of this lemma. Since there is no loss in generality we assume hereafter that we are dealing with the interval $[0, 1]$ and x is in this interval.

Let the decimal expansion of x be $.a_1a_2\cdots$. Then

$$10^{2n}x=a_1a_2\cdots a_{2n}+.a_{2n+1}a_{2n+2}\cdots \text{ and } |\cos 10^{2n}\pi x|=|\cos(.a_{2n+1}a_{2n+2}\cdots)\pi|.$$

Suppose $|\cos 10^{2n}\pi x|<10^{-3}$. Set $.a_{2n+1}a_{2n+2}\cdots=.5+r_n$ where $|r_n|<.5$. Then

$$10^{-3}\geq|\cos(.a_{2n+1}a_{2n+2}\cdots)\pi|=|\sin r_n\pi|=\sin|r_n\pi|\geq\frac{2}{\pi}|r_n\pi|,$$

that is $\frac{1}{2\cdot 10^3}\geq|r_n|$. Hence there is an integer b with $0\leq b\leq 5$ such that $|r_n|=.000b\cdots$. Therefore,

$$|\cos 10^{2(n+1)}\pi x|=|\cos(.0b\cdots)\pi|\geq\left(1-\frac{(.1\pi)^2}{2}\right)>.9\,.$$

Thus for every x and every n_0 there are integers $n>n_0$ such that $|\cos 10^{2n}\pi x|\geq 10^{-3}$. This proves the lemma.

For the function (2) we have

$$\begin{aligned}g(x,\ t)&=\frac{1}{2}[f(x+t)+f(x-t)]-\frac{1}{2t}\int_{x-t}^{x+t}f(s)ds \\ &=\sum_{n=1}^{\infty}\frac{1}{2\cdot 10^{2n}\cdot n^2}[\cos 10^{2n}\pi(x+t)+\cos 10^{2n}\pi(x-t)] \\ &\quad-\frac{1}{2t}\sum_{n=1}^{\infty}\frac{\sin 10^{2n}\pi(x+t)-\sin 10^{2n}\pi(-t)}{10^{2n}\cdot n^3\cdot 10^{2n}\pi}\,.\end{aligned} \tag{3}$$

From elementary trigonometric identities we now obtain

$$g(x,\ t)=\sum_{n=1}^{\infty}\frac{1}{10^{2n}n^2}\cos 10^{2n}\pi x\left[\cos 10^{2n}\pi t-\frac{\sin 10^{2n}\pi t}{10^{2n}\pi t}\right].$$

We investigate in detail the last expression for $g(x, t)$ in (3).

Given x, let $\overline{\lim_{n\to\infty}}|\cos 10^{2n}\pi x|=d$. From the lemma it is clear there are an infinity of integers k with the following properties:

(a) $|\cos 10^{2k}\pi x|>.99d$.

(b) $|\cos 10^{2n}\pi x|<1.01\ d$ for $n\geq[k/3]$

(c) $k\geq 10$.

For these values of k we show that the sign of $g(x, t)$ in (3) is determined by the sign of the k-th term in its series expansion if t is chosen properly. We assume hereafter that k is subject to conditions (a), (b) and (c).

For the given x and subject to conditions (a), (b) and (c) pick k large enough so that for $t=2\cdot 10^{-2k}$, $[x-t, x+t]\subset[0, 1]$. Then

$$(4)\qquad g(x, 10^{-2k})=\sum_{n=1}^{\infty}\frac{\cos 10^{2n}\pi x}{n^2 10^{2n}}\left[\cos 10^{2(n-k)}\pi-\frac{\sin 10^{2(n-k)}\pi}{10^{2(n-k)}\pi}\right]$$

$$=\sum_{n=1}^{k-1}\frac{\cos 10^{2n}\pi x}{n^2 10^{2n}}\left(-\frac{\pi^2}{6}10^{4(n-k)}+\theta_n\cdot 10^{6(n-k)}\right)+(-1)\cdot\frac{\cos 10^{2k}\pi x}{k^2 10^{2k}}+\sum_{n=k+1}^{\infty}\frac{\cos 10^{2n}\pi x}{n^2 10^{2n}}$$

where $|\theta_n|<2$. Now

$$(5)\qquad \left|\sum_{n=1}^{k-1}\frac{\cos 10^{2n}\pi x}{n^2 10^{2n}}\left(-\frac{\pi^2}{6}10^{4(n-k)}+\theta_n 10^{6(n-k)}\right)\right|$$

$$\leq\frac{10}{3}\cdot\frac{1}{10^{2k}}\cdot\sum_{n=1}^{k-1}\frac{|\cos 10^{2n}\pi x|}{n^2}10^{2(n-k)}$$

$$\leq\frac{10}{3}10^{-2k}\left(\sum_{n=1}^{[k/3]-1}\frac{1}{n^2}10^{2(n-k)}\right)\cdot 10^3 d+\left(\frac{10}{3}\right)10^{-2k}\left(\sum_{n=[k/3]}^{k-1}\frac{1}{n^2}10^{2(n-k)}\right)1.01\ d$$

where we have used the lemma and property (b) of k to get the last inequality.

For the first sum in the last inequality of (5) we have

$$(6)\qquad \sum_{n=1}^{[k/3]-1}\frac{1}{n^2}10^{2(n-k)}<\sum_{n=1}^{[k/3]-1}10^{2(n-k)}\leq 10^{-4/3(k-1)}\frac{1-(10^{-2})^{(k/3)}}{1-10^{-2}}$$

$$<(1.01)10^{-4/3(k-1)}.$$

To get an estimate on the second part of the last inequality of (5), recall that if $s_n=\sum_{i=1}^{n}\alpha_i$

then

$$\sum_{n=r}^{m}\alpha_n\beta_n=\sum_{n=r}^{m}s_n(\beta_n-\beta_{n+1})-s_{r-1}\beta_r+s_m\beta_{m+1}.$$

Letting $\alpha_n = 10^{2(n-k)}$, $\beta_n = 1/n^2$ we get

$$(7)\quad \sum_{n=[k/3]}^{k-1} \frac{10^{2(n-k)}}{n^2} = \sum_{n=[k/3]}^{k-1} 10^{-2(k-n)}\left(\frac{1-10^{-2n}}{1-10^{-2}}\right)\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)$$

$$-10^{-2(k-[k/3]-1)}\left(\frac{1-10^{-2[k/3]}}{1-10^{-2}}\right)\cdot 1/[k/3]^2 + 10^{-2}\left(\frac{1-10^{-2(k-1)}}{1-10^{-2}}\right)\cdot\frac{1}{k^2} < \frac{2}{10^2}\frac{1}{k^2}$$

at least for $k \geq 10$. Using the estimates obtained in (6) and (7) we get

$$(8)\quad \frac{10}{3}\cdot 10^{-2k}\sum_{n=1}^{k-1}\frac{|\cos 10^{2n}\pi x|}{n^2 10^{2n}} \leq \frac{d}{10^{2k}}\left[\frac{1.01}{3}10^{-4/3(k-1)+4} + \frac{10}{3}\cdot(1.01)\cdot\frac{2}{10^2 k^2}\right]$$

$$< \frac{2}{10} d\cdot\frac{1}{k^2 10^{2k}} \quad \text{for} \quad k \geq 10\,.$$

Furthermore

$$(9)\quad \left|\sum_{n=k+1}^{\infty}\frac{\cos 10^{2n}\pi x}{10^{2n}n^2}\left[\cos 10^{2(n-k)}\pi - \frac{\sin 10^{2(n-k)}\pi}{10^{2(n-k)}\pi}\right]\right|$$

$$\leq 1.01\, d \sum_{n=k+1}^{\infty}\frac{1}{10^{2n}n^2} < \frac{1.01\, d}{(k+1)^2}\cdot\frac{1}{10^{2(k+1)}}\cdot\frac{1}{1-10^{-2}} < \frac{1}{10}\frac{d}{k^2 10^{2k}}\,.$$

From (8) and (9) we see that the k-th term of the series for $g(x, 10^{-2k})$ is greater in absolute value than the sum of the remaining terms. Hence the signs of $g(x, 10^{-2k})$ and $-10^{-2k}k^{-2}\cos 10^{2k}\pi x$ are the same. For $t = 2\cdot 10^{-2k}$ the k-th term of the series for $g(x, 2\cdot 10^{-2k})$ is $10^{-2k}k^{-2}\cos 10^{2k}\pi x$ and in the same manner as above one can show that the signs of $g(x, 2\cdot 10^{-2k})$ and the k-th term are the same. Since $10^{-2k}k^{-2}\cos 10^{2k}\pi x$ and $-10^{-2k}k^{-2}\cos 10^{2k}\pi x$ are of opposite signs, $g(x, t)$ vanishes for some $t \in (10^{-2k}, 2\cdot 10^{-2k})$. But for $g(x, t)$ to vanish means that $f(x)$ satisfies (1). Since for each x there are an infinity of k's satisfying (a), (b) and (c), there are (for each x) arbitrarily small values of t for which the $f(x)$ of (2) satisfies (1). Hence this $f(x)$ has the property (1).

3. Sufficient conditions for a function to be linear.

LEMMA 1. *If $f(x)$ is continuously differentiable and $f''(x_0) \neq 0$, then $g(x_0, t)$ is of one sign for some open interval $(0, t_0)$ $(t_0 > 0)$.*

Under the stated conditions we may represent $f(x)$ by

$$(10)\quad f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + o[(x-x_0)^2].$$

Using (10) and the definition of $g(x, t)$ gives

$$(11)\quad g(x_0,\ t)=\frac{f(x_0+t)+f(x_0-t)}{2}-\frac{1}{2t}\int_{x_0-t}^{x_0+t}f(u)du$$
$$=\left\{f(x_0)+\frac{f''(x_0)}{2}t^2+o(t^2)\right\}-\frac{1}{2t}\int_{x_0-t}^{x_0+t}\left\{f(x_0)+f'(x_0)(u-x_0)\right.$$
$$\left.+\frac{f'(x_0)}{2}(u-x_0)^2+o[(u-x_0)^2]\right\}du=\frac{1}{3}f''(x_0)t^2+o(t^2).$$

Thus if $f''(x_0)\neq 0$, it is clear $g(x_0,\ t)$ is one-signed for sufficiently small values of t.

THEOREM 1. *If $f(x)$ has property* (1) *and $f'(x)$ is absolutely continuous then $f(x)$ is linear.*

For $f''(x)$ exists almost everywhere and by Lemma 1 it is zero everywhere it exists because $f(x)$ has property (1). Hence $f'(x)$ is a constant and $f(x)$ is linear.

THEOREM 2. *If $f'(x)$ is continuous, monotone increasing and not constant in any sufficiently small symmetric interval about x_0 then $g(x_0,\ t)$ is one-signed in an interval* $(0,\ t_0)$.

One has

$$f(x_0+t)=f(x_0-t)+\int_{x_0-t}^{x_0+t}f'(u)\,du$$

and for any $x\in(x_0-t,\ x_0+t)$ we get

$$(12)\quad f(x)\leq f(x_0-t)+f'(x)(x-x_0+t),\ \ f(x_0+t)\geq f(x)+f'(x)(x_0+t-x)$$

where at least one of the inequalities is strict by the hypothesis of Theorem 2. From (12) one obtains

$$(13)\qquad \frac{(x-x_0+t)f(x_0+t)+(x_0-x+t)f(x_0-t)}{2t}>f(x)\,.$$

It is obvious from (13) that $g(x_0,\ t)$ is positive. Clearly this result with the inequality reversed holds if $f'(x)$ is monotone decreasing.

We do not know if property (1) and bounded variation of $f'(x)$ imply linearity for $f(x)$. In view of the two preceding theorems it seems quite likely.

4. Remarks on the approximation of a function by its averages.

Suppose $f(x)$ is a continuous function defined on the interval $(a-\delta,\ b+\delta)$ $(\delta>0)$. We make some remarks on the approximation to

$f(x)$ by its averages

$$f(x, t)=\frac{1}{2t}\int_{x-t}^{x+t} f(u)du \qquad (0<t<\delta)\,, \quad x\varepsilon[p, b]\,.$$

If $f(x)$ is linear then $f(x, t)\equiv f(x)$. If $f(x)$ is not linear in any subinterval then there is an everywhere dense subset of points x at which the approximating functions are all either above or below $f(x)$. Otherwise the conditions of the theorem of [2, p. 253] are met and $f(x)$ would be linear.

One might ask if there are necessarily points at which $f(x, t)$ approaches $f(x)$ monotonely. From the results of § 2 above this can be seen to be false. For $t>0$, $f(x, t)$ is continuously differentiable function of t and

$$f_t(x, t)=\frac{1}{t}\left\{\frac{f(x+t)+f(x-t)}{2}-\frac{1}{2t}\int_{x-t}^{x+t} f(u)du\right\}=\frac{1}{t}\,g(x, t)\,.$$

From this it is clear the function of § 2 gives an example of a continuously differentiable function which at no point is approximated monotonely by its averages.

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UNIVERSITY OF UTAH

ON A PROBLEM IN THE THEORY OF MECHANICAL QUADRATURES

PHILIP DAVIS

1. Introduction. In the present note we study a scheme of mechanical quadratures of the form

$$\int_{-1}^{+1} f(x)dx \sim \sum_{k=0}^{n} a_{nk} f(\lambda_{nk}) = Q_n(f) \,, \tag{1}$$

as applied to certain distinguished classes of analytic functions on $[-1, +1]$. The question of the convergence of $Q_n(f)$ to the integral in (1) has been solved completely by Pólya [4] when f is selected from the class of continuous functions. There seems to be less discussion of the problem when f is selected from the class of analytic functions on $[-1, +1]$ or from certain of its subclasses.

Let B designate a region in the complex $z=x+iy$ plane which we shall assume contains $[-1, +1]$ in its interior. By $L^2(B)$ we designate the class of functions which are analytic and single valued in B and are such that

$$\iint_B |f|^2 dxdy < \infty \,. \tag{2}$$

With

$$(f, g) = \iint_B f\bar{g}\,dxdy \tag{3}$$

as an inner product, and $||f||^2 = (f, f)$ as a norm, the space $L^2(B)$ becomes a well-known and very useful Hilbert space of functions, possessing a reproducing kernel $K_B(z, \bar{w})$ which is generally referred to as the Bergman kernel for B [1].

Let E be a bounded linear functional over $L^2(B)$; its norm (over the conjugate space of all linear functionals) may be obtained in the following way. Let $\varphi_n(z)$ $(n=0, 1, \cdots)$ be a complete orthonormal system for $L^2(B)$. Then it may be shown that

$$||E||^2 = \sum_{n=0}^{\infty} |E(\varphi_n)|^2 \,. \tag{4}$$

This may be expressed in the alternate but equivalent form,

$$||E||^2 = E_z E_{\bar{w}} K_B(z, \bar{w}) \,, \tag{5}$$

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where the subscripts on the E mean that the functional operation is to be carried out on the variable indicated. We have, then, for all $f \in L^2(B)$,

(6) $$|E(f)| \leq \|E\| \, \|f\|,$$

with the equality sign being attained for some $f \in L^2(B)$. If now, the abscissas λ_{nk} lie in the interior of B, and the segment $[-1, +1]$ lies in the interior of B, then the linear functional

(7) $$E_n(f) = \int_{-1}^{+1} f(x)dx - \sum_{k=0}^{n} a_{nk} f(\lambda_{nk})$$

is bounded (cf. [2]) over $L^2(B)$, so that we have, for all $f \in L^2(B)$,

(8) $$|E_n(f)| \leq \|E_n\| \, \|f\| .$$

2. Uniform convergence. We shall say that the quadrature scheme (1) converges *uniformly in* $L^2(B)$ if, having been given an $\varepsilon > 0$, there is an $n_0 = n_0(\varepsilon)$ such that, for all $f \in L^2(B)$ and $n \geq n_0$, we have

(9) $$\left| \int_{-1}^{+1} f(x)dx - \sum_{k=0}^{n} a_{nk} f(\lambda_{nk}) \right| \leq \varepsilon \|f\| .$$

THEOREM 1. *A necessary and sufficient condition that the quadrature scheme* (1) *converges uniformly in* $L^2(B)$ *is that*

(10) $$\lim_{n\to\infty} \|E_n\|^2 = \lim_{n\to\infty} E_{nz} E_{n\bar{w}} K_B(z, \bar{w}) = 0 .$$

Proof. Suppose that (10) holds. Then given an $\varepsilon > 0$ we can find an $n_0(\varepsilon)$ such that $\|E_n\| \leq \varepsilon$ for all $n \geq n_0(\varepsilon)$. Hence, by (6), the inequality (9) must hold. Conversely, suppose that (9) holds. For each n, it is possible to find a nontrivial function $f_n(z) \in L^2(B)$ such that

(11) $$|E_n(f_n)| = \|E_n\| \, \|f_n\| .$$

By (9), given an $\varepsilon > 0$ we may find an $n = n_0(\varepsilon)$ such that for all $n \geq n_0(\varepsilon)$ and for all $f \in L^2(B)$ we have $|E_n(f)| \leq \varepsilon \|f\|$. Hence, in particular, for the f_n of (11),

(12) $$\|E_n\| \, \|f_n\| = |E_n(f_n)| \leq \varepsilon \|f_n\| .$$

Therefore (10) must follow.

We note that, in view of (4), the condition (10) can, in principle, be converted into a necessary and sufficient condition on the weights a_{nk} and abscissas λ_{nk}.

The following special case is of considerable interest. Let $\mathscr{E}_\rho$, $\rho>1$, designate an ellipse with foci at $(-1, 0)$ and $(1, 0)$ and with semimajor and semiminor axes a and b respectively, and where ρ is given by

$$\rho=(a+b)^2, \quad a=(\rho+1)/2\rho^{1/2}, \quad b=(\rho-1)/2\rho^{1/2}. \tag{13}$$

Observe that as $\rho\to 1$, $\mathscr{E}_\rho$ collapses to $[-1, +1]$. If $U_n(z)$ $(n=0, 1, \cdots)$ designates the Tschebysheff polynomials of the second kind defined by

$$U_n(z)=(1-z^2)^{-1/2}\sin((n+1)\operatorname{arc}\cos z), \tag{14}$$

then it is well known that the system of polynomials

$$\varphi_n(z)=2\sqrt{\frac{n+1}{\pi}}(\rho^{n+1}-\rho^{-n-1})^{-1/2}U_n(z) \quad (n=0, 1, 2, \cdots) \tag{15}$$

will be complete and orthonormal over $L^2(\mathscr{E}_\rho)$. Thus we have:

THEOREM 2. *A necessary and sufficient condition in order that the quadrature scheme* (1) *converge uniformly in* $L^2(\mathscr{E}_\rho)$ *is that*

$$\operatorname*{limit}_{n\to\infty}\frac{4}{\pi}\sum_{k=0}^{\infty}(k+1)\frac{|E_n(U_k)|^2}{\rho^{k+1}-\rho^{-k-1}}=0. \tag{16}$$

3. **Interpolatory quadrature.** An important class of quadrature schemes is formed by those which are of interpolatory type. For such quadratures we have

$$Q_n(f)=\int_{-1}^{+1}f(x)dx, \tag{17}$$

whenever f is a polynomial of degree not larger than n. If the scheme is of interpolatory type then (16) becomes

$$\lim_{n\to\infty}\frac{4}{\pi}\sum_{k=n+1}^{\infty}(k+1)\frac{|E_n(U_k)|^2}{\rho^{k+1}-\rho^{-k-1}}=0. \tag{18}$$

In view of the inequalities

$$\rho^{-1}\cdot\rho^{-k}\leq(\rho^{k+1}-\rho^{-k-1})^{-1}\leq(\rho-\rho^{-1})^{-1}\rho^{-k}, \qquad (\rho>1), \tag{19}$$

condition (18) is equivalent to

$$\lim_{n\to\infty}\sum_{k=n+1}^{\infty}(k+1)\frac{|E_n(U_k)|^2}{\rho^k}=0. \tag{20}$$

If we now define

$$\sigma_k = \begin{cases} 0 & (k \text{ odd}), \\ \dfrac{2}{k+1} & (k \text{ even}), \end{cases} \tag{21}$$

then (20) becomes

$$\lim_{n\to\infty} \sum_{k=n+1}^{\infty} (k+1)\Big(\sigma_k - \sum_{j=0}^{n} a_{nj} U_k(\lambda_{nj})\Big)^2 \rho^{-k} = 0 . \tag{22}$$

The following sufficient condition for the uniform convergence in $L^2(\mathscr{E}_\rho)$ of an interpolatory quadrature scheme can now be obtained. Set

$$M_n = \sum_{j=0}^{n} |a_{nj}| , \tag{23}$$

and observe that for real absissas λ in $[-1, +1]$ we have

$$|U_k(\lambda)| \leq k+1 . \tag{24}$$

Then, using (21) and (23), for fixed $\rho > 1$ we get

$$\begin{aligned} \sum_{k=n+1}^{\infty} (k+1)\Big(\sigma_k - \sum_{j=0}^{\infty} a_{nj} U_k(\lambda_{nj})\Big)^2 \rho^{-k} &\leq \sum_{k=n+1}^{\infty} (k+1)(\sigma_k + (k+1)M_n)^2 \rho^{-k} \\ &\leq 4 \sum_{k=n+1}^{\infty} ((k+1)\rho^k)^{-1} + 4M_n \sum_{k=n+1}^{\infty} (k+1)\rho^{-k} + M_n^2 \sum_{k=n+1}^{\infty} (k+1)^3 \rho^{-k} \\ &< o(1) + C_1 M_n n \rho^{-n} + C_2 M_n^2 n^3 \rho^{-n}, \qquad (n \to \infty) , \end{aligned} \tag{25}$$

where C_1 and C_2 are two positive constants which may depend upon ρ but are independent of n. Thus, we have the following result.

THEOREM 3. *Let*

$$\lim_{n\to\infty} M_n n^{3/2} \rho^{-n/2} = 0 . \tag{26}$$

Then an interpolatory quadrature scheme converges uniformly in $L^2(\mathscr{E}_\rho)$

Pólya [4, p. 285] has remarked that if

$$\lim_{n\to\infty} (M_n)^{1/n} = 1 \tag{27}$$

then an interpolatory quadrature scheme converges for all functions which are analytic in the closed basic interval. Under hypothesis (27), we have

$$M_n = (1+\varepsilon_n)^n , \quad \varepsilon_n \to 0,$$

so that (26) holds with all $\rho > 1$. Thus, under Pólya's hypothesis, we see that the convergence is also uniform in every $L^2(\mathscr{E}_\rho)$, $\rho > 1$.

4. Newton-Cotes quadrature. We turn now to a specific quadrature scheme on $[-1, +1]$, namely, the Newton-Cotes scheme. In this scheme, we have

$$Q_n(f)=a_{n0}f(-1)+a_{n1}f(-1+2/n)+a_{n2}f(-1+4/n)+\cdots+a_{nn}f(1) \quad (n=1, 2, \cdots), \tag{28}$$

where the Cotes numbers a_{nk} have been determined so that

$$Q_n(f)=\int_{-1}^{+1} f\,dx$$

holds for an arbitrary polynomial of degree $\leq n$. We have now the following estimate due to J. Ouspensky [3] (Ouspensky's basic interval is [0, 1]):

$$a_{nk}=-\frac{2}{n(\log n)^2}\binom{n}{k}\left[\frac{(-1)^k}{k}+\frac{(-1)^{n-k}}{n-k}\right](1+\eta_{nk}), \tag{29}$$

where $\eta_{n,k}\to 0$ as $n\to\infty$ uniformly for $k=1, 2, \cdots, n-1$, while

$$a_{n0}=a_{nn}=\frac{2}{n\log n}(1+\varepsilon_n), \qquad \varepsilon_n\to 0. \tag{30}$$

Thus,

$$M_n=\sum_{j=0}^{\infty}|a_{nj}|\leq\frac{4(1+\delta_n)}{n(\log n)^2}\sum_{k=1}^{n-1}\binom{n}{k}+\frac{4}{n\log n}(1+\varepsilon_n), \tag{31}$$

where we have written $\eta_{nk}<\delta_n$ $(k=1, 2, \cdots, n-1)$, $\delta_n\to 0$. Hence,

$$M_n\leq\frac{4(1+\delta_n)2^n}{n(\log n)^2}+\frac{4}{n\log n}(1+\varepsilon_n). \tag{32}$$

Condition (26) now holds with $\rho^{1/2}>2$. We have therefore arrived at the following result:

THEOREM 4. *The Newton-Cotes quadrature scheme converges uniformly in $L^2(\mathscr{E}_\rho)$ whenever $\rho>4$.*

Investigation of the convergence of the Newton-Cotes quadrature scheme has an interesting history which is worth retelling here. T. Stieltjes in 1884 first proved the convergence of the Gauss mechanical quadrature for the class of Riemann integrable functions, and in a letter to Hermite raised the question of the convergence of the Newton-Cotes scheme. In 1925 J. Ouspensky [3] arrived at the asymptotic result (29), and from the growth of Cotes numbers concluded only that the Newton-Cotes scheme is devoid of practical value. In 1933

G. Pólya [4] showed that this scheme is not valid for all continuous functions, and, indeed, is not valid for the class of analytic functions. Pólya's counterexample, referred to the interval $[-1, +1]$ is

$$f(w) = -\sum_{k=4}^{\infty} a^{k!} \frac{\sin k!\{(w+1)/2\}}{\cos \pi\{(w+1)/2\}} \qquad (1/2 < a < 1), \tag{33}$$

for which the Newton-Cotes scheme diverges. The functions $f(w)$ is regular in the strip

$$|\mathscr{I}(w)| < \frac{-2 \log a}{\pi} \tag{34}$$

and has a natural boundary along the sides of the strip. The widest such strip must be less than

$$|\mathscr{I}(w)| < \frac{2 \log 2}{\pi} = 0.4412 .$$

The function (33) cannot, therefore, be continued analytically to $\mathscr{E}_\rho = 4$, for which the semiminor axis is $b = .7500$. Theorem 4, therefore, rehabilitates the Newton-Cotes quadrature scheme for functions which are regular over a sufficiently large portion of the complex plane.

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National Bureau of Standards,
Washington, D. C.

ON CLOSED DIFFERENTIABLE CURVES OF ORDER n IN n-SPACE

DOUGLAS DERRY

1. Introduction. Let C_n be a closed curve in real projective n-space S_n whose coordinates x_i $(1 \leqq i \leqq n+1)$ are given in the parametric form

$$x_i = x_i(s)\,, \quad 1 \leqq i \leqq n+1\,, \quad q \leqq s < q+1\,,$$

where $x_i(s)$ are real continuous periodic functions of period 1, and q is any real number. The point with coordinates $x_i(s)$ $(1 \leqq i \leqq n+1)$ will be designated by its defining number s.

The curve C_n is to satisfy the following *order condition.*

No hyperplane of S_n contains more than n points of C_n.

A simple consequence of the above condition is that any $k+1$ $(0 \leqq k \leqq n)$ distinct curve points $s_1, s_2, \cdots, s_{k+1}$ span a linear k-subspace $[s_1, s_2, \cdots, s_{k+1}]$. (The square-bracket symbol $[A, B, \cdots]$ will be used throughout to designate the linear subspace spanned by the sets A, B, $\cdots$.)

The curve C_n is to satisfy the following *differentiability condition.*

For each point s of C_n and for each integer k $(0 \leqq k \leqq n-1)$ a linear k-subspace (k, s), known as the osculating k-space at s, exists for which $[s_1, s_2, \cdots, s_{k+1}]$ converges to (k, s) as $s_1, s_2, \cdots, s_{k+1}$ all approach s in any way whatsoever.

The curves C_3 were considered by A. Kneser [2] who studied properties which are invariant to certain continuous displacements. One of his results is that the set of planes of the projective space each of which contains exactly k $(k=1$ or $3)$ points of a C_3 builds a connected set. In the present paper the methods used by Kneser are adapted to study the properties of the curves C_n. All the proofs make use of those lines l each point of which is included in n distinct $(n-1, s)$. Thus the paper is, in a sense, a study of this line system. Among

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the results is a generalization of the foregoing Kneser result to n dimensions. This in turn leads to the result that those hyperplanes which contain less than n points of C_n are exactly those hyperplanes which contain at least one line l. This result is related to a result, implicit in a paper of Scherk [4], which states that the above hyperplanes are exactly those hyperplanes which contain certain limiting positions of the lines l.

2. **Multiplicities.** As all the critical boundary cases involve multiple intersection points, these points will have special importance. In this section we record the definition for multiplicity and note some known results which we shall use.

DEFINITION 1. A linear subspace Q is defined to intersect C_n *exactly k-fold* $(0<k\leqq n-1)$ at s if $(k-1, s)\subseteqq Q$, $(k, s)\nsubseteqq Q$, and *n-fold* if $(n-1, s)=Q$.

A point P is defined to be included in $(n-1, s)$ *exactly k-fold* $(0<k\leqq n-1)$ if $P\in(n-k, s)$, $P\notin(n-k-1, s)$, and *n-fold* if $P=(0, s)$.

The following *multiplicity convention* will be assumed throughout. Let $s_1, s_2, \cdots, s_j$ be any point system, and let s_i occur k_i-times $(1\leqq i\leqq j)$ in this system. A linear subspace Q is said to contain this system provided $(k_i-1, s_i)\subseteqq Q$ $(1\leqq i\leqq j)$. A point P is said to be included in the system $(n-1, s_1)$, $(n-1, s_2), \cdots, (n-1, s_j)$ provided $P\in(n-k_i, s_i)$ $(1\leqq i\leqq j)$. Unless otherwise stated the points of any given set are not necessarily all distinct.

For reference we state the easily proved:

LEMMA 1. *For $n\geqq 2$, the projection of C_n from one of its curve points s' is a C_{n-1}. The space (k, s), $s\neq s'$, $0\leqq k\leqq n-2$, projects into the space (k, s) of the projected C_{n-1} and the space (k, s'), $1\leqq k\leqq n-1$, into the space $(k-1, s')$ of C_{n-1}.*

By use of Lemma 1, it can be proved by induction that C_n satisfies the *sharpened order condition*, that no hyperplane cuts C_n in more than n curve points where multiple intersections are now counted with their proper multiplicity. This leads to the fact that the system s_1, $s_2, \cdots, s_{k+1}$ $(0<k\leqq n-1)$ is included in a unique k-space which we designate by $[s_1, s_2, \cdots, s_{k+1}]$. We note without proof that C_n satisfies the *sharpened differentiability condition* that $[s_1, s_2, \cdots, s_{k+1}]$ converges to (k, s) as $s_1, s_2, \cdots, s_{k+1}$ all approach s.

Use will be made of the duality theorem of Scherk [3] which states that all the $(n-1, s)$ build the dual of a C_n. This implies that no point P is contained within more than n $(n-1, s)$ and also that the intersection of $(n-1, s_1)$, $(n-1, s_2)$, $\cdots$, $(n-1, s_k)$ $(1 \leqq k \leqq n)$ approaches $(n-k, s)$ as $s_1, s_2, \cdots, s_k$ all approach s in any way whatsoever.

3. **Notation.** Throughout the paper the symbols l, l^μ will be tacitly assumed to represent lines each of the points of which is within n distinct $(n-1, s)$ of a given C_n; L, L^μ will be assumed to represent the $(n-2)$-spaces with the property that every hyperplane through such a space cuts C_n in n distinct points.

Where a proof involves both C_n and C_{n-1} the symbol $(k, s)_{n-1}$ will be used to designate the osculating k-space of the curve C_{n-1}.

4. A construction for the lines l.

THEOREM 1. *If, for $n \geqq 2$, A and B are any two distinct points of a given line l, then curve points s_i, t_i of C_n exist so that $A \in (n-1, s_i)$, $B \in (n-1, t_i)$ $(1 \leqq i \leqq n)$ and $s_1 < t_1 < s_2 < \cdots < s_n < t_n < s_1 + 1$ $(= s_{n+1})$.*

Conversely if A and B are points for which $A \in (n-1, s_i)$, $B \in (n-1, t_i)$, $s_1 < t_1 < s_2 < t_2 < \cdots < s_n < t_n < s_1 + 1$ $(= s_{n+1})$, then AB is a line l.

PROOF. Let $P(s)$ be the intersection $l \cap (n-1, s)$. Note that $l \not\subseteq (n-1, s)$; for otherwise l would contain a point of $(n-2, s)$, which point would be within $(n-1, s)$ at least twice contrary to the definition of l. Therefore $P(s)$ is defined uniquely for all s. As s moves continuously on C_n in a fixed direction, $P(s)$ moves continuously on l because $(n-1, s)$ is continuous. Also, $P(s)$ moves continuously in a fixed direction; for if $P(s)$ were to experience a reversal of direction at $P(s_0)$ then, in every curve neighborhood of s_0, points s_L, s_R would exist so that $s_L < s_0 < s_R$, $P(s_L) = P(s_R)$. Then, as $P(s)$ is continuous,

$$P(s_0) \in \lim_{s_L \to s_0,\ s_R \to s_0} (n-1, s_L) \cap (n-1, s_R) = (n-2, s_0)$$

and l would contain a point not in n distinct $(n-1, s)$ contrary to the hypothesis. Let $(n-1, s_i)$ $(1 \leqq i \leqq n;\ s_1 < s_2 < \cdots < s_n < s_1 + 1\ (= s_{n+1}))$ be the complete set of $(n-1, s)$ which contain A. As s increases continuously from s_1 to s_2, $P(s)$ makes one complete circuit of l in a fixed direction. Consequently it crosses the point B exactly once. Hence t_1 exists on C_n so that $B \in (n-1, t_1)$ $(s_1 < t_1 < s_2)$. Likewise within each arc $s_i < s < s_{i+1}$ $(2 \leqq i \leqq n)$, a point t_i exists on C_n so that $s_i < t_i < s_{i+1}$, $B \in (n-1, t_i)$. Thus the theorem is proved.

To prove the converse, let C be any interior point of one of the segments AB of the line through A and B, and D any interior point of the other segment. As $P(s)$ is continuous and

$$P(s_1)=A\,,\quad P(t_1)=B\,,$$

at least one solution $P(s)=C$, or $P(s)=D$ must exist for which $s_1<s<t_1$. Likewise each of the $2n$ arcs $s_i<s<t_i$, $t_i<s<s_{i+1}$ $(1\leqq i\leqq n)$ contains at least one solution $P(s)=C$ or $P(s)=D$. But as C is contained in at most n $(n-1, s)$ there must be exactly n solutions $P(s)=C$. As these are all distinct and C is arbitrary, AB is a line l. The proof is now complete.

This proof of the converse, due to Dr. P. Scherk, replaces a more complicated one of my own. I should like to take the opportunity to thank him for many helpful suggestions which have contributed to the readability of the paper.

5. Hyperplanes with a given number of curve points.

LEMMA 2. *If, for $n\geqq 3$, C_{n-1} is the projection of C_n from one of its points s, then a line l of C_n is projected into a line l of C_{n-1}.*

This is proved in [1].

LEMMA 3. *For $n\geqq 3$, the projection of a C_n from a line l is a C_{n-2}.*

PROOF. No hyperplane through l can cut C_n in more than $n-2$ points. This is true for $n=2$ as it is equivalent to the fact that a line l of C_2 cannot contain any curve points. Assume the assertion is true for C_{n-1} $(n>2)$. Let H be a hyperplane which contains l. The result is clear if H contains no points of C_n. Let $\bar{s}$ be a point of C_n within H. Project from s. Then C_n is projected into a C_{n-1} by Lemma 1, and l into a line l of C_{n-1}, by Lemma 2, which is within the projection $\overline{H}$ of H. By the induction assumption $\overline{H}$ contains at most $n-3$ points of C_{n-1}. Therefore H, which contains the points C_n into which these are projected together with $\bar{s}$ contains at most $n-2$ points of C_n.

The space of all 2-spaces through l is an $(n-2)$-space S_{n-2} whose hyperplanes are the hyperplanes of the original space which contain l. The elements $[l, s]$ of S_{n-2} build a curve C, and C has order $n-2$ by the result of the previous paragraph. This implies

$$[l,\ s']\neq[l,\ s''] \qquad \text{if } s'\neq s''\,.$$

Thus there is a one-to-one correspondence between the points of C_n and those of C. Where $0\leqq k\leqq n-2$, let

$$[l,\ s_1],\ [l,\ s_2], \cdots,\ [l,\ s_{k+1}]$$

be given curve points of C. Because of the order condition these points span a $(k+2)$-space Q which contains l. If $s_1, s_2, \cdots, s_{k+1}$ all approach s, then $Q\rightarrow[l,\ (k,\ s)]$ because of the differentiability condition. Thus the set of elements $[l,\ s]$ of S_{n-2} is a C_{n-2} with osculating k-spaces $[l,\ (k,\ s)]$. As this set is equivalent to the projection of C_n from l, the lemma is established.

Most induction proofs for the curves C_n make use of Lemma 1; in the following proof Lemma 3 is used for this purpose.

THEOREM 2. *Where* $0\leqq k\leqq n$, $k\equiv n$ (mod 2), *let* $s_1, s_2, \cdots, s_k$; $t_1, t_2, \cdots, t_k$ *be any points of* C_n; *then:*

(a) *If, for* $n\geqq 1$, H_1, H_2 *be hyperplanes which contain* $s_1, s_2, \cdots, s_k$; $t_1, t_2, \cdots, t_k$ *respectively, and no additional points of* C_n, *then hyperplanes* $H(p)$ $(0\leqq p\leqq 1)$ *exist, continuously dependent on* p, *each of which contains exactly* k *points of* C_n *and for which* $H(0)=H_1$, $H(1)=H_2$;

(b) *If* $s_i=t_i$ $(1\leqq i\leqq k)$, *then* $H(p)$ *can be chosen so that it contains exactly the points* s_i $(1\leqq i\leqq k,\ 0\leqq p\leqq 1)$;

(c) *if* $n\geqq 2$, $0\leqq k\leqq n-2$, *for a given line* l, *a hyperplane* H^l *exists so that it contains exactly the points* $s_1, s_2, \cdots, s_k$, *together with the line* l.

PROOF. We first prove (c). If $n=2$ then $k=0$ and the result is equivalent to the fact that $H^l=l$ does not cut C_2. Assume the result for for all curves C_{n-1} $(n>2)$. Project from l. Thus C_n is projected into a C_{n-2}, by Lemma 3, and $s_1, s_2, \cdots, s_k$ into points of C_{n-2} with the same numerical coordinates. If $k=n-2$, a unique hyperplane

$$H'=[s_1,\ s_2,\ \cdots,\ s_k]$$

exists in the projected $(n-2)$-space through these points. If $k<n-2$, then by the induction assumption a hyperplane H' exists in the projected space which contains exactly the points $s_1, s_2, \cdots, s_k$ of C_{n-2}. Consequently, if H^l is defined to be the hyperplane of the original space which is projected into H', this hyperplane contains exactly the points $s_1, s_2, \cdots, s_k$ of C_n. As $l\subseteqq H^l$, (c) is proved for C_n. The proof

can now be completed by induction.

To prove (a) and (b), consider first the case $k=0$. With this restriction neither H_1 nor H_2 contains points of C_n. As the curve is connected, it lies entirely within one of the two open regions of the projective space whose boundary is the set of points of H_1 and H_2. Hence an affine coordinate system exists so that the equations of H_1, H_2 are $x_1=0$, $x_1=1$, respectively, and C_n contains no points for which $0\leqq x_1\leqq 1$. Now (a) and (b) follow for $k=0$ if $H(p)$ is defined to be the hyperplane with the equation $x_1=p$, $0\leqq p\leqq 1$.

Now let $k=n$; (b) is trivial in this case. Let $f_i(p)$ $(0\leqq p\leqq 1,\ 1\leqq i\leqq n)$ be any real-valued continuous functions for which $f_i(0)=s_i$, $f_i(1)=t_i$. Then (a) follows if $H(p)$ is defined to be the hyperplane spanned by the points with coordinates $f_i(p)$ $(1\leqq i\leqq n)$.

In particular this establishes (a) and (b) for C_1 and C_2. Assume both results for all C_{n-1} $(n>2)$. We may assume $0<k\leqq n-2$. Let l be arbitrary. By (c), hyperplanes H_1^l, H_2^l exist which contain exactly the points $s_1, s_2, \cdots, s_k$; $t_1, t_2, \cdots, t_k$, respectively, together with the line l. Let $\overline{H}_1$, $\overline{H}_1^l$, C_{n-1} be the projections of H_1, H_1^l, C_n, respectively, from s_1. By the induction assumption (b), hyperplanes $\overline{H}(p)$ $(0\leqq p\leqq 1)$ exist in the projected space, continuously dependent on p, each of which contains exactly the points $s_2, \cdots, s_k$ of C_{n-1}, and for which

$$\overline{H}(0)=\overline{H}_1,\ \ \overline{H}(1)=\overline{H}_1^l.$$

Let $H(p)$ $(0\leqq p\leqq(1/3))$ be the hyperplane of the original space which is projected into $\overline{H}(3p)$. Then $H(p)$ depends continuously on p, contains exactly the points $s_1, s_2, \cdots, s_k$ of C_n, and $H(0)=H_1$, $H(1/3)=H_1^l$. Likewise $H(p)$ $((2/3)\leqq p\leqq 1)$ exists so that it depends continuously on p, contains exactly the points $t_1, t_2, \cdots, t_k$ of C_n, and for which

$$H(2/3)=H_2^l,\ \ H(1)=H_2.$$

After a projection from l, a similar argument can be used to construct a hyperplane $H(p)$ $((1/3)\leqq p\leqq(2/3))$ which depends continuously on p, contains exactly k points of C_n, and for which

$$H(1/3)=H_1^l,\ \ H(2/3)=H_2^i.$$

This proves (a) for C_n. Also (b) is clear if $H(p)$ is defined as above with the additional conditions that

$$H_1^l=H_2^l=H(p) \qquad ((1/3)\leqq p\leqq(2/3)).$$

The proof can now be completed by induction.

6. Hyperplanes which do not contain n points of C_n.

DEFINITION 2. $\sum(C_n)$ is the set of all points included in at least one space L of the curve C_n (cf. § 3).

LEMMA 4. *If, for $n\geqq 3$, $\overline{P}\in\sum(C_{n-1})$, where $\overline{P}$ is the projection of a point P from a point s' of C_n, $P\neq s'$, and C_{n-1} that of C_n, then $P\in\sum(C_n)$.*

Proof. If $\overline{P}\in\sum(C_{n-1})$, then points $s_1, s_2, \cdots, s_{n-1}$; $t_1, t_2, \cdots, t_{n-1}$ of the projection C_{n-1} exist so that

$$\overline{P}\in[s_1, s_2, \cdots s_{n-1}]\cap[t_1, t_2, \cdots, t_{n-1}]=L$$

and

$$s_1<t_1<s_2<\cdots<t_{n-1}<s_1+1\ ,$$

by the dual of Theorem 1. Moreover,

$$[s_1, s_2, \cdots, s_{n-1}],\ [t_1, t_2, \cdots, t_{n-1}]$$

may be chosen to be any two distinct hyperplanes through L within the projected $(n-1)$-space. Therefore these hyperplanes may be chosen so that $t_{n-1}<s'<s_1+1$. Let the numbers

$$s_1, s_2, \cdots, s_{n-1}, t_1, t_2, \cdots, t_{n-1}, s'$$

now represent points of C_n. Then $P\in[t_1, t_2, \cdots, t_{n-1}, s']$. As $t_1, t_2, \cdots, t_{n-1}, s'$ are represented by linearly independent vectors the intersection

$$\prod_{i=1}^{i=n-1}[t_1, t_2, \cdots, t_{i-1}, t_{i+1}, \cdots, t_{n-1}, s']=s'.$$

Hence, because $P\neq s'$, at least one value i exists with

$$P\notin[t_1, t_2, \cdots, t_{i-1}, t_{i+1}, \cdots, t_{n-1}, s'] \qquad (1\leqq i\leqq n-1).$$

For such a value i

$$[t_1, t_2, \cdots, t_{i-1}, P, t_{i+1}, \cdots, t_{n-1}, s']=[t_1, t_2, \cdots, t_{n-1}, s'].$$

Let t_n be a point of C_n with $t_n>s'$. Then

$$[t_1, t_2, \cdots, t_{i-1}, P, t_{i+1}, \cdots, t_{n-1}, t_n]$$

approaches $[t_1, t_2, \cdots, t_{n-1}, s']$ as t_n approaches s'. Because of the continuity of the curve points of C_n, $[t_1, t_2, \cdots, t_{i-1}, P, t_{i+1}, \cdots, t_n]$ will contain a point t_i' of C_n for which $s_i<t_i'<s_{i+1}$ provided t_n is sufficiently

close to s'. If t_n is such a point, and s_n is defined as s', then

$$P \in [s_1, s_2, \cdots, s_n] \cap [t_1, t_2, \cdots, t_{i-1}, t'_i, t_{i+1}, \cdots, t_n]$$

and

$$s_1 < t_1 < s_2 < \cdots < s_i < t'_i < s_{i+1} < s_n < t_n < s_1 + 1 .$$

It follows from the dual of Theorem 1 and Definition 2 that $P \in \sum(C_n)$. The lemma is thus established.

COROLLARY. *If, for $n \geqq 3$, P is a point for which $P \in [(k, s_1), s_2]$ $(s_1 \not\asymp s_2,\ 0 \leqq k \leqq n-3,\ P \not\asymp s_2)$ $P \notin (k, s_1)$, then $P \in \sum(C_n)$.*

PROOF. If $n=3$ then $P \in [s_1, s_2]$ $(s_1 \not\asymp s_2,\ P \not\asymp s_1,\ P \not\asymp s_2)$. Let t_1, t_2 be points of C_3 for which $s_1 < t_1 < s_2 < t_2 < s_1 + 1$. Then $P \notin [t_1, t_2]$; for otherwise t_1, t_2, s_1, s_2 would be coplanar in contradiction to the order condition. Hence $[P, t_1, t_2]$ is a plane. This plane must contain a third point t of C_3, as C_3 is closed. Now $P \not\asymp t$ because $[s_1, s_2]$ cannot contain a third curve point. If $\bar{P}$ is the projection of P from t then

$$\bar{P} \in [s_1, s_2] \cap [t_1, t_2],$$

where s_1, s_2, t_1, t_2 now represent curve points of the projection C_2 of C_3 from t. This implies, by the dual of Theorem 1, that $\bar{P} \in \sum(C_2)$, and so by the Lemma that $P \in \sum(C_3)$. Thus the corollary is true for $n=3$. Assume it to be true for all C_{n-1}, $n>3$. The result for C_n then follows from the Lemma by a projection from s_1 if the least possible $k=n-3$ and otherwise by a projection from a point of C_n different from s_1 and s_2.

LEMMA 5. (a) *For $n \geqq 2$, $\sum(C_n)$ is open.* (b) *If a boundary point $\bar{P}$ of $\sum(C_n)$ is approached by a sequence P^μ of points interior to $\sum(C_n)$, and $\bar{L}$ is the limit of a space sequence L^μ for which $P^\mu \in L^\mu$, then* (k, s) $(0 \leqq k \leqq n-2)$ *exists for which $\bar{P} \in (k, s) \subseteqq \bar{L}$.*

PROOF. If $P \in \sum(C_n)$ then a space L exists for which $P \in L$. By the dual of Theorem 1, $s_1, s_2, \cdots, s_n$; $t_1, t_2, \cdots, t_n$ exist so that

$$L \subseteqq [s_1, s_2, \cdots, s_n] \cap [t_1, t_2, \cdots, t_n] \text{ and } s_1 < t_1 < s_2 < \cdots < t_n < s_1 + 1.$$

If P' is sufficiently close to P then it is contained within an $(n-2)$-space L' which is so close to L that it has the form

$$[s'_1, s'_2, \cdots, s'_n] \cap [t'_1, t'_2, \cdots, t'_n] \qquad (s'_1 < t'_1 < s'_2 < \cdots < t'_n < s'_1 + 1).$$

By the dual of Theorem 1, $P' \in \sum(C_n)$, and so (a) is proved.

To prove (b), let H_1^μ, H_2^μ be any two hyperplane sequences with $L^\mu \subseteqq H_1^\mu$, $L^\mu \subseteqq H_2^\mu$, which converge to two distinct limits H_1 and H_2, respectively. By the dual of Theorem 1, $s_1^\mu, s_2^\mu, \cdots, s_n^\mu; t_1^\mu, t_2^\mu, \cdots, t_n^\mu$ exist so that $s_1^\mu < t_1^\mu < s_2^\mu < \cdots < t_n^\mu < s_1^\mu + 1$ and

$$H_1^\mu = [s_1^\mu, s_2^\mu, \cdots, s_n^\mu], \quad H_2^\mu = [t_1^\mu, t_2^\mu, \cdots, t_n^\mu].$$

As H_1^μ, H_2^μ converge, the sequences s_i^μ, t_i^μ $(1 \leqq i \leqq n)$ also converge. If s_i, t_i are the respective limits of these sequences,

$$\overline{L} = [s_1, s_2, \cdots, s_n] \cap [t_1, t_2, \cdots, t_n] \text{ and } s_1 \leqq t_1 \leqq s_2 \leqq \cdots \leqq t_n \leqq s_1 + 1.$$

At least one equality sign must occur in this system, for otherwise $\overline{P} \in \overline{L}$ and so $\overline{P} \in \sum(C_n)$; this is impossible as $\overline{P}$ is a boundary point of the open set $\sum(C_n)$. We may suppose, after a possible adjustment in the notation, $s_1 = t_1$. Hence $s_1 \in \overline{L}$. If $n=2$ this proves the Lemma, as

$$\overline{P} = \overline{L} = s_1 = (0, s_1).$$

Assume it holds for all curves C_{n-1}, $n>2$. If $\overline{P} = s_1$, then it is already true for C_n. If $\overline{P} \neq s_1$, project from s_1. Let C_{n-1} be the projection of C_n and $\overline{P}'$ that of $\overline{P}$. Then $\overline{P}' \notin \sum(C_{n-1})$, for otherwise, by Lemma 4, $\overline{P} \in \sum(C_n)$. Moreover,

$$\overline{P}' \in [s_2, s_3, \cdots, s_n] \cap [t_2, t_3, \cdots, t_n] = \overline{L}'$$

and this space is approached by the system

$$[s_2^\mu, s_3^\mu, \cdots, s_n^\mu] \cap [t_2^\mu, t_3^\mu, \cdots, t_n^\mu],$$

where all the numbers now represent points of C_{n-1}. Thus $\overline{P}'$ is a boundary point of $\sum(C_{n-1})$. Therefore by the induction assumption $(k, s)_{n-1}$ exists so that

$$\overline{P}' \in (k, s)_{n-1} \subseteqq \overline{L} \qquad (0 \leqq k \leqq n-3).$$

Consequently, $\overline{P} \in [s_1, (k, s)] \subseteqq \overline{L}$. Because $\overline{P} \neq s_1$, it now follows from the Corollary to Lemma 4 that $\overline{P} \in (k, s)$, or $s = s_1$ and $P \in (k+1, s)$. Either of these possibilities shows the lemma to be true and so the proof is complete.

LEMMA 6. *If, for $n \geqq 3$, l^μ is a sequence which converges to $\overline{l}$, and p an integer for which $\overline{l} \subseteqq (p, s)$, $\overline{l} \nsubseteq (p-1, s)$ $(0<p<n-1)$ then $[l^\mu, (q, s)] \to (q+2, s)$ $(p-1 \leqq q \leqq n-3)$.*

PROOF. The space $[l^\mu, (q, s)]$ is a $(q+2)$-space because $q<n-1$ while l^μ and (q, s) have no common points. Consider first the case for which $q=n-3, p=n-2$. If the lemma were false then a convergent subsequence of $[l^\mu, (n-3, s)]$ would exist whose limit would be a hyperplane Q for which $Q \not\geqq (n-1, s)$. As $\bar{l}^\mu \to l$,

$$[l, (n-3, s)]=(n-2, s) \leqq Q.$$

Consequently Q would cut C_n in s at least $(n-1)$-fold. As C_n is closed, Q would cut C_n in one additional point s', and $s' \neq s$ as $Q \not\geqq (n-1, s)$. Hence, if l^μ is sufficiently close to $\bar{l}$, $[l^\mu, (n-3, s)]$ would cut C_n in a point s'' so close to s' that $s'' \neq s$. Therefore the hyperplane $[l^\mu, (n-3, s)]$ would cut C_n in more than $n-2$ points in contradiction to Lemma 3. Thus $[l^\mu, (n-3, s)]$ must approach $(n-1, s)$, and the lemma is proved in this case. In particular, it is completely proved for $n=3$. Assume it is established for all C_{n-1}, $n>3$.

Consider next the case for which $q<n-3$. Project from any point t of C_n different from s. As $t \not\in (p, s)$, $\bar{l}$ is projected into a line $\bar{l}'$, and l^μ is projected into a line l'^μ defined for the projection C_{n-1} of C_n by Lemma 2, Clearly

$$\bar{l}' \leqq (p, s)_{n-1} \text{ and } \bar{l}' \not\leqq (p-1, s)_{n-1},$$

for otherwise

$$l \leqq [(p-1, s), t] \cap (p, s)=(p-1, s).$$

Therefore, by the induction assumption, $[l'^\mu, (q, s)_{n-1}] \to (q+2, s)_{n-1}$. This implies $[l^\mu, (q, s), t] \to [(q+2, s), t]$, and, because t is arbitrary, that $[l^\mu, (q, s)] \to (q+2, s)$. Thus the lemma is proved in this case.

Finally let $q=n-3$, $p<n-2$. If $[l^\mu, (n-3, s)]$ does not converge to $(n-1, s)$ this set contains a convergent subsequence with limit Q, $Q \not\geqq (n-1, s)$. Now $1 \leqq p<n-2$, and so $n \geqq 4$. Hence by the result of the previous paragraph $[l^\mu, (n-4, s)] \to (n-2, s)$. Consequently $(n-2, s) \leqq Q$. This leads to the contradiction encountered in the first paragraph. Thus $[l^\mu, (n-3, s)] \to (n-1, s)$, and the lemma is proved for C_n. The proof can now be completed by induction.

DEFINITION 3. $\sigma(C_n)$ is the set of all hyperplanes each of which contains at least one line l of the curve C_n.

$\sigma(C_n)$ is the dual of the space $\sum(C_n)$.

THEOREM 3. *For $n \geqq 2$, $\sigma(C_n)$ consists of all the hyperplanes which do not contain n points of C_n.*

PROOF. By Lemma 3 each member of $\sigma(C_n)$ contains less than n

points of C_n. It remains to show that every hyperplane which contains less than n points of C_n contains at least one line l. Let H be a hyperplane and $s_1, s_2, \cdots, s_h$ be the points of C_n contained in H, where $0 \leqq h < n$. As C_n is closed, $h \equiv n \pmod 2$. By Theorem 2 (c), for a given line l, a hyperplane H^l exists which contains l and exactly the points $s_1, s_2, \cdots, s_h$ of C_n. By Theorem 2 (b), a system $H(p)$ $(0 \leqq p \leqq 1)$ of hyperplanes exists, continuously dependent on p, each of which contains exactly the points $s_1, s_2, \cdots, s_h$ of C_n and for which

$$H(0)=H^l, \quad H(1)=H.$$

By Definition 3, $H(0) \in \sigma(C_n)$. Assume $H \notin \sigma(C_n)$. By the dual of Lemma 5 (a), $\sigma(C_n)$ is open. Therefore a least value $\bar{p}$ of p exists for which $H(\bar{p}) \notin \sigma(C_n)$. Let p^μ be a sequence for which $p^\mu \to \bar{p}$, $p^\mu < \bar{p}$. As $H(p^\mu) \in \sigma(C_n)$, l^μ exists for which $l^\mu \subseteqq H(p^\mu)$. By replacing p^μ by an appropriate subsequence if necessary we may assume l^μ converges. If $\bar{l}$ be the limit of l^μ then, by the dual of Lemma 5 (b), (k, s) exists so that

$$\bar{l} \subseteqq (k, s) \subseteqq H(\bar{p}) \qquad (1 \leqq k < n-1).$$

We may assume $(k+1, s) \notin H(\bar{p})$; for otherwise (k, s) may be replaced by an osculating space of a greater dimension so that this relation holds. Consequently s occurs exactly $(k+1)$-fold in the set $s_1, s_2, \cdots, s_h$, and $k+1 \leqq h \leqq n-2$. This is impossible if $h \leqq 1$ in which case $H \in \sigma(C_n)$. In particular this proves the theorem for $h \leqq 3$. We assume therefore $n > 3$. As $k \leqq n-3$ and $\bar{l} \subseteqq (k, s)$, the number q of Lemma 6 may be specialized to k. It follows then from this Lemma that $[l^\mu, (k, s)] \to (k+2, s)$. Hence, as $[l^\mu, (k, s)] \subseteqq H(p^\mu)$, $(k+2, s) \subseteqq H(\bar{p})$. This contradicts the fact that $s_1, s_2, \cdots, s_h$ are the only points of C_n in $H(\bar{p})$ among which s occurs exactly $(k+1)$-fold. Therefore $H \in \sigma(C_n)$. Thus the theorem is established.

7. A characterization of the lines l.

THEOREM 4. *For $n \geqq 2$, a straight line is a line l if, and only if, every hyperplane through l contains less than n points of C_n.*

PROOF. Let m be a straight line which is not a line l. Then at least one point P exists on m which is not within n distinct $(n-1, s)$. A sequence of points P^μ exists with $P^\mu \to P$ for which each P^μ is within less than n $(n-1, s)$. (This can be conveniently proved by induction in the dual formulation.) If A is a point of m for which $A \neq P$ then $[A, P^\mu] \to m$. By the dual of Theorem 3, L^μ (cf. § 3) exists for which $P^\mu \in L^\mu$. Now $[A, L^\mu]$ contains $[A, P^\mu]$ and also n points of C_n by the definition of L^μ. The limit of a convergent subsequence of $[A, L^\mu]$ is a hyperplane which contains m together with n points of C_n. This proves that if every hyperplane through a straight line contains less than n points of C_n then every point of the straight line is within n distinct $(n-1, s)$ and so must be a line l.

No hyperplane through a line l can contain n points of C_n by Lemma 3. Thus the proof of the theorem is complete.

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UNIVERSITY OF BRITISH COLUMBIA

BOOLEAN ALGEBRAS WITH PATHOLOGICAL ORDER TOPOLOGIES

E. E. FLOYD

If L is a partially ordered set, there are a variety of known ways in which L may be given a topology compatible, in some sense, with its partial ordering (see [1, 6]). Examples, by Northam [3] and Floyd and Klee [2], have very recently appeared of complete lattices which are not Hausdorff in their order topologies. It appears, then, that the various topologies will not be central in the study of *all* complete lattices. The question remains as to whether or not there is some wide and natural class of lattices in which some compatible topology has nice properties. We give a very simple example of a complete Boolean algebra which is not Hausdorff in any topology compatible with the order. We also give an example of a conditionally complete vector lattice in which addition is not continuous in any compatible topology. This is a counterexample to a result of Birkhoff [1, p. 242], who overlooked the possibility that convergence in the order topology differs from order convergence.

DEFINITION. Suppose that $(P, \geq)$ is a partially ordered set, and suppose that T is a topology for the set P (that is, T is a collection of subsets of P closed under arbitrary unions and finite intersections, and with $\phi \in T$, $P \in T$). We say that T is *σ-compatible with* $\geq$ if and only if whenever (x_i) is a sequence in P with

$$x_1 \geq x_2 \geq \cdots \text{ and } \bigwedge_i x_i = x$$

or

$$x_1 \leq x_2 \leq \cdots \text{ and } \bigvee_i x_i = x\,,$$

then the sequence (x_i) T-converges to x.

THEOREM 1. *Let L denote the complete Boolean algebra of all regular open subsets of the unit interval I, partially ordered by inclusion $>$. Suppose that T is a topology for L which is σ-compatible with $>$. Then the topology T is not Hausdorff.*

Proof. Recall that a subset b of I is a regular open set if and only if b is the interior of its closure. L is known to be a complete

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Boolean algebra [1]. Let $\mathscr{U}$ be a T-neighbourhood of the empty set $\phi \in L$. We show that $I \in \overline{\mathscr{U}}$. Suppose that $U_1, U_2, \cdots$, is a basis for the open sets of I, with each U_i nonempty. There exists for each i a sequence $(A^i_j \mid j=1, 2, \cdots)$ in L with

$$A^i_j \subset U_i, \quad A^i_j \neq \phi,$$

so that $A^i_1 > A^i_2 > \cdots$ and

$$\bigwedge (A^i_j \mid j=1, 2, \cdots) = \phi .$$

Since (A^1_i) converges to ϕ, there exists $A^1_i \in \mathscr{U}$. Define $B_1 = A^1_i$. Since the sequence $(B_1 \vee A^2_j)$ converges to B_1, there exists j with $B_1 \vee A^2_j \in \mathscr{U}$. Define $B_2 = B_1 \vee A^2_j$. Similarly there exists $B_3 = B_2 \vee A^3_k \in \mathscr{U}, \cdots$. Now (B_i) is a sequence in $\mathscr{U}$ with $B_1 < B_2 < \cdots$. Moreover, since the only regular open set containing $\bigcup B_i$ is I, we have $\bigvee_i B_i = I$. Hence $I \in \overline{\mathscr{U}}$ and the theorem follows.

The following remark answers Problem 77 of Birkhoff [1, p. 167].

THEOREM 2. *If L is the complete Boolean algebra of Theorem 1, then there exist, for $i=1, 2 \cdots$, sequences $(X_{i,j} \mid j=1, 2, \cdots)$ with $(X_{i,j})$ order-converging to ϕ for each i but such that for no function $j(i)$ is it true that $(X_{i,j(i)})$ order-converges to ϕ.*

Proof. Let $(X_{i,j})$ denote the sequence (A^i_j) of the proof of Theorem 1. Consider any function $j(i)$, then

$$\bigvee_{i \geq k} A^i_{j(i)} = I .$$

Hence

$$\bigwedge_k \bigvee_{i \geq k} A^i_{j(i)} = I .$$

Hence the sequence $(X_{i,j(i)})$ does not order-converge to ϕ.

THEOREM 3. *Let L be the complete Boolean algebra of Theorem 1, and let M be a Stone representation space for L. Let N denote the lattice of all continuous real-valued functions on M. Then N is a conditionally complete vector lattice in which the function $x-y$ is not T-continuous simultaneously in x and y for any T_1-topology T for N which is σ-compatible with $>$.*

Proof. It is known [4, 7] that N is conditionally complete. We may consider L as identical with the algebra of all open and closed subsets of M. There is a function $t: L \to N$ which assigns to $u \in L$ the characteristic function $t(u)$ of the open and closed set u. We show that t is an embedding of L in N. It is seen that t is an isotone one-to-one map of L onto $t(L)$, and t^{-1} is an isotone map of $t(L)$ on L. We prove that if $K \subset L$ then

$$\bigvee t(K) = t(\bigvee K) ,$$

where $\bigvee t(K)$ denotes the least upper bound in N. Clearly

$$t(\bigvee K) \geq \bigvee t(K) .$$

Now $\bigvee t(K)$ is a nonnegative continuous function whose value is ≥ 1 on the set $\bigcup K$, and hence ≥ 1 also on its closure. But the closure of $\bigcup K$ is $\bigvee K$ [7]. Hence

$$t(\bigvee K) \leq \bigvee t(K)$$

and equality holds. The dual also follows. So t embeds L in N. It follows that $t(L)$ is not Hausdorff in the topology T restricted to $t(L)$. Hence N is not Hausdorff in the topology T. But if $x-y$ is T-continuous in x and y, it is known that N is then regular [5, p. 54] and hence Hausdorff.

COROLLARY. *Suppose, in addition to the hypotheses of Theorem* 3, *that the function $y \to -y$ on N is T-continuous. Then $x+y$ is not T-continuous in x and y simultaneously.*

This answers, in the negative, a part of Problem 4 of Rennie [6, p. 51].

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UNIVERSITY OF VIRGINIA

ASYMPTOTIC LOWER BOUNDS FOR THE FUNDAMENTAL FREQUENCY OF CONVEX MEMBRANES

GEORGE E. FORSYTHE

1. Introduction. Let the bounded, simply connected, open region R of the (x, y)-plane have the boundary curve C. If a uniform ideal elastic membrane of unit density is uniformly stretched upon C with unit tension across each unit length, then λ, the square of the fundamental frequency, satisfies the conditions (subscripts denote differentiation)

$$\text{(1a)} \qquad \begin{cases} \Delta u \equiv u_{xx} + u_{yy} = -\lambda u & \text{in } R\,, \\ \lambda = \text{minimum}\,, \end{cases}$$

with the boundary condition

$$\text{(1b)} \qquad u(x, y) = 0 \quad \text{on} \quad C\,.$$

Variational methods of the Rayleigh-Ritz type are frequently used to approximate λ. They always yield upper bounds for λ, and the upper bounds can be made arbitrarily close.

Another common practical method of approximating λ is to calculate the least eigenvalue λ_h of a suitably chosen finite-difference operator Δ_h over a network with small mesh width h. For one choice of Δ_h it was shown by Courant, Friedrichs, and Lewy [3, p. 57] without details that $\lambda_h \to \lambda$ as $h \to 0$. For convex regions R of a special polygonal form the author has shown [4] that a special case of (11) below is valid for a common choice of Δ_h, and hence that λ_h is asymptotically a lower bound for λ as $h \to 0$. For an unusual finite-difference approximation to problem (1) when R is the union of squares of the network, Polya [12] has found that $\lambda_h > \lambda$ for all h, and also for the higher eigenvalues. The author knows of no other study of the sign or order of decrease of $\lambda - \lambda_h$ to 0.

In the present paper the investigation of [4] is extended to a much wider class of regions: those with piecewise analytic boundary curves and convex corners. The new theorems are stated and proved in §§ 3 and 4. Theorem 2 contains the theorem of [4] as a special case. Lemmas used in the proof of Theorem 1 are given in § 5. Identity (31) of Lemma 7 is interesting in itself.

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When C is no longer made up of line segments of the network, it is necessary when using finite-difference methods either to move C or to alter Δ_h near the boundary. The latter procedure is potentially more accurate, and has been adopted in deriving the rather delicate results proved below. The definition of Δ_h given in § 2 is a self-adjoint modification of Mikeladze's approximation [10; 11], and is believed to be new. The cruder approximations to Δ near C proposed by Collatz in 1933 and expounded in [2, p. 357], while easier to compute in practice, appear to introduce an unmanageable term $O(h^2)$ into (19). It is therefore doubted that Theorem 2 would remain valid for these cruder operators.

The technique of the present paper could be applied to study the asymptotic behavior of λ_h also for other difference approximations to Δ in the interior of R—for example, for those associated with a triangular net [2, p. 367].

It is not clear that one could revise the argument of the paper to prove an inequality of the type

$$\frac{\lambda}{\lambda_h} \leq 1 + bh^2 + o(h^2) \ .$$

2. Definitions. Assume the bounded, simply connected, open region R to have a closed boundary curve C: $x(s)+iy(s)$ $(0 \leq s \leq s_m)$ which is *piecewise analytic*. That is, $x(s)$ and $y(s)$ are real analytic functions of the arc length s of C in each of a finite number m of closed intervals

$$0 = s_0 \leq s \leq s_1, \quad s_1 \leq s \leq s_2, \quad \cdots, \quad s_{m-1} \leq s \leq s_m \ .$$

Moreover, we demand that the corners of C be convex; that is, at any point $x(s_j)+iy(s_j)$ $(0 \leq j < m)$ where distinct analytic curves meet, the interior angle of C must be less than π.

For $h > 0$, let a *net* consist of the lines $x = \mu h$, $y = \nu h$ $(\mu, \nu = 0, \pm 1, \pm 2, \cdots)$. The points $(\mu h, \nu h)$ in R are the *interior nodes* R_h of the net. The *boundary nodes* C_h of the net consist of (i) all points $(\mu h, \nu h)$ on C, and (ii) all *isolated* points of intersection of the net with C. Thus each node $(\mu h, \nu h)$ of R_h has two *neighboring nodes* in $R_h \cup C_h$ on the line $x = \mu h$, and two in $R_h \cup C_h$ on the line $y = \nu h$. Moreover, each node in C_h has at least one neighbor in $R_h \cup C_h$.

We now move toward a definition of the difference operator Δ_h. Let us denote the neighboring nodes of the node

(2) $\quad (x, y)$ of R_h by $(x-h_1, y)$, $(x+h_2, y)$, $(x, y-h_3)$, and $(x, y+h_4)$,

where $0 < h_i \leq h$ for $i = 1, 2, 3, 4$. For nodes remote from C_h, all $h_i = h$. Let v be any net function defined on the nodes of $R_h \cup C_h$, vanishing on C_h. Define $D_x^{(h)}v$ as the (constant) second derivative of the quadratic polynomial function of x assuming the three values $v(x-h_1, y)$, $v(x, y)$, and $v(x+h_3, y)$. That is,

$$(3) \qquad D_x^{(h)}v(x, y)=\frac{2}{h_1+h_2}\left[\frac{v(x+h_2, y)-v(x, y)}{h_2}-\frac{v(x, y)-v(x-h_1, y)}{h_1}\right].$$

Also, $D_y^{(h)}v(x, y)$ is defined analogously. We next define

$$(4) \qquad \begin{aligned} \Delta^{(h)}v(x, y) &= D_x^{(h)}v(x, y)+D_y^{(h)}v(x, y) \\ &= -\left(\frac{2}{h_1h_2}+\frac{2}{h_3h_4}\right)v(x, y) \\ &\quad +\frac{2}{h_1(h_1+h_2)}v(x-h_1, y)+\frac{2}{h_2(h_1+h_2)}v(x+h_2, y) \\ &\quad +\frac{2}{h_3(h_3+h_4)}v(x, y-h_3)+\frac{2}{h_4(h_3+h_4)}v(x, y+h_4) . \end{aligned}$$

The operator $\Delta^{(h)}$ is the approximation to Δ recommended in [10]. It linearly transforms the net function v defined over R_h into the net function $\Delta^{(h)}v$, also defined over R_h. But $\Delta^{(h)}$ is not a self-adjoint linear operator; that is, the matrix $A^{(h)}$ of the linear transformation of v into $\Delta^{(h)}v$ is not symmetric.

We define the matrix A_h as the symmetric part of the matrix $A^{(h)}$:

$$(5) \qquad A_h=\tfrac{1}{2}[A^{(h)}+A^{(h)T}] ,$$

where T means transpose. Finally, *we define Δ_h to be the self-adjoint linear operator corresponding to A_h.*

The explicit expressions for Δ_h assume 16 different forms, depending on the location of (x, y) with respect to C_h. Although we shall not need these expressions for the present paper, we describe them briefly. If, in any of the four directions from (x, y), the neighboring node—say $(x-h_1, y)$, for definiteness—is in R_h, then $h_1=h$, and there is another node $(x-h-h_1', y)$ in $R_h\cup C_h$. Then the term $2v(x-h_1, y)/h_1(h_1+h_2)$ of (4) is to be replaced by

$$(6) \qquad \frac{h_1'+2h+h_2}{(h_1'+h)h(h+h_2)}v(x-h, y) .$$

For any (x, y), the expression for Δ_h is obtained from (4) by making replacements like (6) corresponding to all neighbors of (x, y) in R_h.

When (x, y) is more than two nodes away from C_h, so that all $h_i=h_i'=h$, the values of both $\Delta^{(h)}$ and Δ_h reduce to the familiar form used in [4]:

(7) $$\Delta_h v(x, y) = \Delta^{(h)} v(x, y) = \frac{1}{h^2}[v(x-h, y) + v(x+h, y) + v(x, y-h) + v(x, y+h) - 4v(x, y)] .$$

Let λ_h satisfy the following difference equation for a net function v defined in $R_h \cup C_h$:

(8a) $$\begin{cases} \Delta_h v = -\lambda_h v & \text{in } R_h , \\ \lambda_h = \text{minimum} , \end{cases}$$

where v is extended to satisfy the boundary condition

(8b) $$v = 0 \quad \text{on} \quad C_h .$$

It is readily shown that λ_h is the minimum over all net functions v satisfying (8b) of the quotient

$$\rho_h(v) = \frac{-h^2 \sum_{R_h} v \Delta_h v}{h^2 \sum_{R_h} v^2} .$$

(This is simply the minimum principle for a definite quadratic form.) By (5), we can write $\rho_h(v)$ in the following equivalent form, simpler to use:

(9) $$\rho_h(v) = \frac{-h^2 \sum_{R_h} v \Delta^{(h)} v}{h^2 \sum_{R_h} v^2} .$$

The reason for not using the least eigenvalue μ_h of $\Delta^{(h)}$ in this investigation is that μ_h does not have the foregoing minimum property and, in fact, might turn out to be complex. On the other hand, it is known [9, p. 27] that $\lambda_h \leq \mathscr{R}(\mu_h)$, so that when μ_h is real it could conceivably be a better approximation to λ than λ_h is. The relative magnitude of $|\lambda_h - \lambda|$ to $|\mu_h - \lambda|$ is not known.

3. The results.

The following new result will be proved in § 4:

THEOREM 1. *Let R be a bounded, open, simply connected region bounded by a piecewise analytic curve C whose corners are convex in the sense of* § 2. *Let τ be the angle between the tangent to C and the x axis. Let u solve problem* (1) *for R, and let u_n be the normal derivative of u on C. Define λ_h as in* § 2. *Let*

(10) $$a = a(R) = \frac{\iint_R (u_{xx}^2 + u_{yy}^2) dx dy + \int_C u_n{}^2 \sin^2 2\tau \, d\tau}{12 \iint_R (u_x{}^2 + u_y{}^2) dx dy} .$$

Then $-\infty < a < \infty$ *and, as* $h \to 0$, *one has*

$$\frac{\lambda_h}{\lambda} \leq 1 - ah^2 + o(h^2) \qquad (h \to 0) . \tag{11}$$

In Theorem 1 the quantity a can probably be negative for certain nonconvex R, because $d\tau$ in (10) will be negative at some points of C. But if R is convex we get a stronger result, as an immediate consequence of Theorem 1.

THEOREM 2. *Under the hypotheses of Theorem* 1, *if* R *is also convex, then* $0 < a < \infty$, *and there exists* $h_0 > 0$ *such that* $\lambda_h < \lambda$ *for all* $h < h_0$.

For the operator Δ_h of § 2 the methods of [3] can undoubtedly be followed to show that $\lambda_h \to \lambda$ as $h \to 0$; the author has not attempted to carry through the details. When $\lambda_h \to \lambda$ as $h \to 0$, the lower bounds λ_{h_0} can be made arbitrarily close by choice of h_0 sufficiently small. Thus for these R the Rayleigh-Ritz methods and the finite-difference methods (8) are theoretically complementary, and together could confine λ to an arbitrarily short interval if one knew an upper bound for h_0.

The author has not developed an upper bound for h_0 in Theorem 2, although it would be desirable to do so by estimating the term $o(h^2)$. One could always make an intelligent guess based on the behavior of λ_h for certain h.

The constant a of (10) is the best possible for certain rectangular regions; see [4]. That the corners of C be convex seems essential to the validity of Theorem 1. Indeed, for one nonconvex polygon some heuristics and an experiment mentioned in [4] make it appear that $\lambda_h = \lambda + Ah^{4/3} + o(h^{4/3})$, where $A > 0$. It would be interesting to know the sign of a for the general case of Theorem 1, or in particular when C is a nonconvex analytic curve.

Corners of angle π are frequent in engineering practice, and it would be desirable to know how λ_h behaves when R has such corners. For such corners Lemma 2 is no longer valid. Lewy [7] provides new tools for an attack on corners of angle π.

4. Proof of Theorem 1.

Let u henceforth be the solution of problem (1) for the fundamental eigenvalue λ. It is known that

$$\lambda \iint_R u^2 dxdy = \iint_R (u_x^2 + u_y^2) dxdy . \tag{12}$$

The proof of Theorem 1, following [4], consists in setting the values of the function u at the nodes of $R_h \cup C_h$ into the Rayleigh quotient (9) of problem (8). It will be shown that

$$\frac{\rho_h(u)}{\lambda} = 1 - ah^2 + o(h^2) \qquad (h \to 0) . \tag{13}$$

Since $\lambda_h \leq \rho_h(u)$, the theorem follows from (13).

The denominator $h^2 \sum u^2$ of $\rho_h(u)$ differs from a Riemann sum for $\iint_R u^2 dxdy$ at most by the terms corresponding to squares or part-squares at the boundary C. The total contribution of these terms does not exceed the order of magnitude $Lh \max_R u^2$, where L is the length of C. Hence a fortiori

$$h^2 \sum_{R_h} u^2 = \iint_R u^2 dxdy + o(1) \qquad (h \to 0) . \tag{14}$$

Let the nodes of R_h be divided into three classes:

$$\begin{cases} R_h^1: & \text{those within a distance } h \text{ of some corner of } C\,; \\ R_h^2: & \text{those not in } R_h^1 \text{ but within a distance } h \text{ of } C\,; \\ R_h^3: & \text{the other nodes of } R_h\,. \end{cases} \tag{15}$$

Split the numerator of $\rho_h(u)$ accordingly:

$$-h^2 \sum_{R_h} u\Delta^{(h)} u = \sum_{i=1}^{3} \left(-h^2 \sum_{R_h^i} u\, \Delta^{(h)} u \right) \equiv \sum_{i=1}^{3} S_h^i(u) .$$

There are a fixed number of corners, not exceeding m, and at most two nodes of R_h^1 per corner. Moreover $|\nabla u(x, y)|^2 \to 0$ as $(x, y) \to$ a corner of C, by Lemma 1 in § 5. At any node (x, y) of R_h^1 with neighbors denoted as in (2), we find from (3) that

$$h^2 |u\Delta^{(h)} u| \leq \frac{h^2(u-0)}{\min h_i} \sum_{i=1}^{4} \left| \frac{u-u_i}{h_i} \right| \leq 4h^2 \max |\nabla u|^2 ,$$

where the u_i are the values of u at the four neighbors of (x, y), and where the maximum of $|\nabla u|^2$ is taken over all points within a distance $2h$ of some vertex. Hence

$$|S_h^1(u)| \leq 8mh^2 \max |\nabla u|^2 = o(h^2) \qquad (h \to 0) . \tag{16}$$

Using the notation and assertion of Lemma 3, we have

$$S_h^2(u) = -h^2 \sum_{R_h^2} u\Delta u - \frac{2h^3}{3} \sum_{R_h^2} u(\theta_x u'_{xxx} + \theta_y u''_{yyy}) . \tag{17}$$

Since u satisfies (1a),

$$-h^2 \sum_{R_h^2} u\Delta u = \lambda h^2 \sum_{R_h^2} u^2 . \tag{18}$$

By (17), (18), and Lemma 4,

$$|S_h^2(u)-\lambda h^2 \sum_{R_h^2} u^2| \leq \tfrac{2}{3}h^3 \sum_{R_h^2} u(|u'_{xxx}|+|u''_{yyy}|)=o(h^2) \qquad (h\to 0) .$$

Thus

(19) $$S_h^2(u)=\lambda h^2 \sum_{R_h^2} u^2+o(h^2) \qquad (h\to 0) .$$

Similarly, using the notation and assertion of Lemma 5, and by (1a), we have

(20) $$S_h^3(u)=\lambda h^2 \sum_{R_h^3} u^2-\frac{h^4}{12}\sum_{R_h^3} u(u'_{xxxx}+u''_{yyyy}) .$$

Now

(21) $$h^2 \sum_{R_h^2 \cup R_h^3} u^2=h^2 \sum_{R_h} u^2-h^2 \sum_{R_h^1} u^2=h^2 \sum_{R_h} u^2+o(h^2) ,$$

since $u(x, y)\to 0$ as $(x, y)\to C$, and since there are at most $2m$ vertices in R_h^1. Adding (19) and (20), and using (21), we find that

$$\begin{aligned} S_h^2(u)+S_h^3(u) &= \lambda h^2 \sum_{R_h} u^2-\frac{h^4}{12}\sum_{R_h^3} u(u'_{xxxx}+u''_{yyyy})+o(h^2) \\ &= \lambda h^2 \sum_{R_h} u^2-\frac{h^4}{12}\iint_R u(u_{xxxx}+u_{yyyy})dxdy+o(h^2) , \end{aligned}$$

by Lemma 6. Adding $S_h^1(u)$ to the above, and dividing by (14), we find that

(22) $$\begin{aligned} \rho_h(u) &= \frac{\sum_{i=1}^{3} S_h^i(u)}{h^2 \sum_{R_h} u^2} \\ &= \lambda-\frac{h^2}{12}\,\frac{\iint_R u(u_{xxxx}+u_{yyyy})dxdy}{\iint_R u^2 dxdy}+o(h^2) . \end{aligned}$$

Finally, dividing (22) by λ, and applying Lemma 7 and (12), one proves (13) and hence Theorem 1.

5. Some lemmas. The following lemmas are basic to the proof of Theorem 1. In all of them R satisfies the conditions stated at the start of § 2, while $u=u(x, y)$ solves problem (1).

LEMMA 1. *The function u is an analytic function of x and y in $R\cup C$, except possibly at the corners of C. Let r be the distance of (x, y) from a corner P with interior angle π/α, $1<\alpha<\infty$. Then for $m=0, 1, 2, \cdots$, any partial derivative of u of order m has the local representation*

$$\frac{\partial^m u}{\partial x^\mu \partial y^\gamma} = r^{\alpha-m} f_m(x, y) \qquad (\mu+\gamma=m) , \tag{23}$$

where f_m is continuous at P.

Proof. By [1, p. 179], u is analytic in R. The representation (27′) below shows that the interior normal derivative u_n is integrable on C. Then the analyticity of u on C (corners excluded) was shown by Hadamard [5, p. 25].[1]

Let $t=\xi+i\eta$ and $z=x+iy$. For each $t \in R$ let $w=\Phi(z, t)$ map R conformally onto the circle $|w|<1$, with $\Phi(t, t)=0$. We may assume without loss of generality that P is at $z=0$, and that $\Phi(0, t)=1$. Lichtenstein [8, pp. 255–256 and footnote 273] showed[2] that for $m=0, 1, 2, \cdots$, and $z \in R$,

$$\frac{\partial^m \Phi(z, t)}{\partial z^m} = z^{\alpha-m} \varphi_m(z, t) , \tag{24}$$

where φ_m is continuous at $z=0$. It follows from (24) that

$$\frac{\partial^m \log \Phi(z, t)}{\partial z^m} = z^{\alpha-m} \psi_m(z, t) , \tag{25}$$

where ψ_m is continuous at $z=0$. Let $G(z, t)=G(\xi, \eta; x, y)$ be Green's function for Δu in R. Since

$$G(z, t) = -(2\pi)^{-1} \log |f(z, t)| ,$$

it follows from (25) that for $m=0, 1, 2, \cdots$ and $z \in R$,

$$\frac{\partial^m G(z, t)}{\partial x^\mu \partial y^\nu} = r^{\alpha-m} \Psi_m(z, t) \qquad (\mu+\nu=m) , \tag{26}$$

where Ψ_m is continuous at $z=0$.

Now the function u has the integral representation [1, pp. 182–183]

$$u(x, y) = \lambda \iint_R G(x, y; \xi, \eta) u(\xi, \eta) d\xi d\eta .$$

Hence

$$\frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} \tag{27}$$

[1] The author wishes to thank Professor Lewy for this reference.

[2] Lichtenstein actually asserts that (24) is without question true for all α, but that his proof is valid only for irrational α. Warschawski [**13**] has found a simple proof of (24), valid for all α in the range $\frac{1}{2} \leq \alpha < \infty$.

Added in April 1954: For asymptotic expansions of Φ at a corner, see R. Sherman Lehmann, "Development of the mapping function at an analytic corner," Technical Report No. 21, Applied Mathematics and Statistics Laboratory, Stanford University, California, March 31, 1954, 17 pp.

$$=\lambda\iint_R \frac{G(x+\Delta x,\, y;\, \xi,\, \eta)-G(x,\, y;\, \xi,\, \eta)}{\Delta x} u(\xi,\, \eta)d\xi d\eta$$

$$=\lambda\iint_R \frac{\partial G}{\partial x}(x+\theta\Delta x,\, y;\, \xi,\, \eta)u(\xi,\, \eta)d\xi d\eta \ ,$$

where $0<\theta=\theta(x, y, \Delta x)<1$. Since $G(z, t)=G(t, z)$, it is clear that $\partial G/\partial x =\partial G/\partial \xi$ and, as a function of t, $\partial G/\partial x$ behaves like $|t-t_0|^{\alpha-1}$ at any corner t_0 of R, uniformly in z for z bounded away from C. Hence $(\partial G/\partial x)u(\xi, \eta)$ in (27) is dominated by an integrable function of ξ, η, uniformly with respect to Δx. By Lebesgue's convergence theorem, letting $\Delta x\to 0$ in (27) proves that

$$\frac{\partial u}{\partial x}=\lambda\iint_R \frac{\partial G}{\partial x}(x,\, y;\, \xi,\, \eta)u(\xi,\, \eta)d\xi d\eta \ . \tag{27'}$$

Setting the expression (26) for $m=\mu=1$ into the last equation proves the case $m=\mu=1$ of (23).

In a similar way one can prove all the cases $m=0, 1, 2, 3, 4$ of (23), and the lemma is established.

LEMMA 2. *The functions* u_{xx}^2, $u_x u_{xxx}$, $u u_{xxxx}$, u_{yy}^2, $u_y u_{yyy}$, *and* $u u_{yyyy}$ *are Lebesgue integrable in* R. *The Lebesgue integrals* $\int_C u_x u_{xx} dy$ *and* $\int_C u_y u_{yy} dx$ *exist.*

Proof. By Lemma 1 the functions $u_{xx}^2, \cdots, u u_{yyyy}$ are continuous in $R\cup C$ except possibly at the corners, where they are $O(r^{2\alpha-4})$. Since $0<\alpha$, the first sentence follows. The second sentence is proved analogously.

REMARK. The proof of Lemma 2 breaks down for corners of angle $\pi\ (\alpha-1)$, as r^{-2} is not integrable.

LEMMA 3. *At any node* (x, y) *of* R_h *whose neighbors are denoted as in* (2), *one has*

$$\Delta^{(h)}u=\Delta u+\tfrac{2}{3}h[\theta_x u'_{xxx}+\theta_y u''_{yyy}] \ ,$$

where $-1<\theta_x<1$, $-1<\theta_y<1$, *and where*

$$\begin{cases} u'_{xxx}=u_{xxx}(x',\, y) \ , & x-h_1<x'<x+h_2 \ , \\ u''_{yyy}=u_{yyy}(x,\, y') \ , & y-h_3<y'<y+h_4 \ . \end{cases} \tag{28}$$

Proof. By Lemma 1, u_{xxx} is continuous in the open line segment from $(x-h_1, y)$ to $(x+h_2, y)$, but may become infinite if the endpoint is a corner of C. Since u is continuous in $R\cup C$, it nevertheless follows from Taylor's formula as stated in [6, p. 357] that, if we fix y and set $\phi(x)=u(x, y)$,

$$\frac{\phi(x+h_2)-\phi(x)}{h_2}=\phi'(x)+\frac{h_2}{2}\phi''(x)+\frac{h_2^2}{6}\phi'''(x+\theta_2 h_2)\ ,$$

where $0<\theta_2<1$.

Writing a similar formula for h_1 and subtracting, we find in the notation of (3) that

$$D_x^{(h)}\phi(x)=\phi''(x)+\left[\frac{h_2^2}{3}\phi'''(x+\theta_2 h_2)-\frac{h_1^2}{3}\phi'''(x-\theta_1 h_1)\right](h_1+h_2)^{-1}.$$

If one writes $k=\max(h_1, h_2)\leq h$, the last term can be bounded in absolute value by

$$\frac{2k^2}{3k}\max\left[|\phi'''(x+\theta_2 h_2)|,\ |\phi'''(x-\theta_1 h_1)|\right]\ ,$$

and hence can be written in the form $(2h/3)\theta_x u'_{xxx}$. Addition of a similar expression for $D_y^{(h)}u(x, y)$ proves the lemma.

LEMMA 4. *For each node* (x, y) *of* R_h^2 *defined in* (15) *use the notation of* (28). *Then, as* $h\to 0$, *one has*

$$h\sum_{R_h^2} u(|u'_{xxx}|+|u''_{yyy}|)=o(1)\qquad(h\to 0)\ . \tag{29}$$

Proof. The lemma is proved much like Lemma 6 of [4]. The functions $u|u_{xxx}|$ and $u|u_{yyy}|$ are continuous in $R\cup C$, except at a corner of interior angle $\pi\alpha$, where Lemma 1 states that they behave like $r^{2\alpha-3}$ with $2\alpha-3>-1$. The sum (29) can be majorized by the Lebesgue integral of a step function over a polygonal arc in R which converges in length to C as $h\to 0$. The integrability of $r^{2\alpha-3}$ in (0, 1) permits the application of Lebesgue's convergence theorem as $h\to 0$. Since $u=0$ on C, (29) follows. Details are omitted.

LEMMA 5. *At each node in* R_h^3, *defined in* (15), *one has*

$$\Delta^{(h)}u=\Delta u+\frac{1}{12}h^2(u'_{xxxx}+u''_{yyyy})\ ,$$

where

$$\begin{cases} u'_{xxxx}=u_{xxxx}(x+\theta' h,\ y)\ , & -1<\theta'<1\ ,\\ u''_{yyyy}=u_{yyyy}(x,\ y+\theta'' h)\ , & -1<\theta''<1\ . \end{cases} \tag{30}$$

Proof. In [4]; the points of R_h^3 all have four neighbors in R_h^3, each at a distance h.

LEMMA 6. *At each node of R_h^3, defined in* (15), *use the notation of* (30). *Then, as $h\to 0$, one has*

$$h^2 \sum_{R_h^3} u(u'_{xxxx}+u''_{yyyy})=\iint_R u(u_{xxxx}+u_{yyyy})dxdy+o(1) \qquad (h\to 0).$$

Proof. In [4].

LEMMA 7. *Define u_n and τ as in Theorem* 1. *One then has*

$$\iint_R u(u_{xxxx}+u_{yyyy})dxdy=\iint_R (u_{xx}^2+u_{yy}^2)dxdy+\int_C u_n^2 \sin^2 2\tau d\tau\ ,$$

where the latter is a Riemann-Stieltjes integral.

Proof. The proof repeats that of Lemma 7 in [4] down to (29) of that paper. It then remains only to prove for smooth convex curves C that

$$(31) \qquad \int_C u_{yy}(u_y dx+u_x dy)=\int_C u_n^2 \sin^2 2\tau d\tau\ .$$

Let s denote arclength on C, and let primes denote d/ds. Differentiating the relations $u_x=-u_n \sin\tau$, $u_y=u_n\cos\tau$, we find that, on C,

$$(32) \qquad \begin{cases} u_x'=-u_n' \sin\tau-u_n\tau' \cos\tau=u_{xy}\sin\tau+u_{xx}\cos\tau\ , \\ u_y'=\ \ u_n'\cos\tau-u_n\tau'\sin\tau=u_{xy}\cos\tau+u_{yy}\sin\tau\ . \end{cases}$$

Changing u_{xx} to $-u_{yy}$ by (1), we can solve (32) for u_{yy} on C:

$$u_{yy}=u_n'\sin 2\tau+u_n\tau'\cos 2\tau\ .$$

Since $dx=ds\cos\tau$ and $dy=ds\sin\tau$, we obtain

$$(33) \qquad \begin{aligned} \int_C u_{yy}(u_y dx+u_x dy)&=\int_C (u_n'\sin 2\tau+u_n\tau'\cos 2\tau)(u_n\cos 2\tau)ds \\ &=\int_C u_n^2\tau'\cos^2 2\tau\, ds+\int_C u_n u_n'\cos 2\tau\sin 2\tau\, ds\ . \end{aligned}$$

By partial integration, we have

$$(34) \qquad \begin{aligned} \int_C u_n u_n'\cos 2\tau\sin 2\tau\, ds&=\tfrac{1}{4}\int_C (u_n^2)'\sin 4\tau\, ds \\ &=\tfrac{1}{4}[u_n^2\sin 4\tau]_C-\int_C u_n^2\tau'\cos 4\tau\, ds\ . \end{aligned}$$

Since $\cos^2 2\tau-\cos 4\tau\equiv\sin^2 2\tau$, substitution of (34) into (33) shows that

$$\int_C u_{yy}(u_y dx + u_x dy) = \int_C u_n^2 \tau' \sin^2 2\tau \, ds \; .$$

Since $\tau' ds = d\tau$, the identity (31) is proved, and with it, the lemma.

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NATIONAL BUREAU OF STANDARDS, LOS ANGELES.

ON THE DARBOUX PROPERTY

ISRAEL HALPERIN

A function $f(x)$ with a finite real value for each x in the closed interval (a, b) is said to have the Darboux property if $f(x)$ assumes on every sub-interval (c, d) all values between $f(c)$ and $f(d)$. This note discusses *local* conditions which are necessary and sufficient in order that f have the Darboux property (and corresponding conditions for a generalization of the Darboux property).

For each x in (a, b) let $I_r(x)$ denote the open interval with end points

$$f^r(x) = \limsup \{f(t)\,;\, t \geqslant x,\ t \to x\} \text{ and } f_r(x) = \liminf \{f(t)\,;\, t \geqslant x,\ t \to x\};$$

let $I_l(x)$, $f^l(x)$, $f_l(x)$ be defined similarly, using $t \leqslant x$, $t \to x$. Let $\mathscr{N}$ be any family of N-sets with the properties:

(a) Whenever an open interval is an N-set, its closure is also an N-set.

(b) Every subset of an N-set is an N-set.

(c) The union of a countable number of N-sets is an N-set.

We shall say that f is $\mathscr{N}$-Darboux on (a, b) if $f(x)$ assumes on every sub-interval (c, d) all values between $f(c)$ and $f(d)$ with the exception of an N-set. We shall say that f is $\mathscr{N}$-Darboux at x if for every $h > 0$:

(i) the values assumed by $f(t)$ for $x < t < x + h$ include all of $I_r(x)$ with the exception of an N-set;

(ii) the values assumed by $f(t)$ for $x - h < t < x$ include all of $I_l(x)$ with the exception of an N-set, (i) to be omitted when $x = b$, (ii) to be omitted when $x = a$.

We shall prove the theorem:

THEOREM. *f is $\mathscr{N}$-Darboux on (a, b) if and only if f is $\mathscr{N}$-Darboux at every x in the closed interval (a, b).*

The theorem was suggested by a paper by Akos Csaszar [1] who established the theorem for the two special cases: Case 1: the only N-set is the empty set, giving the usual Darboux property; and Case 2: (iii) also holds, every set consisting of a single point is an N-set.

We use the following modification of a lemma of Csaszar:

LEMMA. *If E is not an N-set then E contains a point y_0 such that IE fails to be an N-set for every open interval I containing y_0, and I*

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fails to be an N-set for every open interval I which has y_0 as one of its end points.

To prove the lemma let E_1 be the set of x in E for which $I(x)E$ is an N-set for some open interval $I(x)$ containing x, let E_2 be the set of x in $E-E_1$ such that x is the right end point of some open interval $J(x)$ which is an N-set and let E_3 be the set of x in $E-E_1$ such that x is the left end point of some open interval which is an N-set. Then

$$\begin{aligned} E_1 &= E_1 \sum \{I(x), \text{ all } x \text{ in } E_1\} \\ &= E_1 \sum \{I(x_n\}, \text{ for a suitable sequence of } x_n\} \\ &= \sum (E_1 I(x_n)) = \text{union of a countable collection of } N\text{-sets.} \end{aligned}$$

By (c), E_1 is an N-set. Since the $J(y)$ are clearly disjoint for different y in E_2, they form a countable collection; the closure of $J(y)$ includes y and is an N-set because of (a); it follows that E_2 and similarly E_3, are N-sets. Hence $E_1+E_2+E_3$ is an N-set, thus not identical with E which must therefore contain some y_0 not in $E_1+E_2+E_3$. This proves the lemma.

To prove the theorem, we note that the 'only if' part is an easy consequence of (b) and (c). To prove the 'if' part it is sufficient to assume that the set E of real numbers which lie between $f(a)$ and $f(b)$ but are not assumed by $f(t)$ is *not* an N-set, that y_0 is a point of E as described in the preceding lemma and obtain a contradiction. For this purpose we shall prove:

(*) *For every sub-interval (a_1, b_1) of (a, b) with y_0 between $f(a_1)$ and $f(b_1)$ and for every $m>0$ there is a sub-interval (a_2, b_2) of (a_1, b_1) such that y_0 is between $f(a_2)$ and $f(b_2)$ and*

$$|f(t)-y_0|<1/m \text{ for all } a_2 \leqslant t \leqslant b_2 .$$

Successive application of (*) with $m\to\infty$ will give a nested sequence of closed intervals such that at any of their common points $f(t)-y_0=0$, a contradiction since y_0 is in E, the set of omitted values.

Thus we need only prove (*). Since y_0 is in E, we have $f(x)\neq y_0$ for all x. It is easily seen that if $f(x)>y_0$ then $f_r(y)\geqslant y_0$ and $f_l(x)\geqslant y_0$ (because of the particular properties of y_0) and hence x lies in some open interval $I(x)$ on which $f(t)-y_0>-1/m$. Similarly if $f(x)<y_0$ then x lies in some open interval $J(x)$ on which $f(t)-y_0<1/m$. By the Heine-Borel theorem, a finite number of $I(x)$ and $J(x)$ cover (a_1, b_1) and hence it follows that some $I(x_1)$ and some $J(x_2)$ must contain a common open interval (u, v) say. We may suppose $x_1<u<v<x_2$. If y_0 is between $f(u)$ and $f(v)$ we can choose (u, v) to be the (a_2, b_2) required by (*). Otherwise we may suppose $f(u)>y_0$, $f(x_2)<y_0$. Let a_2 be sup t with $f(x)>y_0$ on $u\geqslant x>t$. Then $f(a_2)<y_0$ is impossible; for if $f(a_2)<y_0$ held, the open interval $(f(a_2), y_0)$ would be contained in $I_l(a_2)$ and yet

omitted from the values of f on (u, a_2), implying that $(f(a_2), y_0)$ is an N-set and thus contradicting the particular properties of y_0. Thus $f(a_2)>y_0$ and $u\leqslant a_2<x_2$. It now follows easily that $f_r(a_2)=y_0$ and that a_2 is the limit of a sequence of t_n with $t_n>a_2$ and $f(t_n)<y_0$. Hence, for sufficiently large n, t_n may be selected as b_2 to give (a_2, b_2) with the properties required by (*).

The example $f(x)=x$ for $x<0$ and $f(x)=1$ for $x\geqslant 0$ with the open subsets of (0, 1) as the class $\mathscr{N}$ shows that the condition (a) cannot be omitted.

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Queen's University.

ON CHAINS OF INFINITE ORDER

T. E. HARRIS

1. Introduction. We consider stationary[1] stochastic processes Z_n, $n=0, \pm 1, \cdots$, where Z_n can take D distinct values, $D \geq 2$. It is convenient to let the values be $Z_n=0, 1, \cdots, D-1$. Let u be any sequence of integers, $u=(u_1, u_2, \cdots)$. Then the transitions of the process are described by the functions $Q_i(u)$,

$$Q_i(u)=P(Z_n=i \mid Z_{n-1}=u_1, Z_{n-2}=u_2, \cdots) , \quad i=0, 1, \cdots, D-1 . \tag{1.1}$$

Our aim will be to relate some stochastic properties of the Z_n-process to functional properties of the $Q_i(u)$. Because of the fact that the future behavior of Z_n depends in general on its complete past history, we shall refer to these processes as stationary infinite-order chains.

The first systematic study of such chains was made by Onicescu and Mihoc [13], and was carried on in further papers [14], [15], and [16] by Onicescu and Mihoc, and [12] by Onicescu. These authors considered chains of a somewhat more special type which they called *chaînes à liaisons complètes*. Further results were obtained by Doeblin and Fortet [6], who applied the term *chaîne à liaisons completes* to any chain for which the relations

$$P(Z_n=i \mid Z_{n-1}=u_1, \cdots, Z_{n-k}=u_k)$$

are specified for every sequence $u_1, \cdots, u_k$, $k=1, 2, \cdots, \infty$. See also Fortet [8] and Ionescu Tulcea and Marinescu [17].

The authors cited prove, under various hypotheses on the functions Q_i of (1.1), that $P(Z_n=i \mid Z_{-1}=u_1, \cdots, Z_{-k}=u_k)$ has a limit as $n \to \infty$, and obtain various other generalizations of the limit theorems for Markov chains. Also, in [6] the case of cyclic motions is considered. We shall not treat this case. The case of infinitely many states, stronger hypotheses, is treated in [17].

Our point of view is somewhat different. We introduce the random variables X_n, $n=0, \pm 1, \cdots$, defined by

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[1] Throughout this paper a "stationary" process, Markov or not, will be a process which not only has transition laws independent of time but also has a stationary absolute distribution.

(1.2) $$X_n = \sum_{j=1}^{\infty} Z_{n-j}/D^j .$$

That is, X_n is the number whose representation in the D-ary numeral system is $.Z_{n-1}Z_{n-2}\cdots$. (If we make the proper conventions, ambiguities will have total probability 0.) Thus X_n contains the complete past history of the Z_n process, and is a Markov process whose transition probabilities are defined in the following way. Let $0 \leq x \leq 1$ be a number whose D-ary expansion is

$$x = .u_1u_2\cdots .$$

Observe that $X_n = x$ is equivalent to $Z_{n-1} = u_1$, $Z_{n-2} = u_2$, $\cdots$. Now if $Z_n = i$, then $X_{n+1} = .iu_1u_2\cdots = (i+x)/D$. Now let $f_i(x)$ be functions of x defined by

(1.3) $$f_i(x) = Q_i(u) , \qquad i = 0, \cdots, D-1$$

where $.u_1u_2\cdots$ is the D-ary expansion of x and the Q_i are defined by (1.1). If $X_n = x$, then X_{n+1} is formed by applying with probability $f_i(x)$ the transformation $[i+(\ \ )]/D$ to X_n; that is

(1.4) $$P\left(X_{n+1} = \frac{i+x}{D} \,\middle|\, X_n = x \right) = f_i(x) .$$

The representation (1.2) was used by Borel [3] for the case where the Z_n are independent and equidistributed. Apparently it has not been systematically exploited for other cases, although an abstract analogue of (1.2) is used in [17]. The representation (1.2) has the advantage that Fourier and Laplace transform methods can be used to deal with the distribution of the complete past history of the Z_n-process.

After making precise the relation between the Z_n- and X_n-processes, we show the existence of a unique stationary Z_n-process whose conditional probabilities

$$P(Z_n = i \mid Z_{u-1} = u_1, \cdots)$$

are equal to specified functions Q_i, provided the latter satisfy certain conditions. This extends a result of Doeblin and Fortet. Next we study the distribution $G(x)$ of X_n. It is shown that this has one of three forms, provided certain general conditions of mixing behavior hold. (1) $G(x)$ has a single jump of magnitude 1 at one of the points $i/(D-1)$, $i = 0, \cdots, D-1$. This is true if and only if $P(Z_n = i) = 1$. (2) $G(x) = x$, $0 \leq x \leq 1$. This is true if and only if the Z_n are independent and equidistributed on 0, 1, $\cdots$, $D-1$. (3) $G(x)$ is continuous and purely singular.[2]

[2] The fact that G is singular if the Z_n are independent and not equidistributed was pointed out to the author by Henry Scheffé.

Next we consider processes which we shall call *grouped Markov chains*. Let Y_n, $n=0, \pm 1, \cdots$, be the variables of a stationary Markov chain whose states are divided into D mutually exclusive and exhaustive nonempty subsets $B_0, B_1, \cdots, B_{D-1}$. Define $Z_n=i$ when $Y_n \in B_i$, $i=0, 1, \cdots, D-1$. We shall refer to this type of Z_n-process as a grouped Markov chain; it is in general not Markovian. We study such chains for the case where Y_n has a finite number of possible states and where each element of the transition matrix of the Y_n-process is positive. Using the Laplace transform, we show how to determine the functions

$$P(Z_n=i \mid Z_{n-1}=u_1, \cdots, Z_{n-k}=u_k)$$

and

$$P(Z_n=i \mid Z_{n-1}=u_1, Z_{n-2}=u_2, \cdots) ,$$

as well as the corresponding functions of a real variable $f_i(x)$ given by (1.3). This may be considered a solution of the prediction problem for grouped Markov chains.

The X_n-process is closely related to models which have recently been used for learning and decision processes by Bush and Mosteller [4], Bales and Householder [1], Flood [7], and others. The author wishes to thank these men for stimulating the present line of work.

Theorem 3 can be extended to certain types of these "learning models." A discussion of certain learning models has been given by Bellman, Harris, and Shapiro [2] and by Karlin [11]. Karlin's work has points of contact with ours.[3]

2. Relation of the Z_n- and X_n-processes. In this section we make explicit the relation between the Z_n- and X_n-processes and give a general condition which implies the existence of a Z_n-process with prescribed Q_i. Later sections will show that this condition is satisfied in many instances.[4]

Let $D \geq 2$ be an integer and let $u=(u_1, u_2, \cdots)$ represent a sequence of integers with $0 \leq u_j \leq D-1$. Let $Q_i(u)$ be functions of u, $i=0, 1, \cdots, D-1$, with

$$Q_i(u) \geq 0 , \qquad i=0, \cdots, D-1 , \tag{2.1}$$

$$\sum_{i=0}^{D-1} Q_i(u) \equiv 1 . \tag{2.2}$$

Now if x is a real number, $0 \leq x \leq 1$, we adopt the following convention about the D-ary expansion of x in the ambiguous cases. The

[3] See § 4.

[4] Further discussion of the relationship follows Theorem 6.

D-ary expansion of $x=1$ will be taken as $x=.(D-1)(D-1)\cdots$. In all other ambiguous cases, an expansion terminating in 0's will be preferred to one terminating in $(D-1)$'s. Thus in the decimal system the expansion of $x=1$ will be $.999\cdots$ while the expansion of $x=1/2$ will be $.5000\cdots$ rather than $.499\cdots$. Thus the D-ary expansion of x is unambiguously defined.

Now define functions $f_i(x)$, $0\leq x\leq 1$, by

$$f_i(x)=Q_i(u) \tag{2.3}$$

where $x=.u_1u_2\cdots$.

THEOREM 1. *Suppose functions* $Q_i(u)$ *are given satisfying* (2.1) *and* (2.2) *and such that the* $f_i(x)$ *defined by* (2.3) *are Borel-measureable; suppose there exists a distribution* $G(x)$, $G(0-)=0$, $G(1)=1$, *which satisfies the functional equation*

$$G(x)=\sum_{j=0}^{D-1}\int_0^{Dx-j} f_j(y)dG(y)\,, \qquad 0\leq x\leq 1\,. \tag{2.4}$$

Then there exists a stationary process $\cdots Z_0, Z_1, \cdots$, *such that* Z_n *has possible values* $0, 1, \cdots, D-1$, *and such that*

$$P(Z_n=i\,|\,Z_{n-1}, Z_{n-2}, \cdots)=Q_i(Z_{n-1}, Z_{n-2}, \cdots) \tag{2.5}$$

with probability 1.

Proof. We consider a real-valued Markov process $\cdots X_n, X_{n+1}, \cdots$ whose transition probabilities are given by

$$P\left(X_n=\frac{i+x}{D}\,\middle|\,X_{n-1}=x\right)=f_i(x)\,, \qquad 0\leq x\leq 1\,, \tag{2.6}$$

where the $f_i(x)$ are the functions defined by (2.3). It can be verified that if G satisfies (2.4), then G is a stationary absolute distribution for this Markov process; we shall suppose that X_n has this distribution.

Define the function $h(x)$, $0\leq x\leq 1$, by

$$h(x)=\text{1st digit in } D\text{-ary expansion of } x. \tag{2.7}$$

Now define random variables Z_n by

$$Z_{n-1}=h(X_n)\,, \qquad n=0, \pm 1, \cdots \tag{2.8}$$

The Z_n then form a stationary process, whose nature is clearly completely determined by $G(x)$. It can be shown that

$$P\left[X_n=\frac{Z_{n-1}}{D}+\frac{Z_{n-2}}{D^2}+\cdots\right]=1\,, \tag{2.9}$$

since $P[X_{n-1}=DX_n-h(X_n)]=1$ for all n. Also

$$(2.10) \qquad Q_i(Z_{n-1}, Z_{n-2}, \cdots) = f_i\left(\frac{Z_{n-1}}{D} + \frac{Z_{n-2}}{D^2} + \cdots\right)$$

holds with probability 1. The only sequences $(Z_{n-1}, Z_{n-2}, \cdots)$ for which the two sides of (2.10) might be different are sequences other than $(D-1, D-1, \cdots)$ which terminate in unbroken $(D-1)$'s, and these can be shown to have probability 0.

It can be shown from this that $Q_i(u)$ is a permissible version of $P(Z_n=i \mid Z_{n-1}=u_1, Z_{n-2}=u_2, \cdots)$.

3. Continuity properties of the Q_i.

We assume that functions Q_i are given satisfying (2.1) and (2.2) and that functions f_i are then defined by (2.3).

We shall refer to a point x whose D-ary expansion terminates in an unbroken sequence of 0's as a *lattice point.*

If $u^n=(u_1^n, u_2^n, \cdots)$ is a sequence for each $n=1, 2, \cdots$, then $u^n \to u$ will mean that for each k, $u_k^n = u_k$ for all n sufficiently large.

CONDITION A. *For each i and u, $u^n \to u$ implies $Q_i(u^n) \to Q_i(u)$ as $n \to \infty$.*

THEOREM 2. *Under Condition* A *the $f_i(x)$ are continuous to the right for each x, $0 \leq x < 1$, and continuous to the left except possibly at lattice points. Left-continuity holds at $x=1$.*

COROLLARY. *Under Condition* A *the $f_i(x)$ are Borel-measurable (in fact, belong to Baire class* 1.).

The proof follows from the definition of the $f_i(x)$. The corollary follows from the well-known fact that a function with only countably many discontinuities belongs to Baire class 1.

4. Existence of stationary Z_n- and X_n-processes.

Our procedure will be as follows. We consider a Markov process X_n with transition probabilities defined by (2.6), where the $f_i(x)$ are given functions. We give conditions on the $f_i(x)$ which insure that the probabilities $P(X_n \leq x \mid X_0=y)$ are C–1 summable to a distribution $G(x)$ which is independent of y. The distribution $G(x)$ satisfies (2.4) and is the only stationary distribution for the X_n-process.

Now let functions $Q_i(u)$ be given satisfying (2.1) and (2.2). Making use of Theorem 1 we show that under certain restrictions on the Q_i there is a uniquely determined stationary process Z_n satisfying (2.5) with probability 1. This process is ergodic. It is discussed in Theorem 6.

Under somewhat stronger conditions Doeblin and Fortet proved essentially that

$$\lim_{n\to\infty} P(Z_n = i \mid Z_{-1}, Z_{-2}, \cdots)$$

exists with probability 1 and is independent of $Z_{-1}, Z_{-2}, \cdots$.[5] We shall show the C–1 anologue of this under the weaker conditions.

The method we use is a development of one used originally by Doeblin in [5].

Now consider given functions $f_i(x) \geq 0$, $i=0, \cdots, D-1$; $0 \leq x \leq 1$; $\sum f_i(x) \equiv 1$. We use the notation

$$(x \equiv y)_m \tag{4.1}$$

to mean that the first m digits in the D-ary expansion of x are the same as the first m in the expansion of y. We define

$$\varepsilon_m = \sup_{i,\, (x \equiv y)_m} |f_i(x) - f_i(y)| , \qquad m=0, 1, \cdots . \tag{4.2}$$

Doeblin and Fortet used a condition which would be equivalent in the present context to

$$\sum_{m=0}^{\infty} \varepsilon_m < \infty . \tag{4.3}$$

We shall use Condition B, expressed by the requirements

$$\lim_{n \to \infty} \varepsilon_m = 0 , \tag{4.4}$$

$$\sum_{m=0}^{\infty} \prod_{k=0}^{m} \left(1 - \frac{1}{2} D \varepsilon_k\right) = \infty . \tag{4.5}$$

We shall understand that any of the factors $\left(1 - \frac{1}{2} D \varepsilon_k\right)$ in (4.5) which is zero or negative will be replaced by 1. As an example, Condition B is satisfied provided we have for sufficiently large k

$$\varepsilon_k \leq \frac{2}{Dk} .$$

In addition to Condition B, some sort of condition of positivity will be required. We shall choose the simplest one.

Condition C. For some i, $f_i(x) \geq \Delta > 0$, $\qquad 0 \leq x \leq 1$.

It is easy to see how C can be replaced by weaker conditions. For example, in the case $D=2$, $f_0(x)=x$, Condition C is not satisfied but it will be clear from the subsequent arguments that a condition sufficiently like C is satisfied.

Theorem 3. *Let $f_i(x)$, $i=0, \cdots, D-1$, be nonnegative functions with*

[5] Simple examples show that the existence of limiting probabilities does not, in general, imply the existence of a stationary distribution. The existence of at least one stationary Z_n-process can be shown under quite weak conditions. The difficulty is to show uniqueness.

$\sum f_i = 1$. *Let* X_n, $n=0, 1, \cdots$, *be the variables of a Markov process with* $X_0 = y$, $0 \leq y \leq 1$ *and with transition law defined by* (2.6).

Define

$$G_n(y\,;x) = P(X_n \leq x \mid X_0 = y) \ . \tag{4.6}$$

Then Conditions B *and* C *imply that* $G_n(y\,;x)$ *is summable* C–1 *to a distribution* $G(x)$ *which is independent of* y. *If* (4.3) *holds then the ordinary limit exists. In either case the limit is uniform in* y.[6]

For the proof of Theorem 3 we require the following lemma about sums of (not necessarily independent) random variables.

LEMMA 1. *Let* $x_1, x_2, \cdots$, *be positive integer-valued random variables. Let* $s_n = x_1 + \cdots + x_n$ *and let* u_m *be the probability that for some* j *we have* $s_j = m$, $m=1, 2, \cdots$. *Suppose*

$$P(x_n > i \mid x_1, x_2, \cdots, x_{n-1}) \geq R_i \tag{4.7}$$

where the R_i *are nonnegative numbers which are independent of* $x_1, \cdots, x_{n-1}$ *and* n *and satisfy*

$$\sum_{i=1}^{\infty} R_i = \infty \ . \tag{4.8}$$

Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} u_m = 0 \ . \tag{4.9}$$

The proof of the lemma, which is closely related to a standard renewal theorem, is simple, and is omitted.

Proof of Theorem 3. The method is related to an idea of Doeblin [5], who proved the ergodic theorem for Markov chains with a finite number of states by considering two particles starting in different states, which move independently until they simultaneously occupy the same state, after which they merge. An idea similar to Doeblin's original one is used in [6], and a related device has been used by Hodges and Rosenblatt [9].

In order not to obscure the main idea by details we give the proof for the case $D=2$. Since Condition C holds we can just as well take $f_0(x) \geq \Delta > 0$. Then

$$\left| \frac{1}{2} D\varepsilon_k \right| = |\varepsilon_k| < 1 \ , \qquad k=0, 1, \cdots \ .$$

Let $t_0, t_1, \cdots$, be independent random variables uniformly distributed

[6] We use conditional probabilities $G_n(y;x)$, etc., to mean those probabilities which are uniquely determined by the Markov transition operator, starting from a given value y. They are thus uniquely defined for all y.

on (0, 1). Define processes X_n and X'_n as follows: $X_0=y$, $X'_0=y'$, $0\leq y, y' \leq 1$. Suppose X_n and X'_n are determined. Then

$$X_{n+1}=\begin{cases} \frac{1}{2}X_n & \text{for} \quad t_n\leq f_0(X_n) \\ \frac{1}{2}+\frac{1}{2}X_n & \text{for} \quad t_n>f_0(X_n)\,; \end{cases}$$

while

$$X'_{n+1}=\begin{cases} \frac{1}{2}X'_n & \text{for} \quad t_n\leq f_0(X'_n) \\ \frac{1}{2}+\frac{1}{2}X'_n & \text{for} \quad t_n>f_0(X'_n)\,. \end{cases}$$

It is convenient to let U_n (U'_n) designate the transformation applied to X_n (X'_n). That is, $U_n=i\,[U'_n=i]$ if $X_{n+1}=(i+X_n)/2\,[X'_{n+1}=(i+X')/2]$. Then

$$P(U_n\neq U'_n\,|\,X_n, X'_n)\leq|f_0(X_n)-f_0(X'_n)|\,. \tag{4.10}$$

From (4.10) we then have

$$P(U_n=U'_n, U_{n+1}=U'_{n+1}, \cdots, U_{n+k}=U'_{n+k})\geq(1-\varepsilon_0)(1-\varepsilon_1)\cdots(1-\varepsilon_k)\,, \tag{4.11}$$

independently of X_n, X'_n.

Now the event $\{U_n=U'_n, \cdots, U_{n+k}=U'_{n+k}\}$ implies[7]

$$(X_{n+k+1}\equiv X'_{n+k+1})_{k+1} \tag{4.12}$$

which in turn implies

$$|X_{n+k+1}-X'_{n+k+1}|\leq 2^{-k-1}\,. \tag{4.13}$$

Let us say that an "engagement" occurs on the nth step if $U_{n-1}\neq U'_{n-1}$, $U_n=U'_n$. If we interpret the random variables $x_1, x_2, \cdots$ of Lemma 1 as the intervals between successive engagements, we see from (4.11), (4.12), Conditions B and C, and Lemma 1 that

$$\lim_{N\to\infty}\frac{\text{Expected no. engagements in 1st } N \text{ steps}}{N}=0\,, \tag{4.14}$$

the limit in (4.14) being uniform in the starting points y and y'.

It can be shown from (4.14) that for any $\varepsilon>0$, we have,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}P(|X_n-X'_n|>\varepsilon)=0\,, \tag{4.15}$$

[7] A slight modification is necessary if $y=1$ or $y'=1$.

uniformly in y and y'. The argument is roughly as follows. Whenever $U_n \neq U'_n$, the length of time till an engagement occurs is small, with uniformly high probability, because of Condition C. Therefore, if only a small number of engagements have occurred in the first N steps, where N is large, then the probability is high that $U_N = U'_N$, $U_{N-1} = U'_{N-1}$, $\cdots$, $U_{N-k} = U'_{N-k}$ where k is large. Thus (4.15) follows from (4.13). (It is easy to make this argument precise.) A simple type of argument then shows that $P(X_n \leq x | X_0 = y)$ is C–1 summable to a distribution $G(x)$ which is independent of y. Moreover the difference

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} P(X_n \leq x \mid X_0 = y) - G(x) \right| \tag{4.16}$$

goes to zero uniformly in y at all points of continuity of $G(x)$.

If the stronger condition (4.3) holds, as well as Condition C, we can replace (4.15) by the stronger statement

$$P(|X_n - X'_n| > \varepsilon) \to 0 \,. \tag{4,17}$$

In fact, with probability 1 we have $U_n = U'_n$ for all sufficiently large n in this case. We then get actual convergence, rather than just C–1 summability, of the distributions to $G(x)$.

THEOREM 4. *Assume that Conditions* B *and* C *hold. Then* $G(x)$ *of Theorem* 3 *either has a single discontinuity of magnitude* 1 *at one of the points* $0, 1/(D-1), 2/(D-1), \cdots, 1$ *or is continuous.*

Proof. First let i in Condition C be 0. If $f_0(0) = 1$ it is clear that $G(x)$ has a jump of magnitude 1 at $x = 0$, and conversely. If $f_0(0) < 1$, $G(x)$ is everywhere continuous. First, $G(x)$ must be continuous at 0. For let K and n be integers, $0 < K < n$. Consider an X_n-process with an arbitrary starting point $X_0 = y$. If the D-ary expansion of X_n begins with K 0's then $U_{n-1} = U_{n-2} = \cdots = U_{n-K} = 0$. Hence

$$P(X_n < D^{-K}) \leq P(U_{n-1} = U_{n-2} = \cdots = U_{n-K} = 0) \,. \tag{4.18}$$

Now no matter what is the value of $X_{n-K} = \bar{x}$, we have

$$P(U_{n-1} = \cdots = U_{n-K} = 0 | X_{n-K} = \bar{x}) = f_0(\bar{x}) f_0(\bar{x}/2) f_0(\bar{x}/2^2) \cdots f_0(\bar{x}/2^{K-1}) \,. \tag{4.19}$$

Because $f_0(0) < 1$ and $f_0(x)$ is continuous at 0, the right side of (4.19) $\to 0$ as $K \to \infty$, uniformly in $\bar{x}$. Using (4.18) and (4.19), we have continuity of $G(x)$ at 0.

Similar arguments show continuity of $G(x)$ at other points x. The argument is almost the same if the i of Condition C is not 0.

THEOREM 5. *Under the conditions of Theorem* 3, $G(x)$ *is a stationary absolute distribution for the* X_n*-process and satisfies* (2.4). *It is the only stationary distribution.*

Simple examples show that if the $f_i(x)$ do not satisfy the proper continuity conditions, there can be a limiting distribution independent of the starting point which is nevertheless not a stationary distribution.

The uniqueness of the stationary distribution, once its existence is known, is an immediate consequence of the existence of a C-1 limiting distribution for X_n uniformly independent of X_0.

In the case $f_i(i/(D-1))=1$ it is readily verified that $P(X_n=i/(D-1))=1$ is a stationary distribution satisfying (2.4). We can thus limit ourselves below to the case where $G(x)$ is continuous. (Theorem 4.)

Instead of starting with a fixed value for X_0 it is now convenient to give X_0 an arbitrary continuous distribution $G_0(x)$ assigning probability 1 to the interval (0, 1). Letting $G_n(x)=P(X_n \leq x)$ we have

$$G_{n+1}(x)=\sum_{j=0}^{D-1}\int_0^{Dx-j} f_j(y)dG_n(y)\,, \qquad n=0, 1, \cdots. \tag{4.20}$$

Then $G_n(x)$ is continuous for each n, and we know from Theorem 3 that $G_n(x)$ is summable to $G(x)$. It follows from Condition B that it is justified to pass to the limit under the integral sign in (4.20) (C-1 limit if necessary), and Theorem 5 follows.

We can now give the main results of the present section. As before u and u' will denote sequences of integers between 0 and $D-1$ inclusive. For convenience we let V denote the set of all sequences which terminate in unbroken $(D-1)$'s, with the single exception of the sequence, each of whose members is $D-1$. For any stationary process whatever it can be shown that

$$\text{Prob}\,[(Z_{-1}, Z_{-2}, \cdots)\in V]=\text{Prob}\,[(Z_1, Z_2, \cdots)\in V]=0\ .$$

We use the notation $(u\equiv u')_m$ to mean that the first m elements in the u sequence are the same as the first m elements in the u' sequence.

Now let $Q_i(u)$ be nonnegative functions of u with $\sum_{i=0}^{D-1} Q_i(u)\equiv 1$ and define

$$\varepsilon_m=\sup_{\substack{i,\,(u\equiv u')_m\\ u\notin V,\, u'\notin V}} |Q_i(u)-Q_i(u')|. \tag{4.21}$$

Then the quantities ε_m defined by (4.21) are identical with those defined by (4.2) if functions $f_i(x)$ are defined by (2.3).

We shall say that the Q_i satisfy Condition B if (4.4) and (4.5) are satisfied. These are requirements that the future is conditioned only slightly by the remote past. Condition C will mean that for some j

$$Q_j(u)\geq \Delta > 0\ , \qquad u\notin V. \tag{4.22}$$

Now let I be a finite sequence of integers,

$$I=(i_0, i_1, \cdots, i_k)$$

and let

$$Q_I^n(u)=P(Z_n=i_0, \cdots, Z_{n+k}=i_k \mid Z_{-1}=u_1, \cdots) . \tag{4.23}$$

The quantities $Q_I^n(u)$ are to be interpreted as defined, relative to the "past" u, by means of the $Q_i(u)$; they thus have meaning even before it is known that there is a stationary absolute distribution.

THEOREM 6. *Let the functions* $Q_i(u)$ *satisfy Conditions* B *and* C. *Then*

a) *there exists a stationary process* Z_n *such that* (1.1) *holds with probability* 1;
b) *this is the only stationary process for which* (1.1) *holds*;
c) *the C–1 limit of* $Q_I^n(u)$ *exists for every* u (*except those in the set* V *defined above*) *for every* I, *and is equal to the stationary measure of* I. *The C–1 limit is approached uniformly in* u.
d) *For every* u *not in* V *we have, for each* $i=0, 1, \cdots, D-1$,

$$\lim_{k\to\infty} P(Z_0=i|Z_{-1}=u_1, \cdots, Z_{-k}=u_k) = Q_i(u) , \tag{4.24}$$

provided the left side of (4.24) *is defined for each* k.

Proof. Define functions $f_i(x)$ by (2.3). From Theorem 3 there is a unique distribution $G(x)$ satisfying (2.4). From Theorem 1 there exists a stationary Z_n-process for which (1.1) holds with probability 1. As remarked in § 2, the nature of the Z_n-process is determined by the distribution G. Hence, since G is uniquely determined, so is the Z_n-process. This proves (a) and (b) above.

The proof of (c) is an immediate consequence of the relation between the Z_n- and X_n-processes, together with Theorems 3 and 5. A slight modification is required if $u=(1, 1, \cdots)$.

The relation in (d) above is, it is well known, true for almost all u. A simple argument shows that it holds for every u not in V.

5. Further properties of $G(x)$. We now change our point of view somewhat. Suppose we are given a stationary infinite-order chain Z_n as defined in the introduction. Define

$$X_n=Z_{n-1}/D+Z_{n-2}/D^2+\cdots , \qquad n=0, \pm 1, \cdots . \tag{5.1}$$

Then X_n is a stationary process.

We shall further suppose throughout § 5 *that the* Z_n*-process is of the mixing type.*[8] *The* X_n*-process then is likewise.*

Let the functions $Q_i(u)$ be defined by

$$Q_i(u)=P(Z_n=i|Z_{n-1}=u_1, Z_{n-2}=u_2, \cdots) .$$

[8] See [**10**, p. 36]. Roughly, if A and B are events, and $B(n)$ is the event B translated n units in time, then $P[AB(n)]\to P(A)P(B)$.

As in §2 we then define functions $f_i(x)$ by

$$f_i(x)=Q_i(u)\ , \qquad i=0, \cdots, D-1\ , \tag{5.2}$$

where $.u_1u_2\cdots$ is the D-ary expansion of x. The functions $Q_i(u)$ are defined at least for almost all u-sequences ("almost all" in the sense of the measure on sequences in the Z_n-process.)

Let $G(x)$ be the distribution of X_n. It is then clear that the functions $f_i(x)$ are defined for almost all x (G-measure). It is also readily seen that the X_n-process is Markovian and that $G(x)$ satisfies (2.4) with the $f_i(x)$ defined by (5.2).

Remark on uniqueness. Let G^* be a distribution satisfying (2.4), with $G^*(0-)=0$, $G^*(1)=1$, and suppose G^* is absolutely continuous with respect to G. Then G and G^* are identical. This follows from the general theory of Markov processes.

LEMMA 2. *Let Z_n be a stationary infinite-order chain as defined in the introduction. Suppose Z_n is mixing. Let*

$$X_n=\sum_{j=1}^{\infty} Z_{n-j}/D^j$$

and let $G(x)$ be distribution of X_n. Then G either has a single discontinuity of magnitude 1 *at one of the points* $0, 1/(D-1), \cdots, 1$ *or is continuous.*

The proof is similar to that of Theorem 4 and is omitted.

LEMMA 3. *Under the conditions of Lemma* 2, *$G(x)$, if it is continous, is either purely singular or purely absolutely continuous.*

Proof. Suppose we have the continuous case. To obtain a contradiction let us suppose

$$G=cG_1+(1-c)G_2\ , \qquad 0<c<1\ ,$$

where G_1 and G_2 are the singular and the absolutely continuous parts of G respectively, neither being identically zero.

If we write (2.4) in the operator form $G=TG$, then we have

$$c(G_1-TG_1)=-(1-c)(G_2-TG_2). \tag{5.3}$$

Now it is easily seen from the nature of T that TG_1 is singular and TG_2 is absolutely continuous. Moreover, neither G_1-TG_1 nor G_2-TG_2 can vanish identically. This follows from the remark above on uniqueness. Thus (5.3) is a contradiction.

LEMMA 4. *Under the conditions of Lemma* 2, *the f_i are determined uniquely by G up to a set of G-measure* 0.

For from (2.4) we have

$$G(x)-G\left(\frac{i}{D}\right)=\int_0^{Dx-1} f_i(y)dG(y)\ , \qquad \frac{i}{D}\le x<\frac{i+1}{D}\ ;\ i=0, \cdots, D-1. \tag{5.4}$$

Uniqueness of the f_i follows from (5.4) and the Nikodym-Radon theorem.

THEOREM 7. *Let Z_n be a stationary infinite-order chain of the mixing type. Let*

$$X_n = \sum_{j=1}^{\infty} Z_{n-j}/D^j$$

and let $G(x)$ be the distribution of X_n. Then $G(x)$ is one of the three following types.

(a) *$G(x)$ has a single jump of magnitude 1 at one of the points $i/(D-1)$, $i=0, \cdots, D-1$. This is true if and only if $P(Z_n=i)=1$.*

(b) *$G(x)=x$, $0\leq x\leq 1$. This is true if and only if the Z_n are independent, each being equidistributed on $0, 1, \cdots, D-1$.*

(c) *$G(x)$ is continuous and purely singular.*

Proof. If $G(x)$ has any discontinuities, then (a) follows from Lemma 2. Next we introduce the moment-generating function (s is any complex number)

$$\phi(s)=\int_0^1 e^{sx}dG(x) .$$

From (2.4) it follows that ϕ satisfies

$$\phi(Ds)=\phi(s)+\sum_{j=1}^{D-1}(e^{js}-1)\int_0^1 e^{sx}f_j(x)dG(x) . \tag{5.5}$$

Setting $s=2\pi ki$, $i=\sqrt{-1}$, we have

$$\phi(2\pi kDi)=\phi(2\pi ki) , \qquad k=\pm 1, \pm 2, \cdots . \tag{5.6}$$

First suppose $\phi(2\pi ki)=0$, $k=\pm 1, \pm 2, \cdots$. Since $\phi(it)$ is the characteristic function of a distribution on (0, 1), it is uniquely determined by its values at the points $2\pi ki$; hence in this case

$$\phi(it)=\frac{e^{it}-1}{it}$$

and $G(x)=x$. It can be verified directly that (2.4) is satisfied with $G(x)=x$ and $f_j(x)=1/D$. From Lemma 4, this is the only case where $G(x)=x$ can occur.

Next suppose that for some integer k we have $\phi(2\pi ki)\neq 0$. Iteration of (5.6) shows that $\phi(it)$ does not $\to 0$ as $t\to\infty$ or $t\to-\infty$ and hence G is not purely absolutely continuous. Thus Lemma 3 shows that G, if continuous and not of type (b), is purely singular.[9]

[9] The fact that G has in general no absolutely continuous component can be seen from a simple argument not involving Fourier transforms.

6. Grouped Markov chains. Let Y_n be the variables of a Markov chain with a finite number of states, which we shall call $1, 2, \cdots, K$. Let the transition matrix be $M=(p_{ij})$, $i, j=1, \cdots, K$. We assume $p_{ij}>0$. Otherwise, even if some power of M has all positive elements, there may be complications. We also assume $K>1$. Now let the states of the chain be divided into D mutually exclusive and exhaustive nonempty subsets $B_0, \cdots, B_{D-1}$. We can define an infinite-order chain Z_n by

$$Z_n=i \rightleftarrows Y_n \in B_i \, . \tag{6.1}$$

We shall call such a process a *grouped Markov chain*. We shall be particularly interested in the case where the Y_n-process, and hence the Z_n-process, is stationary. We show that Conditions B and C are satisfied, determine the distribution of the "past" of the Z_n-process, and show how the functions $Q_i(u)$ and the corresponding $f_i(x)$, can be determined. The Q_i or f_i give the solution to the problem of predicting the future values of Z_n, given the past.

We first give a result about Markov chains.

THEOREM 8. *Let $M=(p_{ij})$ be the transition matrix of a Markov chain, $i, j=1, \cdots, K>1$; $p_{ij}>0$. Let Y_n be the variables of the chain. Let*

$$\lambda = \min_{i,j,k,l} \frac{p_{kj}p_{il}}{K^2 p_{ij}p_{kl}} \tag{6.2}$$

(Note that $0<\lambda<1$.) For each $n=1, 2, \cdots$, let A_n be a nonempty subset of states of the chain. Let g and h be two states. Then

$$\begin{aligned} &\big| P(Y_{n+1} \in A_{n+1} | Y_0=g, Y_1 \in A_1, \cdots, Y_n \in A_n) \\ &\quad - P(Y_{n+1} \in A_{n+1} | Y_0=h, Y_1 \in A_1, \cdots, Y_n \in A_n) \big| \le (1-\lambda)^n \, , \\ &\qquad n=1, 2, \cdots . \end{aligned} \tag{6.3}$$

The proof is omitted. It can be carried out with Doeblin's "two-particle" method.

It is readily shown that for *every* u, $0 \le u_i \le D-1$, the limit

$$\lim_{k\to\infty} P(Z_0=i | Z_{-1}=u_1, \cdots, Z_{-k}=u_k)$$

exists, for the grouped Markov chain. We may take this limit as a permissible version of $Q_i(u)$ for the Z_n-process defined by (6.1). It can also be seen that

$$|Q_i(u)-Q_i(u')| \le (1-\lambda)^{m-1} \, , \qquad m=1, 2, \cdots , \tag{6.4}$$

whenever the first m terms of u and u' coincide. Thus Condition B is satisfied with[10]

[10] The stronger condition (4.3) is of course, also satisfied.

$$\varepsilon_m \leq (1-\lambda)^{m-1}, \qquad m=1, 2, \cdots .$$

Condition C is a consequence of the obvious fact that the $Q_i(u)$ are uniformly positive.

THEOREM 9. *Let Z_n be defined by* (6.1). *Let*

$$X_n = \sum_{j=1}^{\infty} Z_{n-j}/D^j \tag{6.5}$$

and let $G(x)$ be the distribution of X_n. Then $G(x)$ is continuous, $0 \leq x \leq 1$, and strictly increasing, $0 \leq x < 1$.

Theorem 7 is applicable since Z_n is of the mixing type. Since Z_n has a positive probability of taking at least two distinct values (we are assuming $D>1$), continuity follows. The strictly increasing character of G follows from the fact that the event $(Z_{-1}=u_1, \cdots, Z_{-k}=u_k)$ has positive probability for every sequence $0 \leq u_1, \cdots, u_k \leq D-1$.

DEFINITIONS. Let Y_n and Z_n be as in (6.1) and let X_n be defined by (6.5). Define

$$H_j(x) = P(X_n \leq x \mid Y_n = j), \quad 0 \leq x \leq 1, j=1, 2, \cdots, K, \tag{6.6}$$

$$\theta_j(s) = \int_0^1 e^{sx} dH_j(x), \qquad j=1, 2, \cdots, K, \tag{6.7}$$

s any complex number.

Let π_j, $j=1, \cdots, K$ be the (unique) set of stationary probabilities satisfying

$$\pi_j = \sum_{r=1}^{K} \pi_r p_{rj}, \qquad j=1, \cdots, K. \tag{6.8}$$

Let p_{ij}^* be the set of inverse probabilities

$$p_{ij}^* = p_{ji}\pi_j/\pi_i . \tag{6.9}$$

Let $M(s)$ be the matrix defined as follows:

$$M(s) = (p_{ij}^* e^{s\nu(j)}) \tag{6.10}$$

where $\nu(j)=k$ when j belongs to the group of states B_k.

THEOREM 10. (See preceding definitions.) *The function $\theta_j(s)$, $j=1, \cdots, K$, is the sum of the elements in the jth row of the convergent matrix product*

$$M(s/D)\, M(s/D^2)\, M(s/D^3) \cdots . \tag{6.11}$$

Proof. Let Y_n^* be the variables of a stationary inverse Markov chain with transition probabilities given by (6.9) and let $Z_n^*=i$ when $Y_n^* \in B_i$. It is clear that Z_n^* is inverse to the Z_n-process in the sense that the process Z^*_{-n} obeys the same probabilistic laws as the Z_n-process. Define

$$X_n^* = \sum_{r=1}^{\infty} Z_{n+r}^*/D^r .$$

(Incidentally X_n and X_n^* follow the same law, not inverse laws.)

It is clear that $H_j(x)$, as defined in (6.6) above, is also given by

$$H_j(x) = P(X_n^* \le x \mid Y_n^* = j) . \tag{6.12}$$

We now use (6.12) to find the functions $H_j(x)$.

Suppose r is an integer, $0 \le r \le D-1$, and suppose $r/D \le x < (r+1)/D$; that is, $x = .ru_2u_3\cdots$. Then

$$\begin{aligned} H_j(x) &= P(Z_1^*/D + Z_2^*/D^2 + \cdots \le .ru_2 \cdots \mid Y_0^* = j) \\ &= P(Z_1^* < r \mid Y_0^* = j) \\ &\qquad + P(Z_1^* = r,\ Z_2^*/D + Z_3^*/D^2 + \cdots \le .u_2u_3 \cdots \mid Y_0^* = j) \\ &= P(Z_1^* < r \mid Y_0^* = j) + \sum_{m \in B_r} p_{jm}^* H_m(Dx - r) , \qquad r/D \le x < (r+1)D. \end{aligned} \tag{6.13}$$

Next we note that X_n^* has the same distribution $G(x)$ as X_n. Moreover

$$G(x) = P(X_n^* \le x) = \sum_{r=1}^{K} \pi_r H_r(x) . \tag{6.14}$$

Since $G(x)$ is continuous (Theorem 9), the $H_r(x)$ must also be continuous. Now (6.13) implies the differential relationship

$$dH_j(x) = \sum_{m \in B_r} p_{jm}^* \, d[H_m(Dx - r)] , \tag{6.15}$$

$$\frac{r}{D} \le x < \frac{r+1}{D} , \qquad j = 1, \cdots, K.$$

Defining $\theta_j(s)$ by (6.7) and letting $\Theta(s)$ be the column vector whose components are the θ_j, we see that (6.15) implies (multiplying both sides of (6.15) by e^{Dsx} and integrating)

$$\Theta(Ds) = M(s)\Theta(s) , \tag{6.16}$$

where $M(s)$ is defined in (6.10). Iterating (6.16) and replacing s by s/D gives

$$\Theta(s) = M(s/D) \cdots M(s/D^n)\Theta(s/D^n) .$$

Since $\theta_j(0) = 1$, $\Theta(s/D^n)$ approaches the column vector each of whose components is 1 as $n \to \infty$, while $M(s/D^n)$ approaches the stochastic matrix (p_{ij}^*). The powers $(p_{ij}^*)^n$ converge exponentially as $n \to \infty$, and it is readily seen that the elements of the difference $M(s/D^n) - (p_{ij}^*)$ are

$O(D^{-n})$, where O is uniform in s for any bounded s-region. Hence the matrix product in (6.11) converges uniformly in any bounded s-region, and Theorem 10 follows.

The $\theta_j(s)$ and the $H_j(x)$ can be calculated in various ways. One possibility is to determine the coefficients in the power-series expansions of the θ_j by differentiating (6.16) at $s=0$. The values of the θ_j on some interval near 0 on the imaginary axis can be calculated, and (6.16) can then be used to determine the θ_j on the rest of the imaginary axis.

We can now find the functions $Q_i(u)$ and $f_i(x)$ for grouped Markov chains. In theorem 11, Z_n is a grouped Markov chain as defined above.

THEOREM 11. *Let* $u_1, \cdots, u_k$ *be integers,* $0 \leq u_j \leq D-1$. *Then*

$$P(Z_n=i|Z_{n-1}=u_1, \cdots, Z_{n-k}=u_k)=\sum_{j\in B_i} \pi_j[H_j(x_2)-H_j(x_1)]/[G(x_2)-G(x_1)]$$

where

$$x_1=u_1/D+u_2/D^2+\cdots+u_k/D^k , \quad x_2=u_1/D+\cdots+u_k/D^k+D^{-k} ,$$

and

$$G(x)=\sum_{j=1}^{K} \pi_j H_j(x) .$$

The proof is merely a reinterpretation of Theorem 10.

We thus have an expression for the conditional distribution of Z_n if a finite segment of the past is known.

Next we consider the situation when the complete past is known. Consider the X_n-process and the associated functions $f_i(x)$. Then, if $x=.u_1u_2\cdots$,

$$f_i(x)=\frac{d}{dG(x)}\sum_{j\in B_i}\pi_j H_j(x)=P(Z_n=i|\, Z_{n-1}=u_1, Z_{n-2}=u_2, \cdots) \tag{6.17}$$

where (6.17) holds for every x, $0 \leq x < 1$, provided we take right-hand derivatives on the right side. Thus (6.17) gives the conditional distribution for Z_n if the complete past is known.

Example. Suppose

$$(p_{ij})=\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Then $\pi_1=4/11$, $\pi_2=3/11$, $\pi_3=4/11$, and in this case $p_{ij}^*=p_{ij}$. Take states 1 and 2 as B_0, state 3 as B_1, so that $D=2$. The $\theta_j(s)$ then satisfy the equations

$$(6.18) \qquad \begin{aligned} \theta_1(2s) &= \frac{1}{4}\theta_1(s) + \frac{1}{4}\theta_2(s) + \frac{1}{2}e^s\theta_3(s) \\ \theta_2(2s) &= \frac{1}{3}\theta_1(s) + \frac{1}{3}\theta_2(s) + \frac{1}{3}e^s\theta_3(s) \\ \theta_3(2s) &= \frac{1}{2}\theta_1(s) + \frac{1}{4}\theta_2(s) + \frac{1}{4}e^s\theta_3(s) , \end{aligned}$$

which, with the conditions $\theta_j(0)=1$, determine them uniquely. The $H_j(x)$ can then be determined by Fourier inversion.

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Columbia University
and the RAND Corporation

ON CERTAIN SERIES EXPANSIONS INVOLVING WHITTAKER FUNCTIONS AND JACOBI POLYNOMIALS

PETER HENRICI

1. INTRODUCTION

1. 1. Outline of the paper. By substituting polar coordinates in the partial differential equation

$$(1)\qquad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{4\mu+1}{x}\frac{\partial u}{\partial x}+\frac{4\nu+1}{y}\frac{\partial u}{\partial y}+k[4\lambda-k(x^2+y^2)]u=0\ ,$$

and separating variables, one is led in a natural way to certain combinations of Whittaker functions and Jacobi polynomials (called for brevity J.-W. functions in this paper). With a view towards deriving some functional relations involving hypergeometric functions, we develop in the first part of the paper a technique for the construction of expansions of arbitrary regular analytic solutions of (1) in terms of these J.-W. functions. The method of our investigation consists in setting up a one-to-one correspondence between the class of even analytic functions of one complex variable regular in a circle around the origin and a certain class E of regular solutions of (1). This correspondence associates with a solution $u(x, y) \in E$ the function $u(x,-ix)$ obtained by considering $u(x, y)$ on the (imaginary) characteristic $x-iy=0$ of (1).[1] Since the maps of the even powers of a single variable in this correspondence are shown to be the J.-W. functions mentioned above, the expansion problem in question is reduced to the problem of finding the Taylor expansion of a given analytic function of one variable.

Applying this technique to some special solutions of (1), we are led to three expansions involving various kinds of hypergeometric functions. The first of them contains a number of well-known theorems on special functions as special cases, namely, among others, Bateman's addition theorem in the theory of Bessel functions, Ramanujan's formula for the product of two confluent hypergeometric series, and Erdélyi's addition theorem (with respect to the parameters) for the product of two M-functions. The second application gives rise to

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[1] This procedure is related to Bergman's operator method in the theory of elliptic partial differential equations with regular coefficients; see the remark at the end of § 4.

another addition formula (in the ordinary sense) for the product of two M-functions, while the third may be looked at as an alternate formulation of Bailey's decomposition formula for a special case of Appell's function F_4.

1. 2. Definitions. In (1) the parameters μ, ν, λ, k are arbitrary complex numbers with the only exception that μ and ν are subject to the condition

$$(2) \qquad 2\mu+2\nu\neq-2,\ -3,\ -4,\ \cdots .$$

Sets of values (μ, ν, λ, k) satisfying (2) are called *admissible values* of the parameters.

If $\mathscr{B}$ denotes a domain of the complex (x, y)-space which contains the origin, we denote by $E_{\mathscr{B}}$ the class of analytic functions $u(x, y)$ of the two complex variables x and y which

(i) are regular in $\mathscr{B}$,
(ii) are even fuctions of x and of y, and
(iii) satisfy (1) for certain admissible values of the parameters.[2]

We denote by $\mathscr{K}_r$ the circle $|z|<r$ of the complex z-plane, and by $\mathscr{K}_r\times\mathscr{K}_r$ the bicylinder $|z|<r$, $|z^*|<r$ in the space K^2 of the two complex variables z and z^*.

Our notation of special functions follows the traditional lines. For the ordinary and the generalised hypergeometric series we found it convenient to use Bailey's notation [1, p. 8]

2. Jacobi-Whittaker Functions

Our first aim is to construct a set of solutions of (1) by the elementary method of separating variables. Introducing in (1) the new variables

$$(3) \qquad \rho=x^2+z^2, \qquad \tau=\frac{x^2-y^2}{x^2+y^2},$$

we obtain for $v(\rho, \tau)=u(x, y)$ the equation

$$(4) \qquad \rho^2\frac{\partial^2 v}{\partial\rho^2}+2(1+\mu+\nu)\rho\frac{\partial v}{\partial\rho}+(1-\tau^2)\frac{\partial^2 v}{\partial\tau^2} +2[\mu-\nu-(1+\mu+\nu)\tau]\frac{\partial v}{\partial\tau}+k\rho\left(\lambda-\frac{k\rho}{4}\right)v=0 .$$

If $v(\rho, \tau)=R(\rho)\,T(\tau)$ is a solution of (4), one finds by the usual sepa-

[2] We do not investigate the problem of the extent to which the three conditions (i), (ii), and (iii) imply each other.

ration method that $R(\rho)$ and $T(\tau)$ have to satisfy separately the equations

$$\frac{d^2R}{d\rho^2}+2(1+\mu+\nu)\frac{1}{\rho}\frac{dR}{d\rho}+\left[-\frac{s}{\rho^2}+\frac{k\lambda}{\rho}-\frac{k^2}{4}\right]R=0\,, \tag{5}$$

and

$$(1-\tau^2)\frac{d^2T}{d\tau^2}+2[\mu-\nu-(1+\mu+\nu)\tau]\frac{dT}{d\tau}+sT=0\,, \tag{6}$$

where s is a separation parameter. Writing $s=n(2\mu+2\nu+1+n)$, we find that solutions of (5) which are regular near $\rho=0$ are represented for $n=0,\ 1,\ 2,\ \cdots$ by

$$\begin{aligned} R(\rho)&=\rho^{-\mu-\nu-1}M_{\lambda,\ \mu+\nu+\frac{1}{2}+n}(k\rho)\\ &=k^{\mu+\nu+1+n}\rho^n e^{-\frac{k\rho}{2}}\,{}_1F_1\left[\begin{matrix}\mu+\nu+1+n-\lambda;\ k\rho\\ 2\mu+2\nu+2+2n\end{matrix}\right], \end{aligned} \tag{7}$$

where M denotes the Whittaker function of the first kind, while (6) has for the same values of s the polynomial solution

$$\begin{aligned} T(\tau)&=P_n^{(2\nu,\ 2\mu)}(\tau)\\ &=\frac{(2\nu+1)_n}{n\,!}\,{}_2F_1\left[\begin{matrix}-n,\ 2\mu+2\nu+1+n;\ \frac{1-\tau}{2}\\ 2\nu+1\end{matrix}\right], \end{aligned} \tag{8}$$

where P stands for the Jacobi polynomial in the notation of Szegö [11, p. 61]. Provided (2) is valid, solutions of (1) regular near $x=y=0$ are thus given by the functions

$$f_n^{(\mu,\nu)}(\rho,\ \tau;\ \lambda,\ k)=c_nP_n^{(2\nu,\,2\mu)}(\tau)k^{-n}(k\rho)^{-\mu-\nu-1}M_{\lambda,\ \mu+\nu+\frac{1}{2}+n}(k\rho)\ , \tag{9}$$

where

$$c_n=\frac{2^{2n}n!}{(2\mu+2\nu+1+n)_n}=\frac{(2\mu+2\nu+1)_n n!}{(\mu+\nu+\frac{1}{2})_n(\mu+\nu+1)_n} \tag{10}$$

is a normalisation factor introduced for later convenience. We shall call these functions for brevity Jacobi-Whittaker functions (J.-W. functions) of order n. The arguments λ and k in $f_n^{(\mu,\nu)}$ will usually be omitted, if it is not necessary to exhibit them explicitly.

For later reference we note the following special and limiting cases of the functions $f_n^{(\mu,\nu)}$:

(i) For $\lambda=0$ we have [6, p. 13]

$$f_n^{(\mu,\nu)}(\rho,\tau;0;k) \\ =c_n P_n^{(2\nu,2\mu)}(\tau)\Gamma(\mu+\nu+\frac{3}{2}+n)\left(\frac{k}{4}\right)^{-n}\left(\frac{k\rho}{4}\right)^{-\mu-\nu-\frac{1}{2}} I_{\mu+\nu+\frac{1}{2}+n}\left(\frac{k\rho}{2}\right), \tag{11}$$

where I is a modified Bessel function.

(ii) Putting $\lambda=\frac{\kappa}{4k}$ and letting $k\to 0$, we obtain the function

$$f_n^{(\mu,\nu)}(\rho,\tau;\kappa)=\lim_{k\to 0} f_n^{(\mu,\nu)}\left(\rho,\tau,\frac{\kappa}{4k},k\right) \\ =c_n P_n^{(2\nu,2\mu)}(\tau)\Gamma(2\mu+2\nu+2+2n)\left(\frac{\kappa}{4}\right)^{-n}\left(\frac{\kappa\rho}{4}\right)^{-\mu-\nu-\frac{1}{2}} J_{2\mu+2\nu+1+2n}(\sqrt{\kappa\rho}) \tag{12}$$

where J denotes the ordinary Bessel function. Evidently (12) satisfies the differential equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{4\mu+1}{x}\frac{\partial u}{\partial x}+\frac{4\nu+1}{y}\frac{\partial u}{\partial y}+\kappa u=0\,. \tag{13}$$

We will refer to (12) as to the "reduced" case of the functions $f_n^{(\mu,\nu)}$. The limiting values of λ and k leading to it are included in the admissible values of the parameters.

(iii) For $\lambda=k=0$ we have from (9), (11) or (12)

$$f_n^{(\mu,\nu)}(\rho,\tau;0,0)=c_n\rho^n P_n^{(2\nu,2\mu)}(\tau)\,. \tag{14}$$

We study next some properties of the J.-W. functions considered as functions of the two complex variables z and z^* defined by

$$z=x+iy\,,\qquad z^*=x-iy\,.[3] \tag{15}$$

As such they satisfy the differential equation

$$\frac{\partial^2 U}{\partial z\partial z^*}+\frac{2\mu+\frac{1}{2}}{z+z^*}\left\{\frac{\partial U}{\partial z}+\frac{\partial U}{\partial z^*}\right\}-\frac{2\nu+\frac{1}{2}}{z-z^*}\left\{\frac{\partial U}{\partial z}-\frac{\partial U}{z^*}\right\}+k\left[\lambda-k\frac{zz^*}{4}\right]U=0\,, \tag{16}$$

which is readily constructed by inserting in (1) the variables (15). From (3) it is evident that

$$\rho=zz^*\,,\qquad \tau=\frac{z^2+z^{*2}}{2zz^*}\,.$$

Thus we have for

$$F_n^{(\mu,\nu)}(z,z^*)=f_n^{(\mu,\nu)}(\rho,\tau)$$

[3] It is assumed throughout the paper that x and y are independent complex variables, so that also z and z^* take on independent complex values.

the representation

$$(17)\quad F_n^{(\mu,\nu)}(z, z^*)$$
$$= c_n P_n^{(2\nu,2\mu)}\left(\frac{z^2+z^{*2}}{2zz^*}\right) k^{-n}(kzz^*)^{-\mu-\nu-1} M_{\lambda,\mu+\nu+\frac{1}{2}+n}(kzz^*) .$$

Using the relations [11, pp. 58, 61]

$$P_n^{(2\nu,2\mu)}(\tau) = (-)^n P_n^{(2\mu,2\nu)}(-\tau)$$

and

$$P_n^{(2\mu,2\nu)}(\tau) = \frac{(2\mu+2\nu+1+n)_n}{n!}\left(\frac{\tau-1}{2}\right)^n {}_2F_1\left[\begin{matrix}-n,\ -n-2\mu; \\ -2n-2\mu-2\nu\end{matrix}\ \frac{2}{1-\tau}\right]$$

and observing (7), we may write this also as follows:

$$F_n^{(\mu,\nu)}(z, z^*)$$
$$= (z+z^*)^{2n} {}_2F_1\left[\begin{matrix}-n,\ -n-2\mu; \\ -2n-2\mu-2\nu\end{matrix}\ \frac{4zz^*}{(z+z^*)^2}\right] e^{-\frac{kzz^*}{2}} {}_1F_1\left[\begin{matrix}\mu+\nu+1+n-\lambda; \\ 2\mu+2\nu+2+2n\end{matrix}\ kzz^*\right] .$$

From this representation it is easy to draw the following conclusions:

LEMMA 1. *For all admissible values of the parameters,*

$$F_n^{(\mu,\nu)}(z, z^*) \in E_{K^2} .$$

LEMMA 2. *For all admissible values of the parameters,*

$$F_n^{(\mu,\nu)}(z, 0) = z^{2n} .$$

In order to prove Lemma 1 we observe that the last two factors in (18) are entire functions of zz^*, while, since the series ${}_2F_1$ in (18) terminates after at most n terms, the first two factors form together a polynomial in z and z^*. The solution (18) of (16) is thus an entire function of z and z^*. Furthermore the conditions of symmetry imposed on the elements of E, which for functions of z and z^* amount to the relations

$$(19)\qquad F(z, z^*) = F(-z, -z^*) , \qquad F(z, z^*) = F(z^*, z) ,$$

are satisfied by (18). Lemma 2 follows simply from the fact that for $z^*=0$ the last three factors in (18) reduce to 1. It is easy to see that both Lemma 1 and Lemma 2 remain also true in the reduced case.

We come now to a simple equiconvergence property of series of J.-W. functions.

LEMMA 3. *Let $r>0$ and let*

$$f(z)=\sum_{n=0}^{\infty}a_n z^{2n} \tag{20}$$

be regular in $\mathcal{K}_r$. *Then for fixed admissible values of the parameters the series*

$$F(z, z^*)=\sum_{n=0}^{\infty}a_n F_n^{(\mu,\nu)}(z, z^*; \lambda, k) \tag{21}$$

converges uniformly in every closed subregion $\mathcal{D}$ *of* $\mathcal{B}=\mathcal{K}_r\times\mathcal{K}_r$ *and represents there a function* $\in E_{\mathcal{B}}$ *with the property*

$$F(z, 0)=f(z)\ . \tag{22}$$

Proof. Obviously the second statement of the lemma follows immediatly from the first and from Lemma 2. In order to prove the uniform convergence, we again use for the J.-W. functions the representation (18). It follows in the general case from a well-known theorem on M-functions [8, p. 93] and in the reduced case from an analogous theorem on Bessel functions [13, p. 44, formula (1)] that for bounded (z, z^*) and for n large the product of the last two factors in (18) is asymptotically equal to 1. It suffices therefore to consider the case $\lambda=k=0$. We make now use of the well-known generating function of the Jacobi polynomials [11, p. 68. formula (4.4.5)]. Replacing the variables α, β, x, w in Szegö's formula by 2ν, 2μ, $(z^2+z^{*2})/2zz^*$, tzz^* respectively and observing (17), we obtain the power series in t

$$\sum_{n=0}^{\infty}c_n^{-1}F_n^{(\mu,\nu)}(z, z^*; 0, 0)t^n=E(z, z^*; t)\ , \tag{23}$$

where for given $r''>0$, $E(z, z^*; t)$ is a certain analytic function of z, z^* and t regular in $(z, z^*)\in\{\mathcal{K}_{r''}\times\mathcal{K}_{r''}\}\cap\{|t|<r''^{-2}\}$. Let now $\mathcal{D}$ be enclosed in a bicylinder $\mathcal{K}_{r'}\times\mathcal{K}_{r'}$, where $r'<r$, and choose r'', r''' such that $r'<r''<r'''<r$. Applying to (23) Cauchy's estimate for the coefficients of a power series with $|t|=r'''^{-2}$ yields

$$|F_n^{(\mu,\nu)}(z, z^*; 0, 0)|\le K|c_n|r'''^{2n}\ , \tag{24}$$

where

$$K=\max_{\substack{|t|=r'''^{-2}\\ (z, z^*)\in\mathcal{K}_{r'}\times\mathcal{K}_{r'}}}|E(z, z^*; t)|$$

is finite and does not depend on n. Therefore the terms of (21) are dominated in $\mathcal{D}$ by the terms of the series

$$K\sum_{n=0}^{\infty}|a_n||c_n|r''^{2n}\ ,$$

which converges absolutely, since $|c_n/c_{n-1}| \to 1\ (n \to \infty)$ and (20) converges for some $z=r''''$ with $r'''<r''''<r$.[4]

3. A Uniqueness Theorem for Solutions of (1)

Lemma 4. *Let $\mathscr{D}$ be a domain $\subset K^2$, containing the origin, and let $F(z, z^*)$ be a function $\in E$ such that $F(z, 0)=0$. Then $F(z, z^*)=0$ in $\mathscr{D}$.*

Remark. The proof of this lemma does not follow from the general uniqueness theorems for hyperbolic initial value problems (see, for example, [7, p. 321]), since some of the coefficients in (16) are singular.

Proof. In view of the relations (19) the power series expansion of F, which by assumption converges in a certain neighbourhood of the origin, must be of the form

$$F(z, z^*)=\sum_{m=0}^{\infty} \sum_{n=0}^{m} c_{m,n} z^{2m-n} z^{*n}, \tag{25}$$

where

$$c_{m,n}=c_{m,2m-n}. \tag{26}$$

If we call $s+t$ the weight of the monomial $z^s z^{*t}$, we may say that (25) contains only terms of even weight. By assumption and by (26),

$$c_{m,0}=c_{m,2m}=0, \qquad m=0, 1, 2, \cdots. \tag{27}$$

By differentiating (25) and substituting into (16) we obtain, after multiplying by z^2-z^{*2},

$$\begin{aligned}\sum_{m=0}^{\infty} \sum_{n=0}^{m} c_{m,n} \{ & n(2m-n-2\mu-2\nu-1) z^{2m-n+1} z^{*n-1} + 4(m-n)(\mu-\nu) z^{2m-n} z^{*n} \\ & -(m-n)(n+2\mu+2\nu+1) z^{2m-n-1} z^{*n+1} + R_m\} = 0,\end{aligned} \tag{28}$$

where the symbol R_m denotes terms of higher weight than m. We prove now that $c_{m,n}=0$ for all values of m and n in question by induction with respect to the weight.

By (27), $c_{0,0}=0$. Let us assume that we have proved

$$c_{k,n}=0 \quad \text{for} \quad n=0, 1, \cdots, 2k; \quad k=0, 1, \cdots, m-1. \tag{29}$$

Consider now in (28) the terms of fixed weight m. Then the terms R_m will be multiplied by coefficients $c_{k,n}$ with $k<m$, which are zero by

[4] The author is indebted to a referee for the following remark: Using the theorems about the growth of a power series of one complex variable whose coefficients satisfy certain conditions, one could obtain bounds for the functions (21) in terms of the coefficients a_n.

(29). Considering now the fact that the coefficients of each fixed power $z^{2m-n}z^{*n}$ must vanish separately, we are led to the recurrence relations

(30) $$c_{m,1}(m+\mu+\nu)+2c_{m,0}m(\mu-\nu)=0 ,$$

(31) $$c_{m,n+1}(n+1)(2m-n+2\mu+2\nu)+4c_{m,n}(m-n)(\mu-\nu) \\ -c_{m,n-1}(2m-n-1)(n+2\mu+2\nu)=0 , \qquad n=1, 2, \cdots , 2m-1 .$$

Since $c_{m,0}=0$ and since $m+\mu+\nu\neq 0$ for admissible values of the parameters, we have from (30) $c_{m,1}=0$ and hence from (31) $c_{m,n+1}=0$ as long as $2m-n+2\mu+2\nu\neq 0$, for $n=1, 2, \cdots, 2m-1$. It follows that (29) is true for $k=m$ and hence for all k.

4. Expansion Theorem

The following theorem, which will be the principal tool for the special functions work in the later part of this paper, is now easy to prove.

THEOREM. *Let $r>0$, $\mathscr{B}=\mathscr{K}_r\times\mathscr{K}_r$ and let $F(z, z^*)\in E_{\mathscr{B}}$. If*

(32) $$F(z, 0)=\sum_{n=0}^{\infty}a_n z^{2n} ,$$

then the series

(33) $$\sum_{n=0}^{\infty}a_n F_n^{(\mu,\nu)}(z, z^*)$$

(which by Lemma 3 converges in $\mathscr{B}$) *is equal to $F(z, z^*)$ in $\mathscr{B}$.*

Proof. By Lemma 3, (33) represents a function $\in E_{\mathscr{B}}$ which is equal to $F(z, 0)$ for $z^*=0$. By Lemma 4 the function

$$F(z, z^*)-\sum_{n=0}^{\infty}a_n F_n^{(\mu,\nu)}(z, z^*)$$

vanishes identically in $\mathscr{B}$.

The expansion (33) will sometimes be called J.-W. expansion of $F(z, z^*)$. The function (32), the knowledge of which is sufficient for the construction of the J.-W. expansion of $F(z, z^*)$, will be called the generating function of this expansion.

Remark. For fixed admissible values of the parameters Lemma 3 sets up a mapping of the class of even analytic functions of a single complex variable regular in a $\mathscr{K}_r$ on the class $E_{\mathscr{K}_r\times\mathscr{K}_r}$. This mapping is one-to-one by Lemma 4. The inverse mapping is given by the formula

$$f(z)=F(z, 0) ,$$

which is essentially identical with the inversion formula for Bergman's so-called integıal operator of the first kind,[5] whose existence, however, has been established in general only for the case where the coefficients of the differential equation are regular analytic functions in the considered domain. Our theory presents an example of a representation of an operator analogous to that of Bergman in a case where the considered differential equation has singular coefficients.[6]

We proceed now to construct explicity by our method the J.-W. expansions of several special solutions of (1), which are again obtained by the method of separation of variables.

5. Applications of the Expansion Theorem: Cartesian Coordinates

If the function $u(x, y)=X(x)\,Y(y)$ is introduced in (1) (with $k=1$), we find that the differential equation is satisfied if X and Y satisfy separately the equations

$$(34)\qquad \begin{aligned} &\frac{d^2X}{dx^2}+\frac{4\mu+1}{x}\frac{dX}{dx}+(4\alpha-x^2)X=0\,,\\ &\frac{d^2Y}{dy^2}+\frac{4\nu+1}{y}\frac{dY}{dy}+(4\beta-y^2)Y=0\,, \end{aligned}$$

provided $\alpha+\beta=\lambda$.

Solutions of these equations which are regular near $x=0$ and $y=0$ can again be expressed by means of Whittaker functions. In view of the differential equation satisfied by these functions it is readily verified that, provided none of the numbers 2μ and 2ν is a negative integer, one may put

$$(35)\qquad \begin{aligned} X(x)&=x^{-2\mu-1}M_{\alpha,\mu}(x^2)\,,\\ Y(y)&=y^{-2\nu-1}M_{\beta,\nu}(y^2)\,. \end{aligned}$$

Introducing the variables z and z^* and passing to hypergeometric series we have

$$(36)\qquad \begin{aligned} &u\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}\right)=U(z, z^*)\\ &=e^{-\frac{zz^*}{2}}\,{}_1F_1\left[\begin{matrix}\mu+\frac{1}{2}-\alpha;\ \frac{(z+z^*)^2}{4}\\ 2\mu+1\end{matrix}\right]{}_1F_1\left[\begin{matrix}\nu+\frac{1}{2}-\beta;\ -\frac{(z-z^*)^2}{4}\\ 2\nu+1\end{matrix}\right]. \end{aligned}$$

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[5] See [2, p. 117]. Contrary to the situation described there, our operator maps functions $f(z)$ which are real for real z on solntious of (1) which are real for real x and y.

[6] Other cases of differential equations with singular coefficients have been treated by Bergman [3, 4]. The "reduced" equation (13) has in the case $4\mu+1=0$ been considered by the present author in [10], where a different method has been used.

The generating function of the J.-W. expansion of $U(z, z^*)$ is thus given by

$$U(z,0)={}_1F_1\left[\begin{matrix}\mu+\frac{1}{2}-\alpha;-\frac{z^2}{4}\\2\mu+1\end{matrix}\right]{}_1F_1\left[\begin{matrix}\nu+\frac{1}{2}-\beta;-\frac{z^2}{4}\\2\nu+1\end{matrix}\right]$$

(37)
$$=\sum_{m=0}^{\infty}\left(\frac{z^2}{4}\right)^m\sum_{n=0}^{m}\frac{\left(\mu+\frac{1}{2}-\alpha\right)_m(-2\mu-m)_n\left(\nu+\frac{1}{2}-\beta\right)_n(-m)_n}{\left(-\mu+\alpha-m+\frac{1}{2}\right)_n(2\mu+1)_m(2\nu+1)_n m!n!}$$

$$=\sum_{m=0}^{\infty}\frac{\left(\mu+\frac{1}{2}-\alpha\right)_m}{(2\mu+1)_m m!}\,{}_3F_2\left[\begin{matrix}-2\mu-m,\ \nu+\frac{1}{2}-\beta,\ -m;\\-\mu+\alpha-m+\frac{1}{2},\ 2\nu+1\end{matrix}\right]\left(\frac{z^2}{4}\right)^m.$$

Applying the expansion theorem, writing the J.-W. functions in the form (9) and using the relations (following from (3))

$$x^2=\rho\frac{1+\tau}{2},\qquad y^2=\rho\frac{1-\tau}{2}$$

we obtain the following J.-W. expansion for the product of two Whittaker functions with different pairs of indices and arguments, which is valid for unrestricted values of $\rho, \tau, \alpha, \beta$, as long as none of the numbers 2μ, 2ν and $2\mu+2\nu+1$ is a negative integer:

$$\left(\rho\frac{1+\tau}{2}\right)^{-\mu-\frac{1}{2}}M_{\alpha,\mu}\left(\rho\frac{1+\tau}{2}\right)\cdot\left(\rho\frac{1-\tau}{2}\right)^{-\nu-\frac{1}{2}}M_{\beta,\nu}\left(\rho\frac{1-\tau}{2}\right)$$

(38)
$$=\sum_{m=0}^{\infty}\frac{\left(\mu+\frac{1}{2}-\alpha\right)_m}{(2\mu+1)_m(2\mu+2\nu+1+m)_m}\,{}_3F_2\left[\begin{matrix}-2\mu-m,\ \nu+\frac{1}{2}-\beta,\ -m;\\-\mu+\alpha-m+\frac{1}{2},\ 2\nu+1\end{matrix}\right]$$

$$\times P_n^{(2\nu,2\mu)}(\tau)\rho^{-\mu-\nu-1}M_{\alpha+\beta,\ \mu+\nu+\frac{1}{2}+m}(\rho)\ .$$

This mother expansion has a great number of children and grandchildren, of which some are known since long. In the following we list some of those of its special cases where the function ${}_3F_2$ can be expressed in a more closed form, and some other consequences.

5. 1. Bateman's expansion. Putting in (38)

$$\rho=kr^2\ ,\qquad \tau=\cos 2\vartheta\ ,$$

$$\alpha=\cos^2\varphi/4k,\qquad \beta=\sin^2\varphi/4k$$

and letting $k\to 0$, we obtain, using (12) and replacing 2μ and 2ν by μ and ν respectively,

$$(r\cos\varphi\cos\vartheta)^{-\mu}J_\mu(r\cos\varphi\cos\vartheta)\cdot(r\sin\varphi\sin\vartheta)^{-\nu}J_\nu(r\sin\varphi\sin\vartheta)$$

$$(39)\qquad =\sum_{m=0}^{\infty}\frac{(-)^m 2m!(\mu+\nu+1+2m)\Gamma(\mu+\nu+1+m)}{\Gamma(\mu+1+m)\,\Gamma(\nu+1+m)}$$

$$\times P_m^{(\nu,\mu)}(\cos 2\varphi)P_m^{(\nu,\mu)}(\cos 2\vartheta)r^{-\mu-\nu-1}J_{\mu+\nu+1+m}(r)\ ,$$

which is equivalent to Bateman's expansion for the product of two Bessel functions [13, p. 370]. As pointed out by Watson, a great number of theorems on Bessel functions can be considered as special cases of this expansion.

5. 2. Product of Bessel functions, second case. If $\alpha=\beta=0$, we have, using a theorem by Watson [1, p. 16],

$${}_3F_2\left[\begin{matrix}-m,\ -m-2\mu,\ \nu+\frac{1}{2}\\ -\mu-m+\frac{1}{2},\ 2\nu+1\end{matrix}\right]=\begin{cases}\dfrac{\left(\frac{1}{2}\right)_n(\mu+\nu+1+n)_n}{(\nu+1)_n\left(\mu+\frac{1}{2}+n\right)_n}\ , & \text{if } m=2n\\ 0\ , & \text{if } m=2n+1\ ,\end{cases}$$

$$n=0,1,2,\cdots\ ,$$

and thus by (11), after dividing by a numerical factor and replacing ρ by 2ρ,

$$\left(\rho\frac{1+\tau}{2}\right)^{-\mu}I_\mu\left(\rho\frac{1+\tau}{2}\right)\cdot\left(\rho\frac{1-\tau}{2}\right)^{-\nu}I_\nu\left(\rho\frac{1-\tau}{2}\right)$$

$$=\frac{\sqrt{2}\,\Gamma\left(\mu+\nu+\frac{1}{2}\right)}{\Gamma(\mu+1)\Gamma(\nu+1)}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\mu+\nu+\frac{1}{2}+2n\right)\left(\mu+\nu+\frac{1}{2}\right)_n}{(\mu+1)_n(\nu+1)_n}$$

$$\times P_{2n}^{(2\nu,2\mu)}(\tau)\rho^{-\mu-\nu-\frac{1}{2}}I_{\mu+\nu+\frac{1}{2}+2n}(\rho)\ .$$

This Neumann series for the product of two Bessel functions cannot be deduced from Bateman's expansion. The special case $\mu=\nu$ of it has been given by us already earlier [9, p. 333].

5. 3. Product of two Bessel functions, third case. Replacing in (38) ρ, α, β by $k\rho$, $1/4k$, $-1/4k$, respectively and letting $k\to 0$, we obtain in view of (14), writing again μ, ν instead of 2μ, 2ν,

(41)
$$\left(\rho\frac{1+\tau}{2}\right)^{-\frac{\mu}{2}} J_\mu\left(\sqrt{\rho\frac{1+\tau}{2}}\right)\cdot\left(\rho\frac{1-\tau}{2}\right)^{-\frac{\nu}{2}} I_\nu\left(\sqrt{\rho\frac{1-\tau}{2}}\right)$$
$$=2^{-\mu-\nu}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(\mu+n+1)\,\Gamma(\nu+n+1)}\left(\frac{\rho}{4}\right)^n P_n^{(\nu,\mu)}(\tau)\,.$$

Equivalent forms of this formula are well known in the theory of Bessel functions.[7]

5. 4. J.-W. expansion of a single Whittaker function. In the case $\beta=\nu+\frac{1}{2}$ the ${}_3F_2$ in (38) reduces to 1 (one of its numerator parameters being zero) and the second of the two M-functions on the left becomes an exponential function. Thus we have

(42)
$$e^{-\rho\frac{1-\tau}{4}}\left(\rho\frac{1+\tau}{2}\right)^{-\mu-\frac{1}{2}} M_{\alpha,\mu}\left(\rho\frac{1+\tau}{2}\right)$$
$$=\sum_{m=0}^{\infty}\frac{\left(\mu+\frac{1}{2}-\alpha\right)_m}{(2\mu+1)_m(2\mu+2\nu+1+m)_m}P_m^{(2\mu,\nu)}(\tau)\rho^{-\mu-\nu-1}M_{\mu+\nu+\frac{1}{2},\ \mu+\nu+\frac{1}{2}+m}(\rho)\ .$$

An expansion which is equivalent to this one is listed by Buchholz [**6**, p. 130], who gives credit for it to Erdélyi. Buchholz also indicates various special cases of the expansion.[8]

5. 5. Product of two Whittaker functions, Remanujan's case. Another case in which the function ${}_3F_2$ can be summed elementarily is given by the conditions $\alpha=\beta$, $\mu=\nu$. Then we have, upon application of a theorem by Dixon [**1**, p. 13]

$${}_3F_2\left[\begin{matrix}-2\mu-m,\ \mu+\frac{1}{2}-\alpha,\ -m;\\ -\mu+\alpha-m+\frac{1}{2},\ 2\mu+1\end{matrix}\right]=\begin{cases}\dfrac{(2n)!\left(\mu+\frac{1}{2}-\alpha\right)_n\left(\mu+\frac{1}{2}+\alpha\right)_n}{n!\left(\mu+\frac{1}{2}-\alpha\right)_{2n}(2\mu+1)_n}, & \text{if } m=2n,\\ 0\,, & \text{if } m=2n+1,\end{cases}$$
$$n=0,1,2,\cdots,$$

furthermore [**11**, p. 80]

$$P_n^{(2\mu,2\mu)}(\tau)=\frac{(2\mu+1)_n}{(4\mu+1)_n}\,C_n^{2\mu+\frac{1}{2}}(\tau)\,,$$

[7] See, for example, [**13**, p. 148]; or for the special case $\mu=\nu=0$ also [**12**, p. 2].

[8] The case where the M-functions in the summation reduce to Bessel functions has (with $\tau=1$) been rediscovered recently by Slater [**16**].

where $C_n^{2\mu+\frac{1}{2}}$ denotes the Gegenbauer polynomial. Thus (38) becomes[9]

$$\left(\rho^2\frac{1-\tau^2}{4}\right)^{-\mu-\frac{1}{2}}M_{\alpha,\mu}\left(\rho\frac{1+\tau}{2}\right)M_{\alpha,\mu}\left(\rho\frac{1-\tau}{2}\right)$$

$$(43)\qquad =\sum_{n=0}^{\infty}\frac{(2n)!\left(\mu+\frac{1}{2}-\alpha\right)_n\left(\mu+\frac{1}{2}+\alpha\right)_n}{n!(2\mu+1)_n(4\mu+1)_{4n}}C_{2n}^{2\mu+\frac{1}{2}}(\tau)\rho^{-2\mu-1}M_{2\alpha,2\mu+\frac{1}{2}+2n}(\rho)\ .$$

For $\tau=0$ we obtain in view of

$$C_{2n}^{2\mu+\frac{1}{2}}(0)=(-)^n\frac{(2\mu+\frac{1}{2})_n}{n!}$$

after multiplying by $(\rho/2)^{2\mu+1}$ and replacing ρ by 2ρ the series

$$[M_{\alpha,\mu}(\rho)]^2$$

$$(44)\qquad =2^{-2\mu-1}\sum_{m=0}^{\infty}\frac{(-)^n(2n)!\left(\mu+\frac{1}{2}-\alpha\right)_n\left(\mu+\frac{1}{2}+\alpha\right)_n\left(2\mu+\frac{1}{2}\right)_n}{(n!)^2(2\mu+1)_n(4\mu+1)_{4n}}M_{2\alpha,2\mu+\frac{1}{2}+2n}(2\rho)$$

which expresses the square of an M-function as a series of M-functions in which the first index and the argument are duplicated. Expressing ρ and τ on both sides of (43) by z and z^* and putting $z^*=0$, we have in view of Lemma 2, using (37) on the left.

$$ {}_1F_1\left[\begin{matrix}\mu+\frac{1}{2}-\alpha;\ \frac{z^2}{4}\\ 2\mu+1\end{matrix}\right]{}_1F_1\left[\begin{matrix}\mu+\frac{1}{2}-\alpha;\ -\frac{z^2}{4}\\ 2\mu+1\end{matrix}\right]$$

$$(45)\qquad =\sum_{n=0}^{\infty}\frac{\left(\mu+\frac{1}{2}-\alpha\right)_n\left(\mu+\frac{1}{2}+\alpha\right)_n}{(2\mu+1)_{2n}(2\mu+1)_n n!}\left(\frac{z^2}{4}\right)^{2n}$$

$$=\ {}_2F_3\left[\begin{matrix}\mu+\frac{1}{2}-\alpha,\ \mu+\frac{1}{2}+\alpha;\ \frac{z^4}{64}\\ \mu+\frac{1}{2},\ \mu+1,\ 2\mu+1\end{matrix}\right].$$

This result was already found by Ramanujan [**1**, p. 97].

5. 6. Generalisation of Erdély's integral. Assuming $\Re\mu>-\frac{1}{2}$, $\Re\nu>-\frac{1}{2}$, multiplying (38) by

[9] The special case $\mu=\alpha=\frac{1}{4}$ of this formula has been given in a different notation by Rainville [**15**]. (The M-functions on the left can then be expressed in terms of the error function.) Some misprints in [**15**] are pointed out in [**14**].

$$(\rho/2)^{\mu+\nu+1}(1-\tau)^{2\nu}(1+\tau)^{2\mu}P_n^{(2\nu,\,2\mu)}(\tau)\ ,$$

where n is a fixed nonnegative integer and integrating with respect to τ from -1 to $+1$ we obtain in view of the well-known orthogonality properties of the Jacobi polynomials [11, p. 67]

$$\int_{-1}^{1}(1+\tau)^{\mu-\frac{1}{2}}M_{\alpha,\,\mu}\left[\rho\,\frac{1+\tau}{2}\right]\cdot(1-\tau)^{\nu-\frac{1}{2}}M_{\beta,\,\nu}\left[\rho\,\frac{1-\tau}{2}\right]P_n^{(2\nu,\,2\mu)}(\tau)d\tau$$

$$(46)\qquad =\frac{2^{\mu+\nu}\,\Gamma(2\nu+n+1)\,\Gamma(2\mu+1)\left(\mu+\frac{1}{2}-\alpha\right)_n}{n!\,\Gamma(2\mu+2\nu+2n+2)}$$

$$\times\,{}_3F_2\left[\begin{matrix}-2\mu-n,\ \nu+\frac{1}{2}-\beta,\ -n;\\ -\mu+\alpha-n+\frac{1}{2},\ 2\nu+1\end{matrix}\right]\cdot M_{\alpha+\beta,\mu+\nu+\frac{1}{2}+n}(\rho)\ .$$

For $n=0$ this reduces to a formula equivalent to a well-known result due to Erdélyi [8, p. 134]; see also [6, p. 128]. It is then most easily proved by means of the Laplace transformation.

5. 7. Neumann series for the product of two M-functions. We mention finally that the special case obtained by putting $\beta=-\alpha$, $\mu=\nu$ has been given by us already earlier (See [9, p. 329], and [10, p. 270], where also some special cases are discussed). In this case (and also in the more general case $\mu\neq\nu$) the M-functions on the right of (38) reduce to Bessel functions, without this being the case for the M-functions on the left.

6. Applications of the Expansion Theorem: Jacobian Elliptic Coordinates

Other particular solutions of (1) can be found by introducing in (4) or (16) new variables ξ and η defined by

$$(47)\qquad \xi=\tilde{\omega}+\rho\ ,\qquad \eta=\tilde{\omega}-\rho\ ,$$

where

$$(48)\qquad \tilde{\omega}=\sqrt{(\alpha-z^2)(\alpha-z^{*2})}=\sqrt{\alpha^2-2\alpha\rho\tau+\rho^2}\ ,$$

with some real constant α, the square roots being positive for $z=z^*=0$.[10]

[10] These coordinates can be shown to be a special case of the general ellipsoidal coordinates, as investigated by Jacobi. Their use is also suggested by the structure of the generating function of the Jacobi polynomials.

By elementary computations one finds that if $U(z, z^*)=W(\zeta, \eta)$, (16) is transformed into

$$(\xi^2-\alpha^2)\frac{\partial^2 W}{\partial \xi^2}-(\eta^2-\alpha^2)\frac{\partial^2 W}{\partial \eta^2}+2[(1+\mu+\nu)\xi-(\mu-\nu)\alpha]\frac{\partial W}{\partial \xi}$$

$$-2[(1+\mu+\nu)\eta+(\mu-\nu)\alpha]\frac{\partial W}{\partial \eta}+k\left[(\xi+\eta)\frac{\lambda}{2}-(\xi^2-\eta^2)\frac{k}{16}\right]W=0 . \tag{49}$$

This equation can again be separated. One finds by the usual method that if $W(\xi, \eta)=\Xi(\xi)H(\eta)$ is a solution, Ξ and H have to satisfy separately the two ordinary differential equations

$$\begin{aligned}(\xi^2-\alpha^2)\frac{d^2\Xi}{d\xi^2}+2[(1+\mu+\nu)\xi-(\mu-\nu)\alpha]\frac{d\Xi}{d\xi}+k\left[p+\frac{\lambda}{2}\xi-\frac{k}{16}\xi^2\right]\Xi=0 ,\\ (\eta^2-\alpha^2)\frac{d^2H}{d\eta^2}+2[(1+\mu+\nu)\eta+(\mu-\nu)\alpha]\frac{dH}{d\eta}+k\left[p-\frac{\lambda}{2}\eta-\frac{k}{16}\eta^2\right]H=0 ,\end{aligned} \tag{50}$$

where p is a separation parameter. In order to obtain solutions of these equations in terms of known functions, it seems necessary to simplify them by assigning special values to some of the parameters. Two such simplifications will be indicated below, one of them leading again to Whittaker functions, the other to ordinary hypergeometric functions.

6. 1. Addition theorem for Whittaker functions. If in (50) we set $p=0, \nu=-\frac{1}{2}, \lambda=\frac{\alpha k}{8}$, the first equation becomes divisible by $\xi-\alpha$ and the second by $\eta+\alpha$. Cancelling these factors and setting (without essential loss of generality) $k=1$, we obtain the two equations

$$\left.\begin{aligned}(\xi+\alpha)\frac{d^2\Xi}{d\xi^2}+(2\mu+1)\frac{d\Xi}{d\xi}-\frac{\xi}{16}\Xi=0 ,\\ (\eta-\alpha)\frac{d^2H}{d\eta^2}+(2\mu+1)\frac{dH}{d\eta}-\frac{\eta}{16}H=0 ,\end{aligned}\right\} \tag{51}$$

which again can be easily reduced to Whittaker's equation. Carrying out the reduction one finds that solutions which as functions of z and z^* are regular near $z=z^*=0$, that is regular near $\xi=\eta=\alpha$, are given by

$$\left.\begin{aligned}\Xi(\xi)=\left(\frac{\alpha+\xi}{2}\right)^{-\mu-\frac{1}{2}}M_{\frac{\alpha}{8},\mu}\left(\frac{\alpha+\xi}{2}\right) ,\\ H(\eta)=\left(\frac{\alpha-\eta}{2}\right)^{-\mu-\frac{1}{2}}M_{\frac{\alpha}{8},\mu}\left(\frac{\alpha-\eta}{2}\right) ,\end{aligned}\right\} \tag{52}$$

and that, as long as 2μ is not a negative integer, the product of these two functions belongs to $E_{\mathcal{K}_\alpha \times \mathcal{K}_\alpha}$. Since for $z^*=0$

$$\xi=\eta=\sqrt{\alpha^2-\alpha z^2}\ ,$$

the generating function of the J.-W. expansion of $W(\xi, \eta)=\Xi(\xi)\, H(\eta)$ is

$$(53)\qquad f(z)=\left(\frac{\alpha z^2}{4}\right)^{-\mu-\frac{1}{2}} M_{\frac{\alpha}{8},\mu}\left(\frac{\alpha+\sqrt{\alpha^2-\alpha z^2}}{2}\right) M_{\frac{\alpha}{8},\mu}\left(\frac{\alpha-\sqrt{\alpha^2-\alpha z^2}}{2}\right).$$

If we write

$$(54)\qquad f(z)=\sum_{m=0}^{\infty} A_m^{(\mu)}(\alpha) z^{2m}\ ,$$

the required J.-W. expansion, valid in $\mathcal{K}_\alpha \times \mathcal{K}_\alpha$ and provided 2μ is not a negative integer, is

$$(55)\qquad \begin{aligned} &\left(\alpha\rho\,\frac{1+\tau}{2}\right)^{-\mu-\frac{1}{2}} M_{\frac{\alpha}{8},\mu}\left(\frac{\alpha+\rho+\tilde{\omega}}{2}\right) M_{\frac{\alpha}{8},\mu}\left(\frac{\alpha+\rho-\tilde{\omega}}{2}\right) \\ &\qquad =\sum_{m=0}^{\infty} A_m^{(\mu)}(\alpha)\frac{2^{2m} m!}{(2\mu+m)_m} P_m^{(-1,2\mu)}(\tau)\rho^{-\mu-\frac{1}{2}} M_{\frac{\alpha}{8},\mu+m}(\rho)\ , \end{aligned}$$

where $\tilde{\omega}$ is given by (48). It does not seem possible to express the coefficients $A_m^{(\mu)}$ in any closed form. Using a result of the previous section it is however not difficult to derive for them a series whose general term is again a M-function and whose coefficients can be exhibited explicitly. If in (43) we replace ρ, τ, α by $\alpha, \sqrt{1-z^2/\alpha}, \alpha/8$ respectively we obtain on the left just (53) and have therefore

$$(56)\qquad \begin{aligned} &f(z) \\ &=\sum_{n=0}^{\infty}\frac{(2n)!\left(\mu+\frac{1}{2}-\frac{\alpha}{8}\right)_n\left(\mu+\frac{1}{2}+\frac{\alpha}{8}\right)_n}{n!(2\mu+1)_n(4\mu+1)_{4n}} C_{2n}^{2\mu+\frac{1}{2}}(\sqrt{1-z^2/\alpha})\ \alpha^{-2\mu-1} M_{\frac{\alpha}{4},2\mu+\frac{1}{2}+2m}(\alpha). \end{aligned}$$

Now, by Gauss' quadratic transformation,

$$\begin{aligned} C_{2n}^{2\mu+\frac{1}{2}}(\sqrt{1-z^2/\alpha}\,) &=\frac{(4\mu+1)_{2n}}{(2n)!}\,{}_2F_1\left[\begin{matrix}-2n,\ 4\mu+1+2n; \dfrac{1-\sqrt{1-z^2/\alpha}}{2}\\ 2\mu+1\end{matrix}\right] \\ &=\frac{(4\mu+1)_{2n}}{(2n)!}\,{}_2F_1\left[\begin{matrix}-n, 2\mu+\frac{1}{2}+n; z^2/\alpha\\ 2\mu+1\end{matrix}\right] \end{aligned}$$

Inserting this in (56) and rearranging the series by collecting equal powers of z (which is permissible in view of Weierstrass' theorem) we obtain in view of (54) the desired series representation

$$A_m^{(\mu)}(\alpha)=\frac{(-\alpha)^{-m}\left(2\mu+\frac{1}{2}\right)_n}{(2\mu+1)_m m!}\sum_{n=m}^{\infty}\frac{2^{2n}\left(\mu+\frac{1}{2}-\frac{\alpha}{8}\right)_n\left(\mu+\frac{1}{2}+\frac{\alpha}{8}\right)_n\left(2\mu+\frac{1}{2}+m\right)_n}{(n-m)!\ (4\mu+1)_{4n}} \tag{57}$$
$$\times\,\alpha^{-2\mu-1}M_{\frac{\alpha}{4},\,2\mu+\frac{1}{2}+2n}(\alpha)\,.$$

Since in virtue of this formula (56) expresses the product of the two functions

$$M_{\frac{\alpha}{8},\,\mu}\left(\frac{\alpha+\rho+\varepsilon\tilde{\omega}}{2}\right), \qquad \varepsilon=\pm1\,,$$

in terms of products of M-functions with the arguments ρ and α respectively, (55) may be looked at as an addition theorem for the functions on the left in analogy to a similar situation in the case of the well-known addition theorems of Graf and Gegenbauer in the theory of Bessel functions [13, p. 358].

For $\tau=-1$ we obtain from (55) the following addition theorem of a more elementary character:

$$(\alpha+\rho)^{-\mu-\frac{1}{2}}M_{\frac{\alpha}{8},\,\mu}(\alpha+\rho)=\sum_{m=0}^{\infty}\frac{(-)^m 2^{2m}(2\mu+1)_m}{(2\mu+m)_m}A_m^{(\mu)}(\alpha)\rho^{-\mu-\frac{1}{2}}M_{\frac{\alpha}{8},\,\mu+m}(\rho) \tag{58}$$

6. 2. Hypergeometric functions. Inserting in (51) the special values $p=q/k$, $\alpha=1$, and letting $k\to0$ yields the two differential equations

$$\begin{cases}(\xi^2-1)\dfrac{d^2\Xi}{d\xi^2}+2[(1+\mu+\nu)\xi-(\mu-\nu)]\dfrac{d\Xi}{d\xi}-q\Xi=0\,,\\[2ex](\eta^2-1)\dfrac{d^2H}{d\eta^2}+2[(1+\mu+\nu)\eta+(\mu-\nu)]\dfrac{dH}{d\eta}-qH=0\,,\end{cases} \tag{59}$$

which are of hypergeometric type. Using this fact it is readily proved by substitution that solutions of these equations regular near $z=z^*=0$ (that is regular near $\xi=\eta=1$) are given by

$$\begin{cases}\Xi(\xi)={}_2F_1\left[\begin{matrix}\mu+\nu+\frac{1}{2}-\gamma,\ \mu+\nu+\frac{1}{2}+\gamma;\ \frac{1-\xi}{2}\\ 2\nu+1\end{matrix}\right],\\[3ex] H(\eta)={}_2F_1\left[\begin{matrix}\mu+\nu+\frac{1}{2}-\gamma,\ \mu+\nu+\frac{1}{2}+\gamma,\ \frac{1-\eta}{2}\\ 2\mu+1\end{matrix}\right],\end{cases} \tag{60}$$

where

$$\gamma=\sqrt{(2\mu+2\nu+1)^2-q}$$

is arbitrary, since q was arbitrary. Writing $U(z, z^*)=\Xi(\xi)H(\eta)$ anb

$$U(z, 0)=\sum_{n=0}^{\infty} a_n z^{2n} , \tag{61}$$

we have, according to the expansion theorem,

$$U(z, z^*)=\sum_{n=0}^{\infty} a_n F_n^{(\mu,\nu)}(z, z^*; 0, 0) . \tag{62}$$

Since here the parameters λ and k are both zero, the coefficients a_n can be easily determined by putting $z=z^*$. From (47) one has in this case $\xi=1$, $\eta=1-2z^2$ and from (14), since now $\tau=1$, $\rho=z^2$,

$$F_n^{(\mu,\nu)}(z, z; 0, 0)=\frac{(2\mu+2\nu+1)_n(2\nu+1)_n}{\left(\mu+\nu+\frac{1}{2}\right)_n(\mu+\nu+1)_n} z^{2n} . \tag{63}$$

Thus (61) reduces to

$${}_2F_1\left[\begin{matrix}\mu+\nu+\frac{1}{2}-\gamma,\ \mu+\nu+\frac{1}{2}+\gamma;\ z^2\\ 2\mu+1\end{matrix}\right]=\sum_{n=0}^{\infty} a_n\frac{(2\mu+2\nu+1)_n(2\nu+1)_n}{\left(\mu+\nu+\frac{1}{2}\right)_n(\mu+\nu+1)_n} z^{2n} ,$$

which yields

$$a_n=\frac{\left(\mu+\nu+\frac{1}{2}-\gamma\right)_n\left(\mu+\nu+\frac{1}{2}+\gamma\right)_n\left(\mu+\nu+\frac{1}{2}\right)_n(\mu+\nu+1)_n}{(2\mu+1)_n(2\nu+1)_n(2\mu+2\mu+1)_n n!} . \tag{64}$$

Thus (62) may now be stated more explicitly as follows:

$$\begin{aligned}&{}_2F_1\left[\begin{matrix}\mu+\nu+\frac{1}{2}-\gamma,\ \mu+\nu+\frac{1}{2}+\gamma;\ \frac{1-\rho+\tilde{\omega}}{2}\\ 2\nu+1\end{matrix}\right]\\ &\times{}_2F_1\left[\begin{matrix}\mu+\nu+\frac{1}{2}-\gamma,\ \mu+\nu+\frac{1}{2}+\gamma;\ \frac{1+\rho-\tilde{\omega}}{2}\\ 2\mu+1\end{matrix}\right]\\ &=\sum_{n=0}^{\infty}\frac{\left(\mu+\nu+\frac{1}{2}-\gamma\right)_n\left(\mu+\nu+\frac{1}{2}+\gamma\right)_n}{(2\mu+1)_n(2\nu+1)_n}\rho^n P_n^{(2\nu, 2\mu)}(\tau) .\end{aligned} \tag{65}$$

A result equivalent to this was derived by Brafman[11] from Bailey's decomposition formula for a special case of Appell's hypergeometric function F_4 of two variables [1, p. 81.] Since it is also possible to derive Bailey's formula from (65) simply by replacing the Jacobi polynomial by its hypergeometric definition and inserting appropriate values of ρ and τ, our proof of (65) contains also a new proof of that formula.

Restating (61) with the explicit value of a_n given by (64) we obtain

$$\begin{aligned}
&{}_2F_1\left[\begin{matrix}\mu+\nu+\frac{1}{2}-\gamma,\ \mu+\nu+\frac{1}{2}+\gamma;\ \frac{1-\sqrt{1-z^2}}{2}\\ 2\mu+1\end{matrix}\right]\\
&\times{}_2F_1\left[\begin{matrix}\mu+\nu+\frac{1}{2}-\gamma,\ \mu+\nu+\frac{1}{2}+\gamma;\ \frac{1-\sqrt{1-z^2}}{2}\\ 2\nu+1\end{matrix}\right]\\
&={}_4F_3\left[\begin{matrix}\mu+\nu+\frac{1}{2}-\gamma,\ \mu+\nu+\frac{1}{2}+\gamma,\ \mu+\nu+\frac{1}{2},\ \mu+\nu+1;\ z^2\\ 2\mu+1,\ 2\nu+1,\ 2\mu+2\nu+1\end{matrix}\right],
\end{aligned}\tag{66}$$

which is equivalent to a result proved by Bailey [1, p. 88, formula (3)] by means of transformations of terminating generalized hypergeometric series.

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NATIONAL BUREAU OF STANDARDS, WASHINGTON, D.C.

THE SOLUTION OF CAUCHY'S PROBLEM FOR A THIRD-ORDER LINEAR HYPERBOLIC DIFFERENTIAL EQUATION BY MEANS OF RIESZ INTEGRALS

JOHN G. HERRIOT

1. Introduction. M. Riesz [3] solved Cauchy's problem for the wave equation by means of a generalization of the Riemann-Liouville integral and a consideration of Lorentz space. L. Gårding [1] solved Cauchy's problem for two linear hyperbolic differential equations arising from a consideration of spaces of symmetric and Hermitian matrices by means of similar generalizations of the Riemann-Liouville integral. Gårding [2] also proved some general results for the solution of Cauchy's problem for general linear hyperbolic partial differential equations with constant coefficients again using Riesz-type integrals.

In the present paper the explicit solution of Cauchy's problem for the third-order partial differential equation

$$\Delta u = h(x_1, x_2, x_3) \ , \tag{1.1}$$

where Δ denotes the operator $\partial^3/(\partial x_1\, \partial x_2\, \partial x_3)$, is given by means of a similar generalization of the Riemann-Liouville integral. We restrict our attention to the case in which u and its first and second derivatives are given on the plane S whose equation is $x_1+x_2+x_3=0$. We verify in detail that the solution given actually satisfies the differential equation (1.1), and also that it and its derivatives assume the proper values on S.

Before proceeding to a study of (1.1), we give a brief discussion of the Riemann-Liouville integral and Riesz's generalization of it. (We use mainly the notation of Gårding [1].) Let p be a complex variable, and consider the Riemann-Liouville integral

$$I^p f(x) = \frac{1}{\Gamma(p)} \int_a^x f(t)(x-t)^{p-1} dt \qquad (a \leq x < b \leq \infty) \ , \tag{1.2}$$

where $\mathscr{R}(p) > 0$,[1] and $f(x)$ is a continuous function when $a \leq x < b \leq \infty$. This integral diverges if $\mathscr{R}(p) \leq 0$. If p and q are such that $\mathscr{R}(p) > 0$, $\mathscr{R}(q) > 0$ we have

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[1] $\mathscr{R}(p)$ denotes the real part of p.

$$(1.3)\qquad I^p I^q f(x) = I^{p+q} f(x)$$

and

$$(1.4)\qquad \frac{d}{dx} I^{p+1} f(x) = I^p f(x)\ .$$

Clearly $I^p f(x)$ is an analytic function of p, regular for $\mathscr{R}(p) > 0$, and depending on the parameter x. It can, however, be continued analytically beyond this region provided that $f(x)$ has a sufficient number of continuous derivatives. Let us write

$$(1.5)\qquad f(t) = \sum_{j=0}^{k-1} \frac{f^{(j)}(x)(t-x)^j}{j\,!} + r(x, t, k)\ ,$$

so that $r(x, t, k)/(t-x)^k$ is bounded when $a < t < x$. Then on substituting in equation (1.2) we find that

$$(1.6)\qquad I^p f(x) = \frac{1}{\Gamma(p)} \int_a^x r(x, t, k)(x-t)^{p-1} dt + \sum_{j=0}^{k-1} \frac{(-1)^j f^{(j)}(x)(x-a)^{p+j} p(p+1)\cdots(p+j-1)}{j\,!\,\Gamma(p+j+1)}\ .$$

Here the integral converges for $\mathscr{R}(p) > -k$, and (1.6) provides an analytic continuation of $I^p f(x)$ for such values of p. In particular,

$$(1.7)\qquad I^{-j} f(x) = f^{(j)}(x) \qquad (j = 0, 1, 2, \cdots)\ .$$

By successive integrations by parts we can find another formula which is also useful for the analytic continuation of $I^p f(x)$. We have

$$(1.8)\qquad I^p f(x) = I^{p+m} f^{(m)}(x) + \sum_{j=0}^{m-1} \frac{f^{(j)}(a)(x-a)^j}{\Gamma(p+j+1)}$$

If we let $p \to 0$ we find that

$$(1.9)\qquad f(x) = I^m f^{(m)}(x) + \sum_{j=0}^{m-1} \frac{f^{(j)}(a)(x-a)^j}{j\,!}$$

The right member of (1.9) gives the solution of the differential equation

$$(1.10)\qquad \frac{d^m u(x)}{dx^m} = f^{(m)}(x)$$

whose derivatives of order less than m assume the values $f(a), \cdots, f^{(m-1)}(a)$ when $x = a$.

When generalizing (1.2), Riesz considers Lorentz space L with points $x = (x_1, x_2, \cdots, x_n)$. The square of the distance of $x = (x_1, x_2, \cdots, x_n)$ from

$\xi=(\xi_1, \xi_2, \cdots, \xi_n)$ is

$$r(x-\xi)=(x_1-\xi_1)^2-(x_2-\xi_2)^2-\cdots-(x_n-\xi_n)^2 .$$

The interior of the light cone with its vertex at a fixed point x is characterized by $r(x-\xi)>0$ where ξ is variable. It consists of two parts, the direct and the retrograde cone, characterized by

$$r(x-\xi)>0 , \quad \xi_1-x_1>0 \quad \text{and} \quad r(x-\xi)>0 , \quad \xi_1-x_1<0 ,$$

respectively. It is the retrograde cone denoted by $D(x)$ which is mainly considered by Riesz. The domain of integration used is the bounded domain $D_S(x)$ limited by the nappe $C(x)$ of the retrograde cone $D(x)$ and a certain sufficiently regular surface S having the property that every straight line in L with a direction of nonnegative square length meets S in at most one point. The volume element in L is $d\xi=d\xi_1 d\xi_2\cdots d\xi_n$. Let $f(x)=f(x_1, x_2, \cdots, x_n)$ de a real function defined in the region consisting of all points x whose retrograde cones $D(x)$ intersect S. Then Riesz's generalization of (1.2) is[2]

$$I^p f(x)=\frac{1}{H_n(p)}\int_{D_S(x)} f(\xi)[r(x-\xi)]^{p-(1/2)n}d\xi , \tag{1.11}$$

with

$$H_n(p)=2^{2p-1}\left[\Gamma\left(\frac{1}{2}\right)\right]^{n-2}\Gamma(p)\Gamma\left(p-\frac{n-2}{2}\right) .$$

If $f(x)$ is bounded, the integral is a regular analytic function of p for $\mathscr{R}(p)>(n-2)/2$. It can be shown that (1.3) is valid and, corresponding to (1.4),

$$\Delta_W I^{p+1} f(x)=I^p f(x) , \tag{1.12}$$

where Δ_W is the wave operator

$$(\partial/\partial x_1)^2-(\partial/\partial x_2)^2-\cdots-(\partial/\partial x_n)^2 .$$

If $f(x)$ has derivatives of sufficiently high order it is possible to continue $I^p f(x)$ beyond the region in which the integral converges. The generalizations of (1.7) are found to be

$$I^0 f(x)=f(x) , \quad I^{-j} f(x)=\Delta^j f(x) \qquad (j=1, 2, 3, \cdots) . \tag{1.13}$$

By means of Green's formula it is found that

[2] To get uniform notations in this paper, as in Gårding [**1**], Riesz's variable α is replaced by $2p$ here.

(1.14) $I^{p}f(x)=I^{p+1}\Delta f(x)$

$$+\frac{1}{H_n(p+1)}\int_{S(x)}\left\{\frac{df(\xi)}{d\nu}[r(x-\xi)]^{p+1-\frac{1}{2}n}-f(\xi)\frac{d[r(x-\xi)]^{p+1-\frac{1}{2}n}}{d\nu}\right\}dS,$$

where $S(x)$ is the portion of S interior to the cone $D(x)$, $d/d\nu$ is taken in the direction of the Lorentzian normal to the surface S, and dS is the Lorentzian element of surface area.

If we let $p\to 0$ in (1.14), the right side gives the solution of the differential equation

$$\Delta_W u(x)=h(x) , \tag{1.15}$$

$u(x)$ and its (Lorentzian) normal derivative being given on S.

In the present paper we consider three-dimensional Euclidean space with points $x=(x_1, x_2, x_3)$. In this case the retrograde light cone $D(x)$ with its vertex at a fixed point x is characterized by $x_1-\xi_1>0$, $x_2-\xi_2>0$, $x_3-\xi_3>0$, where $\xi=(\xi_1, \xi_2, \xi_3)$ is variable. We denote by S the plane $\xi_1+\xi_2+\xi_3=0$. The domain of integration used is the bounded domain $D_S(x)$ limited by the boundary of $D(x)$ and the plane S. Then our generalization of (1.2) is

$$I^{p}f(x)=\frac{1}{[\Gamma(p)]^3}\iiint_{D_S(x)}f(\xi)[r(x-\xi)]^{p-1}d\xi , \tag{1.16}$$

where $r(x-\xi)=(x_1-\xi_1)(x_2-\xi_2)(x_3-\xi_3)$ and $d\xi=d\xi_1 d\xi_2 d\xi_3$. If $f(x)$ is bounded, the integral is a regular analytic function of p for $\mathscr{R}(p)>0$. We show that (1.3) is valid and, corresponding to (1.4),

$$\Delta I^{p+1}f(x)=I^{p}f(x) . \tag{1.17}$$

As before, $I^{p}f(x)$ can be continued analytically if $f(x)$ is sufficiently differentiable. The generalizations of (1.7) which we prove are

$$I^{0}f(x)=f(x) , \quad I^{-1}f(x)=\Delta f(x) . \tag{1.18}$$

In § 3 we apply Green's formula to discover a formula similar to (1.14), namely,

$$I^{p}f(x)=I^{p+1}\Delta f(x)+I_{*}^{p+1}f(x) , \tag{1.19}$$

where $I_{*}^{p+1}f(x)$ is an integral over $S(x)$, the portion of S interior to $D(x)$, involving f and its first and second derivatives. If we let $p\to 0$ in (1.19), we obtain the solution of Cauchy's problem for the equation (1.1). The verification of the solution is carried out in § 5 making use of a series of lemmas developed in § 4.

The methods of this paper can be applied to the solution of the nth order partial differential equation

$$\frac{\partial^n u}{\partial x_1 \partial x_2 \cdots \partial x_n} = h(x_1, x_2, \cdots, x_n) .$$

However, the formulas required are very cumbersome to write and for this reason the present discussion has been limited to equations of third order.

2. Generalization of the Riemann-Liouville integral. Since we wish to consider the differential equation

$$\Delta u \equiv \partial^3 u/(\partial x_1 \partial x_2 \partial x_3) = h(x_1, x_2, x_3) , \tag{2.1}$$

the appropriate formula for the cube of the distance between points $x=(x_1, x_2, x_3)$ and $\xi=(\xi_1, \xi_2, \xi_3)$ is

$$r(x-\xi) = (x_1-\xi_1)(x_2-\xi_2)(x_3-\xi_3) . \tag{2.2}$$

The retrograde light cone $D(x)$ with vertex at a fixed point x is characterized by $x_1-\xi_1>0$, $x_2-\xi_2>0$, $x_3-\xi_3>0$, where ξ is variable. We do not make any use of the geometry of the space based on this distance formula but in finding volume elements and surface elements we regard the space as ordinary three-dimensional Euclidean space. It is only in determining the proper generalizations of the Riemann-Liouville integral that (2.2) plays a role. We first consider an integral extended over the whole of $D(x)$. We suppose $f(x)$ defined in a region such that if this region contains a certain point x it contains also the retrograde cone $D(x)$. In order to assure the absolute convergence of the integral considered we suppose among other things that $f(x)$ tends toward zero sufficiently rapidly when $x_1, x_2, x_3 \to -\infty$. We then define, for complex values of p such that $\mathscr{R}(p)>0$,

$$I^p f(x) = \frac{1}{H_3(p)} \iiint_{D(x)} f(\xi)[r(x-\xi)]^{p-1} d\xi . \tag{2.3}$$

We should like to have

$$\Delta I^{p+1} f(x) = I^p f(x) \tag{2.4}$$

and

$$I^p I^q f(x) = I^{p+q} f(x) . \tag{2.5}$$

In order to find the correct form of $H_3(p)$ to accomplish this we consider the particular function

$$f_1(x) = \exp(x_1+x_2+x_3) .$$

Clearly $\Delta f_1(x) = f_1(x)$, so we should have $I^p f_1(x) = f_1(x)$. Introducing this function into (2.3) we easily find that we should choose $H_3(p) = [\Gamma(p)]^3$.

With this choice of $H_3(p)$, it is easy to verify that (2.4) holds by merely carrying out the necessary differentiations. We proceed to verify that also (2.5) holds with this choice of $H_3(p)$. After interchanging the order of integration we find that

(2.6)
$$I^p I^q f(x) = \frac{1}{[\Gamma(p)\Gamma(q)]^3} \iiint_{D(x)} f(\eta)\, d\eta \int_{\eta_1}^{x_1}\int_{\eta_2}^{x_2}\int_{\eta_3}^{x_3} [r(\xi-\eta)]^{q-1}[r(x-\xi)]^{p-1} d\xi \ .$$

If we make use of the well-known formulas

$$\int_a^b (\xi-a)^{\alpha-1}(b-\xi)^{\beta-1} d\xi = (b-a)^{\alpha+\beta-1} B(\alpha, \beta) \tag{2.7}$$

and

$$B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta) \ , \tag{2.8}$$

we find that the right member of (2.6) reduces to $I^{p+q}f(x)$. Thus (2.5) is established.

In the applications to follow, the domain $D(x)$ will be replaced by a bounded domain $D_S(x)$ which is limited by the boundary of the retrograde cone $D(x)$ and by the plane S whose equation is $\xi_1+\xi_2+\xi_3=0$. We shall therefore in all that follows use the following definition of $I^p f(x)$:

$$I^p f(x) = \frac{1}{[\Gamma(p)]^3} \iiint_{D_S(x)} f(\xi)[r(x-\xi)]^{p-1} d\xi \ . \tag{2.9}$$

Since this is the same as (2.3) if only we assume that $f(\xi)=0$ when $\xi_1+\xi_2+\xi_3<0$, it is clear that the relations (2.4) and (2.5) hold also when $I^p f(x)$ is defined by (2.9).

In the application of (2.9) to the solution of Cauchy's problem we shall be concerned with the limit of $I^p f(x)$ as $p \to 0$. We therefore prove:

THEOREM 2.1. *If $f(x)$ is continuous in the region $x_1+x_2+x_3 \geq 0$ then $I^p f(x)$ defined by* (2.9) *is a regular analytic function of p for $\mathscr{R}(p)>0$, and*

$$\lim_{p\to 0} I^p f(x) = f(x) \tag{2.10}$$

in the region $x_1+x_2+x_3>0$.

Proof. That $I^p f(x)$ is analytic when $\mathscr{R}(p)>0$ follows at once from its definition by equation (2.9).

In order to prove (2.10) we make a change of variables by writing, in (2.9),

$$x_1-\xi_1=d\sigma\cos^2\theta_1\ ,\quad x_2-\xi_2=d\sigma\sin^2\theta_1\cos^2\theta_2\ ,\quad x_3-\xi_3=d\sigma\sin^2\theta_1\sin^2\theta_2\ ,$$

where $d=x_1+x_2+x_3>0$. If we also make use of (2.8) and the well-known formula

$$B(\alpha,\beta)=2\int_0^{\pi/2}\sin^{2\alpha-1}\theta\cos^{2\beta-1}\theta\,d\theta\ , \tag{2.11}$$

we find that

$$I^pf(x)-\frac{d^{3p}}{\Gamma(3p+1)}f(x)=\frac{d^{3p}}{[\Gamma(p)]^3}\int_0^1\int_0^{\pi/2}\int_0^{\pi/2}[F(\sigma,\theta_1,\theta_2)-F(0,\theta_1,\theta_2)] \cdot 4\sigma^{3p-1}\sin^{4p-1}\theta_1\cos^{2p-1}\theta_1\sin^{2p-1}\theta_2\cos^{2p-1}\theta_2 d\theta_2 d\theta_1 d\sigma\ , \tag{2.12}$$

where

$$F(\sigma,\theta_1,\theta_2)=f(x_1-d\sigma\cos^2\theta_1,\ x_2-d\sigma\sin^2\theta_1\cos^2\theta_2,\ x_3-d\sigma\sin^2\theta_1\sin^2\theta_2)\ .$$

But since $f(x)$ is continuous, if $\varepsilon>0$ is assigned we can find a δ such that $0<\delta<1$ and such that $|F(\sigma,\theta_1,\theta_2)-F(0,\theta_1,\theta_2)|<\varepsilon$ when $0<\sigma<\delta$, uniformly in θ_1 and θ_2. We now break the integral in (2.12) into two parts J_1 and J_2; in J_1, σ goes from 0 to δ, and in J_2 from δ to 1, while θ_1 and θ_2 assume all values between 0 and $\pi/2$ in both J_1 and J_2. We see at once that

$$|J_1|<\varepsilon d^{3p}/\Gamma(3p+1)\ .$$

If M is the maximum of $F(\sigma,\theta_1,\theta_2)$ in the region of integration, an easy calculation shows that

$$|J_2|\leq 2d^{3p}M\delta^{3p-1}/\Gamma(3p)$$

if $0<p<1/3$. By choosing p sufficiently close to zero, we can make J_2 arbitrarily small, and it follows that

$$\lim_{p\to 0}\,[I^pf(x)-\frac{d^{3p}}{\Gamma(3p+1)}f(x)]=0\ .$$

Equation (2.10) follows at once from this since $d^{3p}/\Gamma(3p+1)\to 1$ as $p\to 0$.

3. Green's formula for $I^pf(x)$. We shall find it convenient to make use of the function

$$v=v(x,\xi)=\frac{[r(x-\xi)]^p}{[\Gamma(p+1)]^3}=\frac{[(x_1-\xi_1)(x_2-\xi_2)(x_3-\xi_3)]^p}{[\Gamma(p+1)]^3}\ . \tag{3.1}$$

We wish to transform the volume integral

(3.2) $$\iiint_{D_S(x)} (f\Delta_\xi v + v\Delta_\xi f)d\xi$$

into a surface integral. Here Δ_ξ denotes the operator Δ with respect to the variable ξ.

The function to be integrated must first be transformed into the form of a divergence. We easily find that

$$f\Delta_\xi v = (fv_{\xi_1\xi_2})_{\xi_3} - (f_{\xi_3}v_{\xi_1})_{\xi_2} + (f_{\xi_3\xi_2}v)_{\xi_1} - v\Delta_\xi f \ .$$

By permutation of ξ_1, ξ_2, ξ_3 we obtain altogether a total of 3! such equations. The left member and the last term of the right member are unaltered by such permutations. Adding these 3! equations and dividing by 3! we obtain

(3.3) $$\begin{aligned} f\Delta_\xi v + v\Delta_\xi f = & \left[\frac{1}{3}(fv_{\xi_2\xi_3} + vf_{\xi_2\xi_3}) - \frac{1}{6}(f_{\xi_2}v_{\xi_3} + v_{\xi_2}f_{\xi_3})\right]_{\xi_1} \\ & + \left[\frac{1}{3}(fv_{\xi_3\xi_1} + vf_{\xi_3\xi_1}) - \frac{1}{6}(f_{\xi_3}v_{\xi_1} + v_{\xi_3}f_{\xi_1})\right]_{\xi_2} \\ & + \left[\frac{1}{3}(fv_{\xi_1\xi_2} + vf_{\xi_1\xi_2}) - \frac{1}{6}(f_{\xi_1}v_{\xi_2} + v_{\xi_1}f_{\xi_2})\right]_{\xi_3} . \end{aligned}$$

We note that if $\mathscr{R}(p) > 0$, v vanishes on the boundary of the retrograde cone $D(x)$, v_{ξ_i} vanishes for $\xi_j = x_j$, $(j \neq i)$, and $v_{\xi_i\xi_j}$ vanishes for $\xi_k = x_k$, $(k \neq i,\ k \neq j,\ i \neq j)$.

Applying the divergence theorem and noting that

$$\Delta_\xi v = -[r(x-\xi)]^{p-1}/[\Gamma(p)]^3 \ ,$$

we obtain

(3.4) $$\begin{aligned} I^p f(x) = & I^{p+1}\Delta f(x) \\ & + \frac{1}{\sqrt{3}}\iint_{S(x)} \left\{ \frac{1}{3}\left[f(v_{\xi_2\xi_3} + v_{\xi_3\xi_1} + v_{\xi_1\xi_2}) + v(f_{\xi_2\xi_3} + f_{\xi_3\xi_1} + f_{\xi_1\xi_2})\right] \right. \\ & \left. - \frac{1}{6}\left[(f_{\xi_2} + f_{\xi_3})v_{\xi_1} + (f_{\xi_3} + f_{\xi_1})v_{\xi_2} + (f_{\xi_1} + f_{\xi_2})v_{\xi_3}\right]\right\} dS \ , \end{aligned}$$

where $S(x)$ is the portion of S included in the retrograde cone $D(x)$, and dS is the surface area element on $S(x)$. If $f(x)$ is continuous, then by Theorem 2.1 the left member of (3.4) becomes $f(x)$ when we let $p \to 0$. If $\Delta f(x)$ is given in $D_S(x)$, and f together with its first and second derivatives are given on S, then the right member of (3.4) can be calculated. We are going to show that it yields the solution of Cauchy's problem for the differential equation $\Delta u = h(x)$.

It is clear that if u and its first and second derivatives are prescribed on S, then these derivatives cannot be prescribed arbitrarily but certain relations exist between u and its derivatives. Only a complete independent set can be prescribed arbitrarily on S. For example, one may prescribe u and its first and second normal derivatives on S, or one may prescribe u, u_{ξ_1}, and $u_{\xi_1\xi_1}$ on S. It is easily shown that it is always possible to determine a function $g(\xi_1, \xi_2, \xi_3)$ which agrees with u on S and whose derivatives agree with the corresponding derivatives of u on S. This being the case, it is reasonable to introduce the following definition:

$$(3.5)\quad I_*^{p+1}f(x)=\frac{1}{\sqrt{3}}\iint_{S(x)}\left\{\frac{1}{3}[f(v_{\xi_2\xi_3}+v_{\xi_3\xi_1}+v_{\xi_1\xi_2})+v(f_{\xi_2\xi_3}+f_{\xi_3\xi_1}+f_{\xi_1\xi_2})]\right.$$
$$\left.-\frac{1}{6}[(f_{\xi_2}+f_{\xi_3})v_{\xi_1}+(f_{\xi_3}+f_{\xi_1})v_{\xi_2}+(f_{\xi_1}+f_{\xi_2})v_{\xi_3}]\right\}dS\,,$$

where v is defined by (3.1). We can then write (3.4) in the form

$$(3.6)\qquad I^p f(x)=I^{p+1}\Delta f(x)+I_*^{p+1}f(x)\,.$$

If we are to solve the differential equation $\Delta u=h(x)$ subject to the conditions that u and its first and second derivatives agree with g and its corresponding derivatives on S, then according to (3.6) and Theorem 2.1 we must have

$$(3.7)\qquad u(x)=I^1h(x)+\lim_{p\to 0} I_*^{p+1}g(x)$$

as the solution. We write the limit as $p\to 0$ in the second term on the right because some of the integrals fail to exist if $p=0$.

4. Lemmas for the evaluation of the surface integrals. The surface integral in (3.5) which is required for the solution of Cauchy's problem converges for $\mathscr{R}(p)>0$. In order to find the solution of Cauchy's problem according to equation (3.7) we need to show that the limit of $I_*^{p+1}g(x)$ exists when $p\to 0$. To verify that u and its derivatives assume the prescribed values on S it is necessary to differentiate (3.7). This is trivial for the first term on the right but not so simple for the second term. But if $\mathscr{R}(p)$ is sufficiently large the differentiation of $I_*^{p+1}g(x)$ is very easy. The resulting integrals fail to exist near $p=0$, and an analytic continuation is required. We wish to show how this analytic continuation can be accomplished and that instead of differentiating the second term on the right of (3.7) after letting $p\to 0$ we can differentiate $I_*^{p+1}(g)$ first and then let $p\to 0$. We, of course, make suitable assumptions concerning the differentiability of g.

We note that all of the integrals occurring in (3.5) are of the form

$$J^{\alpha,\beta,\gamma}f(x) \tag{4.1}$$
$$=\frac{1}{\sqrt{3}\,\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\iint_{S(x)} f(\xi_1,\xi_2,\xi_3)(x_1-\xi_1)^{\alpha-1}(x_2-\xi_2)^{\beta-1}(x_3-\xi_3)^{\gamma-1}dS\ ,$$

where we assume that $f(\xi_1, \xi_2, \xi_3)$ possesses continuous derivatives up to the first, second, or third order. We note that the integral in (4.1) converges when the real parts of α, β, and γ are greater than zero. We proceed to a study of this integral, proving a number of lemmas some of which are of interest in themselves.

The first lemma which we need is similar to one given by Riesz [3, p. 60].

LEMMA 4.1. *Let $G(u,v)$ be a function defined for $0\leq u\leq a<\infty$, $0\leq v<b\leq\infty$, and let it have continuous derivatives to the qth order. Then it may be written in the form*

$$G(u,v)=\pi(u,v)+\sum_{r=0}^{q-2}h_r(v)\frac{u^r}{r!}+k_0(u)+m(u,v)\ , \tag{4.2}$$

where

$$\pi(u,v)=\sum_{r=0}^{q-1}\sum_{s=0}^{q-r-1}\frac{G^{(r,s)}(0,0)}{r!\,s!}u^rv^s \tag{4.3}$$

and

$$h_r(v)=O(v^{q-r})\ ,\quad k_0(u)=O(u^q)\ ,\quad m(u,v)=O(u^{q-1}v)\ . \tag{4.4}$$

Here $G^{(r,s)}(u,v)=\partial^{r+s}G(u,v)/(\partial u^r\partial v^s)$.

Proof. If $G(u,v)$ could be expanded in a Maclaurin's series for sufficiently small u and v, the result would be obvious. Since we do not assume this we proceed as Riesz does. We write

$$h_r(v)=G^{(r,0)}(0,v)-\sum_{s=0}^{q-r-1}G^{(r,s)}(0,0)\frac{v^s}{s!}$$
$$=\frac{1}{(q-r-1)!}\int_0^v G^{(r,q-r)}(0,\eta)(v-\eta)^{q-r-1}d\eta\ .$$

and

$$k_0(u)=G(u,0)-\sum_{r=0}^{q-1}G^{(r,0)}(0,0)\frac{u^r}{r!}=\frac{1}{(q-1)!}\int_0^u G^{(q,0)}(\xi,0)(u-\xi)^{q-1}d\xi\ .$$

Then

$$m(u, v)=G(u, v)-\pi(u, v)-\sum_{r=0}^{q-2} h_r(v)\frac{u^r}{r!}-k_0(u)$$

$$=G(u, v)-\sum_{r=0}^{q-2} G^{(r,0)}(0, v)\frac{u^r}{r!}-G(u, 0)+\sum_{r=0}^{q-2} G^{(r,0)}(0, 0)\frac{u^r}{r!}$$

$$=\frac{1}{(q-2)!}\int_0^u\int_0^v G^{(q-1,1)}(\xi, \eta)(u-\xi)^{q-2}d\eta d\xi .$$

The equalities are verified by integrations by parts, and the order relations are now evident.

Clearly the roles of u and v may be interchanged in equations (4.2) and (4.4). Moreover, other similar lemmas may be found giving different powers of u and v in the estimate of $m(u,v)$.

The second lemma is an immediate consequence of equations (2.7) and (2.8).

LEMMA 4.2. *If* $d=x_1+x_2+x_3>0$, *we have*

$$J^{\alpha,\beta,\gamma}1=d^{\alpha+\beta+\gamma-1}/\Gamma(\alpha+\beta+\gamma) . \tag{4.5}$$

If the real parts of α, β, *or* γ *are less than or equal to zero, this formula provides an analytic continuation of the left member.*

The next three lemmas provide the principal tools for use in § 5.

LEMMA 4.3. *Suppose that* $f(\xi_1, \xi_2, \xi_3)$ *has continuous derivatives up to the third order. Let* $d=x_1+x_2+x_3>0$. *Then* $J^{\alpha,\beta,\gamma}f(x)$, *defined by* (4.1), *can be continued analytically throughout the region* R *in* α, β, γ *space, where* R *is defined by the fact that one of the following three conditions holds:*

(a) $\mathscr{R}(\alpha)>-2$, $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-1$,

or

(b) $\mathscr{R}(\alpha)>-1$, $\mathscr{R}(\beta)>-2$, $\mathscr{R}(\gamma)>-1$,

or

(c) $\mathscr{R}(\alpha)>-1$, $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-2$.

Moreover, $J^{\alpha,\beta,\gamma}f(x)$ *assumes the following special values. (In all cases, if* $\alpha_0, \beta_0, \gamma_0$ *is on the boundary of* R, *the formula is to be interpreted as meaning the limit as* $\alpha\to\alpha_0$, $\beta\to\beta_0$, $\gamma\to\gamma_0$.)

$$J^{1,1,1}f(x)=3^{-1/2}\iint_{S(x)} f(\xi_1, \xi_2, \xi_3)dS , \tag{4.6}$$

$$(4.7)\qquad J^{1,1,0}f(x)=\int_{x_1-d}^{x_1} f(\xi_1,\ -\xi_1-x_3,\ x_3)d\xi_1\ ,$$

$$(4.8)\qquad J^{1,1,-1}f(x)=\int_{x_1-d}^{x_1}[f_{\xi_3}(\xi_1,\ -\xi_1-x_3,\ x_3)-f_{\xi_2}(\xi_1,\ -\xi_1-x_3,\ x_3)]d\xi_1 + f(x_1-d,\ x_2,\ x_3)\ ,$$

$$(4.9)\qquad J^{1,1,-2}f(x)=\int_{x_1-d}^{x_1}[f_{\xi_3\xi_3}(\xi_1,\ -\xi_1-x_3,\ x_3)-2f_{\xi_3\xi_2}(\xi_1,\ -\xi_1-x_3,\ x_3) + f_{\xi_2\xi_2}(\xi_1,\ -\xi_1-x_3,\ x_3)]d\xi_1 + 2f_{\xi_3}(x_1-d,\ x_2,\ x_3)-f_{\xi_1}(x_1-d,\ x_2,\ x_3)-f_{\xi_2}(x_1-d,\ x_2,\ x_3)\ .$$

$$(4.10)\qquad J^{1,0,0}f(x)=f(x_1-d,\ x_2,\ x_3)=f(-x_2-x_3,\ x_2,\ x_3)\ ,$$

$$(4.11)\qquad J^{1,0,-1}f(x)=f_{\xi_3}(x_1-d,\ x_2,\ x_3)-f_{\xi}\,(x_1-d,\ x_2,\ x_3)\ ,$$

$$(4.12)\qquad J^{1,0,-2}f(x)=f_{\xi_3\xi_3}(x_1-d,\ x_2,\ x_3)-2f_{\xi_1\xi_3}(x_1-d,\ x_2,x_3)+f_{\xi_1\xi_1}(x_1-d,x_2,x_3),$$

$$(4.13)\qquad J^{1,-1,-1}f(x)=f_{\xi_3\xi_2}(x_1-d,\ x_2,\ x_3)-f_{\xi_1\xi_3}(x_1-d,\ x_2,\ x_3) - f_{\xi_1\xi_2}(x_1-d,\ x_2,\ x_3)+f_{\xi_1\xi_1}(x_1-d,\ x_2,\ x_3)\ ,$$

$$(4.14)\qquad J^{0,0,0}f(x)=0\ ,$$

$$(4.15)\qquad J^{0,0,-1}f(x)=0\ ,$$

$$(4.16)\qquad J^{0,-1,-1}f(x)=0\ .$$

Formulas analogous to these can be obtained by permuting the superscripts.

Proof. Since $J^{\alpha,\beta,\gamma}f(x)$ is defined by (4.1) and is analytic for $\mathscr{R}(\alpha)>0$, $\mathscr{R}(\beta)>0$, $\mathscr{R}(\gamma)>0$, equation (4.6) is immediate. To obtain equations (4.7), (4.8), and (4.9) we have

$$(4.17)\qquad J^{1,1,\gamma}f(x)=\frac{1}{\Gamma(\gamma)}\int_{-x_1-x_2}^{x_3} F(\xi_3)(x_3-\xi_3)^{\gamma-1}d\xi_3\ ,$$

where

$$(4.18)\qquad F(\xi_3)=\int_{-\xi_3-x_2}^{x_1} f(\xi_1,\ -\xi_1-\xi_3,\ \xi_3)d\xi_1\ .$$

Then (4.17) is an ordinary Riemann-Liouville integral which can be continued analytically for $\mathscr{R}(\gamma)>-3$ since $F(\xi_3)$ has a continuous third derivative. Also, by (1.7), we have

$$(4.19)\qquad J^{1,1,\gamma}f(x)=F^{(-\gamma)}(x_3)\ ,\qquad (\gamma=0,\ -1,\ -2)\ .$$

Equation (4.7) follows by setting $\xi_3=x_3$ in (4.18). Also from (4.18) we have

$$(4.20)\quad \frac{dF}{d\xi_3}=\int_{-\xi_3-x_2}^{x_1}[f_{\xi_3}(\xi_1,\,-\xi_1-\xi_3,\,\xi_3)-f_{\xi_2}(\xi_1,\,-\xi_1-\xi_3,\,\xi_3)]d\xi_1 + f(-\xi_3-x_2,\,x_2,\,\xi_3)\ ,$$

and

$$(4.21)\quad \frac{d^2F}{d\xi_3^{\,2}}$$

$$=\int_{-\xi_3-x_2}^{x_1}[f_{\xi_3\xi_3}(\xi_1,\,-\xi_1-\xi_3,\,\xi_3)-2f_{\xi_3\xi_2}(\xi_1,\,-\xi_1-\xi_3,\,\xi_3)+f_{\xi_2\xi_2}(\xi_1,\,-\xi_1-\xi_3,\,\xi_3)]d\xi_1$$
$$+2f_{\xi_3}(-\xi_3-x_2,\,x_2,\,\xi_3)-f_{\xi_2}(-\xi_3-x_2,\,x_2,\,\xi_3)-f_{\xi_1}(-\xi_3-x_2,\,x_2,\,\xi_3)\ .$$

Equations (4.8) and (4.9) follow by setting $\xi_3=x_3$ in (4.20) and (4.21).

Turning our attention to equations (4.10)–(4.13), we shall express the integral (4.1) in terms of the variables ξ_2 and ξ_3 and use Lemma 4.1 with $q=3$ to expand $f(\xi_1, \xi_2, \xi_3)$ in the form

$$(4.22)\quad f(\xi_1,\xi_2,\xi_3)=f(x_1-d,\,x_2,\,x_3)+(x_2-\xi_2)(f_{\xi_1}-f_{\xi_2})_0+(x_3-\xi_3)(f_{\xi_1}-f_{\xi_3})_0$$
$$+\frac{(x_2-\xi_2)^2}{2}(f_{\xi_1\xi_1}-2f_{\xi_1\xi_2}+f_{\xi_2\xi_2})_0+\frac{(x_3-\xi_3)^2}{2}(f_{\xi_1\xi_1}-2f_{\xi_1\xi_3}+f_{\xi_3\xi_3})_0$$
$$+(x_2-\xi_2)(x_3-\xi_3)(f_{\xi_1\xi_1}-f_{\xi_1\xi_3}-f_{\xi_1\xi_2}+f_{\xi_2\xi_3})_0+L(\xi_2,\xi_3)$$

where

$$(4.23)\qquad L(\xi_2,\xi_3)=L_1(\xi_2)+(x_3-\xi_3)L_2(\xi_2)+L_3(\xi_3)+L_4(\xi_2,\xi_3)$$

with

$$(4.24)\qquad L_1(\xi_2)=O((x_2-\xi_2)^3)\ ,\qquad L_2(\xi_2)=O((x_2-\xi_2)^2)\ ,$$
$$L_3(\xi_3)=O((x_3-\xi_3)^3)\ ,\qquad L_4(\xi_2,\xi_3)=O((x_2-\xi_2)(x_3-\xi_3)^2)\ .$$

Here the subscript 0 indicates that the values of the derivatives are calculated at the point (x_1-d, x_2, x_3).

Considering the first six terms of (4.22), we deal with the term involving $(x_2-\xi_2)^\lambda(x_3-\xi_3)^\mu/(\lambda!\,\mu!)$ where $\lambda+\mu\leq 2$. The contribution to $J^{\alpha,\beta,\gamma}f(x)$ of this term is found to be

$$\frac{\Gamma(\beta+\lambda)\Gamma(\gamma+\mu)d^{\alpha+\beta+\gamma+\lambda+\mu-1}}{\lambda!\,\mu!\,\Gamma(\beta)\Gamma(\alpha)\Gamma(\alpha+\beta+\gamma+\lambda+\mu)}\ ,$$

by Lemma 4.2. We note that this function is analytic for all values of α, β, γ. When $\alpha=1$, $\beta=\gamma=0$, it reduces to 1 if $\lambda=\mu=0$ and to zero otherwise. When $\alpha=1$, $\beta=0$, $\gamma=-1$, it reduces to -1 if $\lambda=0$, $\mu=1$, and to zero otherwise. When $\alpha=1$, $\beta=0$, $\gamma=-2$, it reduces to 1 if $\lambda=0$, $\mu=2$, and to zero otherwise. When $\alpha=1$, $\beta=\gamma=-1$, it reduces to 1 if $\lambda=\mu=1$, and to zero otherwise. Thus these terms yield the values stated in equations (4.10)–(4.13). We have only to show that the contribution of $L(\xi_2, \xi_3)$ to $J^{\alpha,\beta,\gamma}f(x)$ can be continued analytically throughout R and reduces to zero when α, β, γ assume the values needed in (4.10)-(4.13).

We first show that $J^{\alpha,\beta,\gamma}f(x)$ can be continued analytically throughout the region R_1 where $\mathscr{R}(\alpha)\geq 2$, $\mathscr{R}(\beta)> -1$, $\mathscr{R}(\gamma)>-2$. We consider in turn the contributions arising from the four terms of $L(\xi_2, \xi_3)$ given in (4.23).

We have, for $L_1(\xi_2)$,

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\int_{-x_1-x_3}^{x_2} L_1(\xi_2)(x_2-\xi_2)^{\beta-1}d\xi_2\int_{-\xi_2-x_1}^{x_3}(x_3-\xi_3)^{\gamma-1}(x_1+\xi_2+\xi_3)^{\alpha-1}d\xi_3$$

$$=\frac{1}{\Gamma(\beta)\Gamma(\alpha+\gamma)}\int_{-x_1-x_3}^{x_2} L_1(\xi_2)(x_2-\xi_2)^{\beta-1}(x_1+x_3+\xi_2)^{\alpha+\gamma-1}d\xi_2$$

on using (2.7) and (2.8). On taking account of (4.24) we see that the integral is analytic in R_1. Moreover, the expression is zero if $\beta=0$ or -1 even when $\gamma\to-2$.

The contribution of $(x_3-\xi_3)L_2(\xi_2)$ is similarly

$$\frac{\gamma}{\Gamma(\beta)\Gamma(\alpha+\gamma+1)}\int_{-x_1-x_3}^{x_2} L_2(\xi_2)(x_2-\xi_2)^{\beta-1}(x_1+x_3+\xi_2)^{\alpha+\gamma}d\xi_2\ ,$$

which is also analytic in R_1. It is also zero if $\beta=0$ or -1 even when $\gamma=-2$.

The contribution of $L_3(\xi_3)$ is

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\int_{-x_1-x_2}^{x_3} L_3(\xi_3)\,(x_3-\xi_3)^{\gamma-1}d\xi_3\int_{-\xi_3-x_1}^{x_2}(x_2-\xi_2)^{\beta-1}(x_1+\xi_2+\xi_3)^{\alpha-1}d\xi_2$$

$$=\frac{1}{\Gamma(\gamma)\Gamma(\alpha+\beta)}\int_{-x_1-x_2}^{x_3} L_3(\xi_3)(x_3-\xi_3)^{\gamma-1}(x_1+x_2+\xi_3)^{\alpha+\beta-1}d\xi_3\ .$$

The integral is again analytic in R_1. This contribution is zero if $\gamma=0$, -1, or -2 even when $\beta=-1$.

On taking account of (4.24) it is at once evident that the contribution of $L_4(\xi_2, \xi_3)$ is analytic in R_1 and vanishes when $\beta=0$ or $\gamma=-1$ even on the boundary of R_1.

Thus we have shown that $J^{\alpha,\beta,\gamma}f(x)$ can be continued analytically throughout R_1. Since the roles of α, β, γ may be interchanged it can also be continued analytically throughout five similar regions obtained by permuting α, β, γ in the definition of R_1.

We note that $d=(x_1-\xi_1)+(x_2-\xi_2)+(x_3-\xi_3)$ on $S(x)$ and we multiply equation (4.1) through by these expressions to obtain

$$dJ^{\alpha,\beta,\gamma}f(x)=\alpha J^{\alpha+1,\beta,\gamma}f(x)+\beta J^{\alpha,\beta+1,\gamma}f(x)+\gamma J^{\alpha,\beta,\gamma+1}f(x)\ . \tag{4.25}$$

We use (4.25) to show that $J^{\alpha,\beta,\gamma}f(x)$ can be continued analytically throughout R.

We first suppose that $\mathscr{R}(\alpha)\geq 1$. If $\mathscr{R}(\beta)>0$, $\mathscr{R}(\gamma)>0$, $J^{\alpha,\beta,\gamma}f(x)$ is clearly analytic on using the integral definition in (4.1). If $\mathscr{R}(\beta)>1$, $\mathscr{R}(\gamma)>-1$, then $J^{\alpha+1,\beta,\gamma}f(x)$ is analytic since $(\alpha+1,\beta,\gamma)$ belongs to R_1, $J^{\alpha,\beta+1,\gamma}f(x)$ is analytic since $(\alpha,\beta+1,\gamma)$ belongs to a region similar to R_1, and $J^{\alpha,\beta,\gamma+1}f(x)$ is analytic by (4.1). Thus $J^{\alpha,\beta,\gamma}f(x)$ is analytic if $\mathscr{R}(\beta)>1$, $\mathscr{R}(\gamma)>-1$. We proceed in this way using (4.25) to show the possibility of continuing analytically $J^{\alpha,\beta,\gamma}f(x)$ in turn into the regions $\mathscr{R}(\beta)>1$, $\mathscr{R}(\gamma)>-2$; $\mathscr{R}(\beta)>0$, $\mathscr{R}(\gamma)>-1$; $\mathscr{R}(\beta)>0$, $\mathscr{R}(\gamma)>-2$; $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-1$; $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-2$. At any stage we remember that the roles of β and γ can be interchanged where necessary. We conclude that $J^{\alpha,\beta,\gamma}f(x)$ can be continued analytically throughout the region $\mathscr{R}(\alpha)\geq 1$, $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-2$ and throughout five similar regions obtained by permuting α, β, γ.

We next suppose $\mathscr{R}(\alpha)\geq 0$, We proceed as before using (4.25) to show the possibility of continuing $J^{\alpha,\beta,\gamma}f(x)$ analytically in turn throughout the regions $\mathscr{R}(\beta)>0$, $\mathscr{R}(\gamma)>0$; $\mathscr{R}(\beta)>0$, $\mathscr{R}(\gamma)>-1$; $\mathscr{R}(\beta)>0$, $\mathscr{R}(\gamma)>-2$; $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-1$; $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-2$. We conclude that $J^{\alpha,\beta,\gamma}f(x)$ can be continued analytically throughout the region $\mathscr{R}(\alpha)\geq 0$, $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-2$, and throughout five similar regions obtained by permuting α,β,γ.

We next suppose $\mathscr{R}(\alpha)>-1$. We have already shown that $J^{\alpha,\beta,\gamma}f(x)$ can be continued analytically throughout the region $\mathscr{R}(\beta)>0$, $\mathscr{R}(\gamma)>-2$. We then use (4.25) to show that $J^{\alpha,\beta,\gamma}f(x)$ can be continued analytically in turn throughout the regions $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-1$; $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-2$. We conclude that $J^{\alpha,\beta,\gamma}f(x)$ can be continued analytically throughout the region $\mathscr{R}(\alpha)>-1$, $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-2$, and throughout the two similar regions obtained by permuting α,β,γ. Thus we have shown that $J^{\alpha,\beta,\gamma}f(x)$ can be continued analytically throughout R.

We have yet to show that the contribution of $L(\xi_2,\xi_3)$ to $J^{\alpha,\beta,\gamma}f(x)$ reduces to zero when α,β,γ assume the values needed in (4.10)–(4.13). If α were 2 instead of 1, and β and γ were as in (4.10)–(4.13), our analyticity discussion would show that this contribution is zero. If we apply (4.25) using L instead of f we find that the desired result follows easily. This completes the proof of formulas (4.10)–(4.13).

The formulas (4.14)–(4.16) follow immediately from equation (4.25).

If $f(x)$ has continuous derivatives up to only the second or first order we can still get results similar to Lemma 4.3, but the region into which $J^{\alpha,\beta,\gamma}f(x)$ can be continued will be smaller; however, those of formulas (4.6)–(4.16) which are still valid are unchanged. The method of proof is the same as for Lemma 4.3 and the results can be expressed in the form of two lemmas:

LEMMA 4.3.1. *If* $f(\xi_1, \xi_2, \xi_3)$ *has continuous derivatives up to the second order, then Lemma* 4.3 *holds if the region* R *is replaced by the region* R^* *in which* (a) $\mathscr{R}(\alpha)>-1$, $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>-1$, *or* (b) $\alpha=\beta=1$, $\mathscr{R}(\gamma)>-2$, *or* (c) $\alpha=\gamma=1$, $\mathscr{R}(\beta)>-2$, *or* (d) $\beta=\gamma=1$, $\mathscr{R}(\alpha)>-2$, *and if formulas* (4.12), (4.13), *and* (4.16) *are deleted.*

LEMMA 4.3.2. *If* $f(\xi_1, \xi_2, \xi_3)$ *has continuous derivatives of first order, then Lemma* 4.3 *holds if the region* R *is replaced by the region* R^{**} *in which* (a) $\mathscr{R}(\alpha)>-1$, $\mathscr{R}(\beta)>0$, $\mathscr{R}(\gamma)>0$, *or* (b) $\mathscr{R}(\alpha)>0$, $\mathscr{R}(\beta)>-1$, $\mathscr{R}(\gamma)>0$, *or* (c) $\mathscr{R}(\alpha)>0$, $\mathscr{R}(\beta)>0$, $\mathscr{R}(\gamma)>-1$, *and if only formulas* (4.6), (4.7), (4.8), (4.10), and (4.14) *are retained.*

From equation (4.1) it follows immediately that

$$\frac{\partial}{\partial_{x_1}} J^{\alpha+1,\beta,\gamma} f(x) = J^{\alpha,\beta,\gamma}(x) \tag{4.26}$$

as long as $\mathscr{R}(\alpha)>0$, $\mathscr{R}(\beta)>0$, $\mathscr{R}(\gamma)>0$. Similar formulas hold, of course, for derivatives with respect to x_2 and x_3. By analytic continuation the validity of (4.26) follows as long as (α, β, γ) lies in the interior of a region into which $J^{\alpha,\beta,\gamma}f(x)$ can be continued analytically. But even if (α, β, γ) should lie on the boundary of such a region, if it assumes one of the sets of values occurring in equations (4.6)–(4.16) then (4.26) remains valid, as is easily verified by carrying out the appropriate differentiation of the right members of equations (4.6)–(4.16).

The importance of this lies in the fact that it shows that in finding the derivative of $u(x)$ as given by (3.7) we may interchange the order of the limiting procedure $p\rightarrow 0$ and the differentiation in the term $I_*^{p+1}g(x)$. This simplifies materially the task of verifying that (3.7) gives the solution of Cauchy's problem for the differential equation (1.1).

5. The solution of Cauchy's problem for the equation $\Delta u=h(x)$. It has already been pointed out in § 3 that if the Cauchy problem for the differential equation (1.1) is to have a solution, this solution must be given by (3.7). We are now able to prove the following theorem which gives the solution of Cauchy's problem.

THEOREM 5.1. *Let* $h(x)$ *be continuous and let* $g(x)$ *have continuous derivatives up to the third order in the region* $x_1+x_2+x_3\geq 0$. *Then, in the notation of equations* (2.9) *and* (3.5),

$$u(x)=I^1h(x)+\lim_{p\to 0} I_*^{p+1}g(x) \tag{5.1}$$

is, when $x_1+x_2+x_3\geqq 0$, *a solution of the equation* $\varDelta u=h(x)$; *moreover, when* $x_1+x_2+x_3=0$, *we have* $u(x)=g(x)$, *and all the derivatives of* $u(x)$ *of first and second order equal the corresponding derivatives of* $g(x)$.

Proof. We first note that

$$I^1h(x)=\iiint_{D_S(x)} h(\xi)d\xi=\int_{-x_1-x_2}^{x_3}\int_{-\xi_3-x_1}^{x_2}\int_{-\xi_2-\xi_3}^{x_1} h(\xi_1, \xi_2, \xi_3)d\xi_1 d\xi_2 d\xi_3 , \tag{5.2}$$

and

$$\begin{aligned} I_*^{p+1}g(x)= &\frac{1}{3}[J^{p+1,p,p}\, g(x)+J^{p,p+1,p}\, g(x)+J^{p,p,p+1}g(x) \\ &+J^{p+1,p+1,p+1}(g_{x_2x_3}+g_{x_3x_1}+g_{x_1x_2})] \\ &+\frac{1}{6}[J^{p,p+1,p+1}(g_{x_2}+g_{x_3})+J^{p+1,p,p+1}(g_{x_3}+g_{x_1})+J^{p+1,p+1,p}(g_{x_1}+g_{x_2})] , \end{aligned} \tag{5.3}$$

by (3.5), (3.1), and (4.1).

We now verify that (5.1) satisfies the differential equation $\varDelta u=h(x)$. We have

$$\varDelta u=\varDelta I^1h(x)+\lim_{p\to 0}\varDelta I_*^{p+1}g(x) \tag{5.4}$$

on account of the remark at the end of § 4. If (5.2) is used, an elementary calculation shows that $\varDelta I^1h(x)=h(x)$. It follows directly from (3.5) and (3.1) that

$$\varDelta I_*^{p+1}g(x)=I_*^p g(x) \tag{5.5}$$

if $\mathscr{R}(p)>1$, and a suitable analytic continuation as indicated in § 3 establishes the validity of (5.5) for $\mathscr{R}(p)>0$. If we now let $p\to 0$ and make use of (5.5), (5.3), and (4.14)–(4.16), we find that

$$\lim_{p\to 0}\varDelta I_*^{p+1}g(x)=\lim_{p\to 0} I_*^p g(x)=\lim_{p\to -1} I_*^{p+1}g(x)=0 . \tag{5.6}$$

This completes the verification that (5.1) satisfies the differential equation $\varDelta u=h(x)$.

Next we show that $u(x)$ assumes the correct value $g(x)$ on the plane S whose equation is $x_1+x_2+x_3=0$. We consider $u(x)$ at the point $x=(x_1, x_2, x_3)$, where $x_1+x_2+x_3=d>0$, and let $d\to 0$. From (5.1), (5.3), and (4.10), we find that

$$\begin{aligned} u(x)=&I^1h(x) \\ &+\frac{1}{3}[g(x_1-d, x_2, x_3)+g(x_1, x_2-d, x_3)+g(x_1, x_2, x_3-d)+J^{1,1,1}(g_{x_2x_3}+g_{x_3x_1}+g_{x_1x_2})] \\ &+\frac{1}{6}[J^{0,1,1}(g_{x_2}+g_{x_3})+J^{1,0,1}(g_{x_3}+g_{x_1})+J^{1,1,0}(g_{x_1}+g_{x_2})] . \end{aligned} \tag{5.7}$$

Since $h(x)$ is continuous, equation (5.2) shows that $I^1h(x)=O(d^3)$. On account of Lemma 4.2, we see that if $f(x)$ is continuous, and α, β, γ are real and nonnegative, then

$$J^{\alpha,\beta,\gamma}f(x)=O(d^{\alpha+\beta+\gamma-1}) . \tag{5.8}$$

Thus when x approaches S, that is, when $d\to 0$, (5.7) shows that $u(x)\to g(x)$.

If it is desired, $u(x)$ can be written explicitly in terms of $h(x)$ and $g(x)$ and its derivatives by using (4.6) and (4.7).

Next we consider $\partial u/\partial x_1$. On account of the remark at the end of § 4 we have, from (5.1),

$$\frac{\partial u}{\partial x_1}=\frac{\partial I^1h(x)}{\partial x_1}+\lim_{p\to 0}\frac{\partial I_*^{p+1}g(x)}{\partial x_1} . \tag{5.9}$$

We calculate $\partial I_*^{p+1}g(x)/\partial x_1$ by differentiating (5.3) and using (4.26). We then let $p\to 0$ and make use of equations (4.14), (4.11), (4.7), (4.8), and (4.10). On using equation (5.2) it is easily verified that $\partial I^1h(x)/\partial x_1=O(d^2)$, and hence tends to zero with d. We also note that the integrals in (4.7) and (4.8) tend to zero with d. We thus find that $\partial u/\partial x_1\to g_{x_1}(x_1, x_2, x_3)$ when x approaches S. In the same way we can consider $\partial u/\partial x_2$ and $\partial u/\partial x_3$.

In a similar manner we treat $\partial^2 u/\partial x_i^2$ $(i=1, 2, 3)$. We have only to use equations (4.15), (4.12), (4.8), (4.9), and (4.11) and observe that $\partial^2 I^1h(x)/\partial x_i^2=O(d)$.

The treatment of $\partial^2 u/\partial x_i\partial x_j$ $(i, j=1, 2, 3\,;\ i\neq j)$ is also similar and makes use of equations (4.15), (4.13), (4.10), (4.11), and (4.14).

This completes the verification of the solution.

In Theorem 2.1 we showed that, if $f(x)$ is continuous, $I^pf(x)$ is analytic for $\mathscr{R}(p)>0$ and $I^pf(x)\to f(x)$ when $p\to 0$. The following theorem shows that $I^pf(x)$ can be continued analytically when $f(x)$ is sufficiently differentiable.

THEOREM 5.2. *If $f(x)$ has continuous derivatives up to the third order in the region $x_1+x_2+x_3\geq 0$ then $I^pf(x)$ can be continued analytically throughout the region $\mathscr{R}(p)>-1$, and*

$$\lim_{p\to -1} I^pf(x)=\Delta f(x) \tag{5.10}$$

if $x_1+x_2+x_3>0$.

Proof. We make use of equations (3.6) and (5.3) with $g(x)$ replaced by $f(x)$. Then if $\mathscr{R}(p)>-1$, Theorem 2.1 shows that $I^{p+1}\Delta f(x)$ is analytic, and Lemmas 4.3, 4.3.1, and 4.3.2 show that $I_*^{p+1}f(x)$ is analytic. If we let $p \to -1$, equation (5.10) is a consequence of Theorem 2.1 and the last equality in equations (5.6) with $f(x)$ in place of $g(x)$.

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Stanford University

APPLICATIONS OF THE RAYLEIGH RITZ METHOD TO VARIATIONAL PROBLEMS

J. INDRITZ

Introduction. Let R be a bounded either simply or multiply connected plane region with boundary Γ, consisting of a finite number of non-intersecting simply closed regular arcs of class c^k. A plane curve is a regular arc if the defining functions $x(t)$, $y(t)$, $a \leq t \leq b$ have continuous derivatives with $x'(t)^2 + y'(t)^2 \neq 0$ on $a \leq t \leq b$. A regular arc is of class c^k if the defining functions $x(s)$, $y(s)$, s being arc length, have continuous derivatives of order k. We shall say a function $h(x, y)$ defined on $\bar{R} = R + \Gamma$ is of class c^k if the partial derivatives of h of order r, $0 \leq r \leq k$ exist in R and have limits on Γ so as to define functions continuous on $\bar{R}$. Let $g(x, y)$ be a given function of class c^k on $\bar{R}$. The main problem considered is that of finding the function ψ_0 which yields minimum value to the functional

$$I[\psi] = \iint_R (a\psi_x^2 + b\psi_y^2 + c\psi^2 + 2f\psi)\,dx\,dy$$

defined over the admissible class of functions ψ which are of class c^k on $\bar{R}$ and assume the values of g on Γ.

We shall assume $a(x, y) > 0$, $b(x, y) > 0$, $c(x, y) \geq 0$ on $\bar{R}$; a, b, c bounded and integrable in $\bar{R}$; $f(x, y)$ integrable in $\bar{R}$. In the sequel, unless otherwise specified, integrations will be taken over R and the symbol R omitted.

Let $G(x, y)$ be of class c^k on $\bar{R}$, vanishing on Γ, positive in R, with normal derivative $\partial G/\partial \nu$ on Γ different from 0. We show that, if $k \geq 3$, every admissible function ψ has a uniformly convergent expansion on $\bar{R}$

$$\psi = g + \sum_{i=1}^{\infty} b_i f_i(x, y)$$

where f_i are obtained by a Gram-Schmidt process from the functions $\{G x^i y^j\}$ $i, j = 0, 1, 2, \cdots$ and b_i are generalized Fourier coefficients connected with the quadratic functional

$$D[\psi] = \iint (a\psi_x^2 + b\psi_y^2 + c\psi^2)\,dx\,dy\,.$$

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In fact, $b_i=D[\psi-g, f_i]$ where

$$D[\xi, \eta]=\iint(a\,\xi_x\,\eta_x+b\,\xi_y\,\eta_y+c\,\xi\eta)dx\,dy\,.$$

An estimate of the error obtained by using for ψ only the first n terms of the expansion is given in terms of n and k. Sufficient conditions are obtained for the convergence of

$$\nabla\left[g+\sum_{i=1}^{n}b_i f_i\right]$$

to $\nabla\psi$ and an estimate is given for the rate of convergence.

In particular, if ψ_0 is an admissible function minimizing $I[\psi]$, then the expansion

$$\psi_0=g+\sum_{i=1}^{\infty}a_i f_i$$

yields an explicit solution for ψ_0, since the coefficients a_i are given, in this case, by

$$a_i=-\iint f f_i\,dx\,dy-D[g, f_i]$$

which are independent of ψ_0.

The problem of minimizing the functional $I[\psi]$, with $g\equiv0$, has been studied by Kryloff and Bogoliubov [4] and by Kantorovitch [2], both obtaining estimates for convergence to ψ_0 of functions obtainable by the Rayleigh Ritz method. The first paper deals with convex regions R, the second with regions R bounded by $x=0$, $x=1$, $y=g(x)$, $y=h(x)$; $h>g$ on $0\leq x\leq1$. Neither obtains an explicit solution for ψ_0 nor studies the convergence of the derivatives.

In the final section of this paper, we assume the existence of a function ψ_0 yielding minimum value, for $p\geq1$, to

$$D^p[\psi]=\iint_R(a\psi_x^2+b\psi_y^2+c\psi^2)^p\,dx\,dy\,,\qquad \psi=g\quad\text{on}\quad\Gamma$$

and obtain an estimate for the rate of convergence to ψ_0 of functions obtained by the Rayleigh Ritz method.

§ 1. **Preliminary Considerations.** A variation v shall mean a function of class c^k on $\bar{R}$ vanishing on Γ. Form the Hilbert space H by completing the linear manifold V of variations v using the positive definite quadratic form $D[v]$ as the square of the norm of a variation. If $h\in H$, we represent the norm of h by $\|h\|$. If ξ and η are variations, the inner product will be

$$(\xi, \eta)=D[\xi, \eta]\,.$$

Let f_i be any complete orthonormal set of variations in H. If ψ is admissible, then $\psi - g$ is a variation and thus expressible in H as

$$\psi - g = \sum_{i=1}^{\infty} b_i f_i$$

with $b_i = D[\psi - g, f_i]$.

If ψ_0 is an admissible function yielding a minimum value to $I[\psi]$, if λ is real, and v is a variation, then $\psi_0 + \lambda v$ is admissible, and

$$I[\psi_0] \leq I[\psi_0 + \lambda v] = I[\psi_0] + \lambda(2D[\psi_0, v] + \iint 2fv\,dx\,dy) + \lambda^2 D[v].$$

This implies that the coefficient of λ must vanish so that

$$D[\psi_0, v] = -\iint fv\,dx\,dy$$

and

$$I[\psi_0 + \lambda v] = I[\psi_0] + \lambda^2 D[v]$$

for every variation v.

The first relation shows that the Fourier coefficients of $\psi_0 - g$,

$$a_i = D[\psi_0 - g, f_i] = D[\psi_0, f_i] - D[g, f_i] = -\iint f f_i\,dx\,dy - D[g, f_i]$$

are independent of ψ_0.

The second relation implies that if ψ is admissible,

$$I[\psi] = I[\psi_0 + \psi - \psi_0] = I[\psi_0] + D[\psi - \psi_0].$$

Thus if

$$\phi_n = g + \sum_{i=1}^{n} a_i f_i\,,$$

then

$$0 = \lim_{n\to\infty} \left\| \psi_0 - \left(g + \sum_{i=1}^{n} a_i f_i \right) \right\|^2 = \lim_{n\to\infty} D[\psi_0 - \phi_n] = \lim_{n\to\infty} I[\phi_n] - I[\psi_0]$$

so that ϕ_n is a minimizing sequence.

Moreover,

$$D\left[\psi_0 - g - \sum_{i=1}^{n} c_i f_i\right]$$

is a minimum when $c_i = a_i$ implying that ϕ_n are chosen to yield minimum value to $I[\psi_n] - I[\psi_0]$ and hence to $I[\psi_n]$ in the class of functions

$$\psi_n = g + \sum_{i=1}^{n} c_i f_i\,.$$

Thus ϕ_n may be obtained by the Rayleigh Ritz process applied to the functional $I[\psi]$.

We will prove, in Theorem 1, that the class of functions $\{GP\}$ where P is a polynomial in x and y, is dense in H. This class is the linear manifold determined by the set $\{G x^i y^j\}$, a set linearly independent in H. For, if

$$v=\sum_{i=1}^{m}\sum_{i=1}^{n}\alpha_{ij}\, G\, x^i\, y^j,$$

then $D[v]=0$ implies $\alpha_{ij}=0$.

It follows that we can obtain an orthonormal set f_i complete in H by orthonormalizing the set $\{G x^i y^j\}$. Let

$$v_1=G\,x^0\,y^0$$

$$v_2=G\,x^1\,y^0\,,\quad v_3=G\,x^0\,y^1$$

$$\vdots$$

$$v_{\frac{k(k+1)}{2}+1}=G\,x^k\,y^0,\cdots,\ v_{\frac{k(k+1)}{2}+k+1}=G\,x^0\,y^k\,.$$

Then

$$f_n=\frac{v_n-\sum_{j=1}^{n-1}f_j(v_n\,,\,f_j)}{\left\| v_n-\sum_{j=1}^{n-1}f_j(v_n,f_j)\right\|}$$

$$=\begin{vmatrix}(v_1,\ v_1)\cdots(v_1,\ v_{n-1}) & v_1\\ \vdots\qquad\quad \vdots & \\ (v_n,\ v_1)\cdots(v_n,\ v_{n-1}) & v_n\end{vmatrix}\cdot\begin{vmatrix}(v_1,\ v_1) & & \\ & \ddots & \\ & & (v_{n-1}\,,\ v_{n-1})\end{vmatrix}^{-1/2}\cdot\begin{vmatrix}(v_1,\ v_1) & & \\ & \ddots & \\ & & (v_n\,,\ v_n)\end{vmatrix}^{-1/2}.$$

The function f_n is of the form GP_n, where the degree of the polynomial P_n is that of v_n/G. If $v_n=G\,x^r\,y^s$ with $r+s=k$, then

$$\frac{k(k+1)}{2}+1\leq n\leq\frac{k(k+1)}{2}+k+1$$

so that $k^2<k(k+1)<2n-2$ and the degree k of P_n is less than $\sqrt{2n-2}$. Similarly k is greater than $\sqrt{2n}-2$.

§2. The Minimizing sequence. We shall use certain approximation theorems which can be derived by methods used by Mickelson [5][1]. To simplify the notation, let

$$x=(x_1,\cdots x_s)\,,$$

$$x^{(1)}=(x_1^{(1)},\cdots x_s^{(1)})\,,$$

[1] For detailed proofs of Lemmas 1, 2 see J. Indritz "Applications of the Rayleigh Ritz method to the solutions of partial differential equations" Ph. D. Thesis, U. of Minnesota, 1953.

$$f(x)=f(x_1,\cdots,x_s)\ ,$$

$$f_r(x)=\frac{\partial^{r_1+\cdots+r_s}}{\partial x_1^{r_1}\cdots\partial x_s^{r_s}}[f(x_1,\cdots,x_s)]\ ,$$

$$\| x^{(1)}-x^{(2)}\|=\sqrt{\sum_{i=1}^{s}(x_i^{(1)}-x_i^{(2)})^2}\ .$$

The modulus of continuity for a function f defined over a closed set A: $-1\leq x_i\leq 1$ $(i=1,\cdots,s)$ is

$$\Omega(\delta,\ f)=\sup\ |f(x^{(1)})-f(x^{(2)})|$$

for all points $x^{(1)}$, $x^{(2)}$ in A with $\| x^{(1)}-x^{(2)}\|\leq\delta$. The uniform modulus of continuity of a finite number of functions $f_1,\cdots,f_N$ is the largest of the moduli of each f_i for each δ.

LEMMA 1. *Let $F(\theta)$ be a continuous periodic function of period 2π in each θ_i and of class c^k. Let $\omega(\delta)$ be the uniform modulus of continuity of the partial derivatives of F of order 1 to k for $\delta\leq\pi\sqrt{s}$. Let $j\leq k$. Then, corresponding to every set $m_1,\cdots,m_s$ of positive integers, there is a trigonometric sum T^m of order at most m_i in θ_i such that*

$$|F_r(\theta)-T_r^m(\theta)|\leq K_1\left(\sum_{i=1}^{s}\frac{1}{m_i}\right)^{k-j}\sum_{i=1}^{s}\omega\left(\frac{1}{m_i}\right)\quad \textit{for}\quad 0\leq r_1+\cdots+r_s\leq j$$

where K_1 is a constant independent of F, s, m_i.

If the partial derivatives of order 1 to k satisfy

$$|F_r(\theta^{(1)})-F_r(\theta^{(2)})|\leq L\left(\sum_{i=1}^{s}|\theta_i^{(1)}-\theta_i^{(2)}|\right)$$

then

$$|F_r(\theta)-T_r^m|\leq L\,K_2\left(\sum_{i=1}^{s}\frac{1}{m_i}\right)^{k-j+1}\qquad 0\leq r_1+\cdots+r_s\leq j$$

where K_2 is also a constant independent of F, s, m_i.

If F is even in each θ_i separately, T contains only cosine terms.

LEMMA 2. *Let $f(x)$ be of class c^k in the set A: $-1\leq x_i\leq 1$ $(i=1,\cdots,s)$. Let M be the maximum of the absolute values of the derivatives of order 1 to k, and $\Omega(\delta)$ the uniform modulus of continuity of the derivatives of order k. Let B denote a closed set interior to A. Let $j\leq k$. Then, for every set of positive integers $m_1,\cdots,m_s$ with $m_i\geq k$ there is a polynomial P^m of order at most m_i in x_i such that*

$$|f_r(x)-P_r^m(x)|\leq K_3\Big(\sum_{i=1}^{s}\frac{1}{m_i}\Big)^{k-j}\sum_{i=1}^{s}\Omega\Big(\frac{1}{m_i}\Big)$$

for x in B and $0\leq r_1+\cdots+r_s\leq j$. Here K_3 is a constant independent of f and m_i.

If also, the k-th partial derivatives of $f(x)$ satisfy a Lipschitz condition with parameter λ, then, for x in B,

$$|f_r(x)-P_r^m(x)|\leq K_4\Big(\sum_{i=1}^{s}\frac{1}{m_i}\Big)^{k-j+1}\quad for\quad 0\leq r_1+\cdots+r_s\leq j,$$

and where K_4 is a constant independent of f and m_i.

To apply the lemmas to a function defined over the region R, we shall extend the domain of definition of the function. The question arises whether the differentiability properties of the function are maintained under the extension. The answer depends upon the properties of the boundary Γ of R. For example, Hirschfeld [1] has shown that even a cusp in the complementary region may prevent c^1 extension of a function of class c^∞ on a closed set through a continuous boundary arc. Whitney [6] has given a different definition for a function to be of class c^k in a closed set A. If f is of Whitney class c^k in A, then there exists an extension F to the whole plane E_2 which is of class c^k in the ordinary sense on E_2 and is analytic in E_2-A. The derivatives of F of order $\leq k$ coincide with those of f at any interior point of A. Moreover Whitney [7] has shown the following: Let (a) f be of class c^k on $R+\Gamma$, where R is a region, Γ its boundary, in the sense we have defined in the introduction, and (b) R have the property "P", that any two points P_1, P_2 in R, whose linear distance apart may be represented by $|P_1-P_2|$, can be joined by a rectifiable curve in R of length L, with $L/|P_1-P_2|$ bounded uniformly with respect to P_1 and P_2; then f is also of Whitney class c^k and thus can be extended to E_2 to be of class c^k on E_2.

For our purposes we assume R to be a bounded region with boundary Γ consisting of a finite number of non-intersecting simply closed regular arcs Γ_i and we will show R has property "P".

Choose, for each Γ_i, a $\delta>0$ such that no two tangents to Γ_i on any portion of arc length $<\delta$ make with each other an angle greater than 5°. We may choose δ independent of i and smaller than one-fourth the distance between any two Γ_i. Now fix i, and let P_1, P_2 be points on Γ_i on a subarc of length $<\delta$. There is a point Q on that subarc between P_1 and P_2 such that the tangent line at Q is parallel to the chord P_1P_2. Set up an (x, y) coordinate system at Q, using the tangent line as x-axis, the normal as y-axis, and note that the subarc considered has an equation $y=y(x)$ of class c^1 in view of the implicit function theorems. Let $P_1=(x_1, y_1)$, $P_2=(x_2, y_2)$, $\|P_1-P_2\|=$ distance between P_1 and P_2, $\|\widehat{P_1P_2}\|=$length of the subarc joining P_1 to P_2. Then $\|P_1-P_2\|=|x_1-x_2|$ and $|y'(x)|\leq 1$ so that

$$(1)\qquad \|P_1-P_2\|\leq\|\widehat{P_1P_2}\|=\left|\int_{x_1}^{x_2}\sqrt{1+y'^2}\,dx\right|$$
$$\leq\sqrt{2}\,|x_1-x_2|=\sqrt{2}\,\|P_1-P_2\|.$$

Moreover, since $\tan 5°<1/10$, the mean value theorem shows that $\sup|y(x)|\leq\|P_1-P_2\|/10$.

We shall also use the well known property that if Γ_i is a regular arc, there is an $\omega_1>0$ such that for any subarc joining points P_3, P_4 on Γ_i, we have $\|\widehat{P_3P_4}\|/\|P_3-P_4\|\leq\omega_1$. ω_1 can be chosen independent of i.

Now suppose S_1, S_2 are any two points interior to the region R. If the segment S_1S_2 is interior to R, we of course have $\|\widehat{S_1S_2}\|/\|S_1-S_2\|=1$ by using the segment as the arc. Otherwise, let Q_1 be the first intersection of the directed line S_1S_2 with the boundary, say with Γ_1. Let Q_1^1 be a point on S_1Q_1 in R. Let Q_2 be the first point of intersection of the directed line S_2S_1 with Γ_1 and Q_2^1 a point in R on S_2Q_2 such that the open interval $Q_2Q_2^1$ is also in R. Note that Q_1 and Q_2 may coincide. If $Q_2^1S_2$ is not in R, let Q_3 be the first point of intersection of the directed line $Q_2^1S_2$ with the boundary, say with Γ_2 and Q_3^1 in R and on $Q_2^1Q_3$. Let Q_4 be the first point of intersection of the directed line $S_2Q_2^1$ with Γ_2 and Q_4^1 a point in R, on Q_4S_2, with the open interval $Q_4Q_4^1$ in R. Continuing in this way, after at most n steps, we form a finite sequence of points $Q_0^1=S_1$, Q_1^1, $Q_2^1,\cdots,Q_{2m}^1$, $Q_{2m+1}^1=S_2$ such that Q_{2k-1} and Q_{2k} are on the same regular arc, and the lines joining Q_{2k}^1 to Q_{2k+1}^1, $k=0,\cdots,m$ are in R. If we can show there is an $\omega>0$, independent of the points, and arcs λ_1 in R joining consecutive points Q_j^1 to Q_{j+1}^1 such that $\|\widehat{Q_j^1Q_{j+1}^1}\|\leq\omega\|Q_j^1-Q_{j+1}^1\|$, then we can attain the desired results by addition. It suffices to show that Q_1^1 and Q_2^1 and an arc λ joining Q_1^1 to Q_2^1 and in R may be chosen so that $\|\widehat{Q_1^1Q_2^1}\|\leq\omega\|Q_1^1-Q_2^1\|$. Suppose first that Q_1 and Q_2 coincide. A sufficiently small circle with Q_1 as center will have one of the arcs cut off by S_1S_2 entirely in R and we may choose Q_1^1 and Q_2^1 as the intersections of S_1S_2 with this circle. In this case

$$\|\widehat{Q_1^1Q_2^1}\|=\frac{\pi}{2}\|Q_1^1-Q_2^1\|.$$

Otherwise, let L be the length of an arc on Γ_1 joining Q_1 to Q_2.

Divide this arc into N equal segments of length $\beta=L/N$ where N is sufficiently large so that $\beta<\delta$. Draw circles of radius $r=\beta/\sqrt{2}$ about each of the division points and the end points. We first show that consecutive circles intersect. If R_1 and R_2 are two consecutive centers, (1) implies

$$\| R_1-R_2 \|\leq\beta\leq\sqrt{2}\ \| R_1-R_2 \|$$

so that

$$\frac{\| R_1-R_2 \|}{2}\leq\frac{\| R_1-R_2 \|}{\sqrt{2}}\leq\frac{\beta}{\sqrt{2}}\leq\| R_1-R_2 \| ,$$

and the circles must intersect.

Moreover, since $r\geq\| R_1-R_2 \|/\sqrt{2}$, the semi-length τ of the common chord is

$$\tau=\sqrt{r^2-\frac{\| R_1-R_2 \|^2}{4}}\geq\frac{\| R_1-R_2 \|}{2} ,$$

whereas the arc joining R_1 to R_2 has distance $<\| R_1-R_2 \|/10$ from the chord. Hence the arc lies entirely within the circles.

Now let Q_1^1 be an intersection of S_1S_2 with the circle whose center is Q_1 and Q_2^1 an intersection of S_1S_2 with the circle whose center is Q_2, the points being chosen to lie in R and have the desired properties. Starting from Q_1^1 we may proceed to Q_2^1 via the circumferences of the circles. The total length of the curve thus formed will be less than

$$(N+1)2\pi\frac{\beta}{\sqrt{2}}=\frac{N+1}{N}\frac{2\pi}{\sqrt{2}}L\leq\frac{4\pi}{\sqrt{2}}L$$

and

$$\frac{\| \widehat{Q_1^1Q_2^1} \|}{\| Q_1^1-Q_2^1 \|}\leq\frac{4\pi}{\sqrt{2}}\frac{L}{\| Q_1^1-Q_2^1 \|}\leq\frac{4\pi}{\sqrt{2}}\frac{L}{\| Q_1-Q_2 \|}\leq\frac{4\pi}{\sqrt{2}}\omega_1 .$$

This concludes the proof that R has property "P".

We will be particularly interested in extending a function of the form $v(x, y)/G(x, y)$ where $G(x, y)>0$ in R, $\partial G/\partial\nu>0$ on Γ, $G=v=0$ on Γ and we seek differentiability conditions on v and G which insure that v/G is of class c^k on $R+\Gamma$. Here again the nature of the boundary is of importance. The next two lemmas deal with this problem. The letter P will refer to a point in R and Q to a point on Γ, the boundary of R. By a neighborhood $N(Q)$ in $R+\Gamma$ we will mean a set of points S in $R+\Gamma$ such that for some sufficiently small circle with center at Q, every point of the circle which lies in $R+\Gamma$ also lies in S.

LEMMA 3. a) *Let R be a region bounded by Γ, a finite number of closed Jordan curves, no two having a point in common. Let γ be a regular subarc of Γ, and Q_0 an interior point of γ. Let N be the normal to γ at Q_0. Then there is a neighborhood $N(Q_0)$ in $R+\Gamma$ such that through each point P in $RN(Q_0)$, the line parallel to N cuts $\gamma_0=\gamma N(Q_0)$ in one and only one point Q, PQ lies in $N(Q_0)$, and Q ranges over γ_0.*

b) *Let $\psi(x, y)$ be of class c^1 in $RN(Q_0)$ and suppose ψ, ψ_x, ψ_y have continuous limits on γ_0. Define $(\partial\psi/\partial s)(P)$ to be the derivative at $P\in RN(Q_0)$ in the direction of the tangent at the corresponding point Q on γ_0. The derivative $(\partial\psi/\partial s)(P)$ has continuous limits on γ_0 which we will denote by $(\partial\psi/\partial s)(Q)$.*

If $\psi=0$ on γ_0, then $(\partial\psi/\partial s)(Q)=0$ for Q on γ_0.

Proof. Let γ be given by $x(t)$, $y(t)$ and Q_0 defined by the parameter value t_0. Let (ξ, η) be rectangular axes along the tangent and normal at Q_0. In a suitable neighborhood of t_0, $t_1<t<t_2$, defining an arc λ_0 containing Q_0, γ admits a representation $\eta=\eta(\xi)$. We may assume λ_0 so small that no two tangents to it make with each other an angle greater than 5°. There is a positive distance d between $\Gamma-\lambda_0$ and the arc λ_1 defined by the parameter range $(t_1+t_0)/2<t<(t_0+t_2)/2$. Take

$$\delta<\min[d,\ |\xi(t_0)-\xi((t_0+t_2)/2)|,\ |\xi(t_0)-\xi((t_0+t_1)/2)|]$$

and draw a square T of side δ with sides parallel to the (ξ, η) axes and center at Q_0. Let $\gamma_0=\gamma T$, the projection of RT on γ by lines parallel to N, and let γ_h be the arcs formed by displacing γ_0 a distance h parallel to itself into R along N. For $h<h_1$ sufficiently small, $\gamma_h\subset T$. The regular arc γ_0 may be given a representation $x=x(s)$, $y=y(s)$, $0\leq s\leq L$, in terms of arc length s, where L is the length of γ_0. Then γ_h is given by

$$x=x(s)+h\cos\alpha, \qquad y=y(s)+h\cos\beta,$$

where $\cos\alpha$, $\cos\beta$ are the direction cosines of the line N directed inward into R. The neighborhood $N(Q_0)$ may be chosen as given by these equations with $0<s<L$. $0\leq h<h_1$.

It is clear that

$$\frac{\partial\psi}{\partial s}=\frac{\partial\psi}{\partial x}\cdot\frac{dx}{ds}+\frac{\partial\psi}{\partial y}\frac{dy}{ds}$$

has continuous limits on γ_0. Write

$$\frac{\partial\psi}{\partial s}(P)=\frac{\partial\psi}{\partial s}(x(s)+h\cos\alpha,\ y(s)+h\cos\beta)=F(s,\ h).$$

If λ is any closed subarc of γ_0, we have

$$\lim_{h\to 0} F(s,\ h)=\frac{\partial\psi}{\partial s}(Q)$$

uniformly in s.

Along γ_h we have

$$\psi(P_2)-\psi(P_1)=\int_{s_1}^{s_2} F(s,\ h)\,ds$$

where P_1 and P_2 are points on γ_h corresponding to points Q_1 and Q_2 on λ with parameter values s_1 and s_2. As h approaches 0, the limits on the integral remain fixed. Since $\psi=0$ on λ, we find, by letting $h\to 0$,

$$0=\int_{s_1}^{s_2}\frac{\partial\psi}{\partial s}(Q)\,ds$$

for arbitrary s_1, s_2. Thus $(\partial\psi/\partial s)(Q)=0$ on λ and hence on γ_0.

LEMMA 4. *Let R, γ, Q_0, $N(Q_0)$, N, γ_0 be defined as in Lemma 3. Let $v(x,\ y)$ and $G(x,\ y)$ be of class c^0 on $N(Q_0)$ and of class c^1 on $N(Q_0)[R+Q_0]$. Let $v=G=0$ on γ_0, $G>0$ in $RN(Q_0)$, $(\partial G/\partial\nu)(Q_0)\neq 0$. Then there exists $\lim_{P\to Q_0} v(P)/G(P)$ for $P\in R$.*

If γ is of class c^{k+1} on $N(Q_0)$ and v, G are of class c^k in $N(Q_0)$ and of class c^{k+1} on $N(Q_0)[R+Q_0]$, then v/G is of class c^k on $N(Q_0)[R+Q_0]$.

Proof. Denote differentiation along a line parallel to N by $\partial/\partial h$. By the mean value theorem one finds that $(\partial G/\partial\nu)(Q_0)$ is the limiting value of $(\partial G/\partial h)(P)$ as $P\in RN(Q_0)$ approaches Q_0 along the normal at Q_0, and hence $(\partial G/\partial\nu)(Q_0)$ is the limiting value of $(\partial G/\partial h)(P)$ as P approaches Q_0 by any approach in $RN(Q_0)$. A similar statement is true for $(\partial v/\partial\nu)(Q_0)$.

Let P_n be any sequence of points in $RN(Q_0)$ converging to Q_0 and let Q_n be the points on γ_0 associated, by projection along N, with P_n. By the generalized mean value theorem,

$$\frac{v(P_n)}{G(P_n)}=\frac{v(P_n)-v(Q_n)}{G(P_n)-G(Q_n)}=\frac{(\partial v/\partial h)(P'_n)}{(\partial G/\partial h)(P'_n)}$$

where P'_n is interior to the line segment P_nQ_n.

Thus

$$\lim_{P_n\to Q_0}\frac{v(P_n)}{G(P_n)}=\frac{(\partial v/\partial\nu)(Q_0)}{(\partial G/\partial\nu)(Q_0)}\ .$$

It is clear from the construction of $N(Q_0)$ that the equations

$$x=X(s,\ h)=x(s)+h\cos\alpha\ ,\quad y=Y(s,\ h)=y(s)+h\cos\beta$$

yield a one to one transformation of $N(Q_0)$ into $N^*(Q_0)$: $0\leq h<h_1$, $0<s<L$ and γ_0 into γ_0^*: $h=0$, $0<s<L$ and Q_0 into Q_0^*: $h=0$, $s=s_0$. In fact, in view of the restriction on the slope of the tangent to γ_0, the Jacobian of the transformation is

$$J=x'(s)\cos\beta-y'(s)\cos\alpha\neq 0\ .$$

If $x(s)$, $y(s)$ are of class c^{k+1} on $0<s<L$ then so are $X(s,\ h)$, $Y(s,\ h)$ in $N^*(Q_0)-\gamma_0^*$. Any partial derivative of $X(s,\ h)$, $Y(s,\ h)$ of order $r\leq k+1$ converges, as $h\to 0$, uniformly on any closed subinterval of γ_0^* and thus this derivative has a continuous limit on γ_0^*. By the implicit function theorems, the inverse functions $s=S(x,\ y)$, $h=H(x,\ y)$ are of class c^{k+1} in $RN(Q_0)$. Moreover, the partial derivatives of S, H of order $r\leq k+1$ have continuous limits on γ_0, for the relationships

$$1=\frac{\partial X}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial X}{\partial h}\frac{\partial h}{\partial x}=x'(s)\frac{\partial s}{\partial x}+\cos\alpha\frac{\partial h}{\partial x}$$

$$0=\frac{\partial Y}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial Y}{\partial h}\frac{\partial h}{\partial x}=y'(s)\frac{\partial s}{\partial x}+\cos\beta\frac{\partial h}{\partial x}$$

can be solved for $\partial s/\partial x$, $\partial h/\partial x$, $\partial s/\partial y$, $\partial h/\partial y$ and the resulting equations indicate that these derivatives and their derivatives of order $\leq k$ have continuous limits on γ_0.

Under this transformation $v(x,\ y)$ becomes $v^*(s,\ h)$ and $G(x,\ y)$ becomes $G^*(s,\ h)$. It is sufficient to show v^*/G^* is of class c^k at Q_0^* since any partial derivative of order $r\leq k$ of $v(x,\ y)/G(x,\ y)$ is a polynomial in the derivatives of v^*/G^* and in the derivatives of s and h with respect to x and y of order $\leq r$.

By the hypothesis and comments above, $v^*(s,\ h)$ and $G^*(s,\ h)$ are of class c^k on $N^*(Q)$ and of class c^{k+1} on $(N^*(Q_0)-\gamma_0^*)+Q_0^*$. In view of the continuity of $\partial G/\partial h$ at Q_0. there is a neighborhood of Q_0 where $(\partial G/\partial h)(P)>\delta>0$, It is no loss of generality to assume $(\partial G/\partial h)>\delta>0$ in $N(Q_0)$ and we shall do so. By Lemma 3, $\partial v/\partial s$ and $\partial G/\partial s$ vanish on γ_0. By repeated application of Lemma 3, $\partial^r v/\partial s^r$ and $\partial^r G/\partial s^r$ $(0\leq r\leq k)$ vanish on γ_0.

The proof is greatly facilitated by an auxiliary transformation. Let $t=s$, $z=G^*(s,\ h)$ carrying Q_0^* into Q_0^{**}, γ_0^* into γ_0^{**}, $N^*(Q_0)$ into $N^{**}(Q_0)$. For each s, z is a monotone increasing function of h and the inverse function $h=H^*(t,\ z)$ is a monotone increasing function of z for each t. As above, we see that $v^*(s,\ h)=v^{**}(t,\ z)$ is of class c^k on

$N^{**}(Q_0)$ and of class c^{k+1} on $(N^{**}(Q_0)-\gamma_0^{**})+(Q_0^{**})$. Moreover, it suffices to prove that $v^{**}(t, z)/z$ is of class c^k at Q_0^{**}. For notational simplicity, let $w(t, z)=v^{**}(t, z)$. Note that $N^{**}(Q_0)$ is the set $0\leq z<G(t, h_1)$, $0<t<L$.

By induction, we verify

$$\frac{\partial^r}{\partial z^r}\left(\frac{w}{z}\right)=\frac{r!(-1)^r}{z^{r+1}}\left(w-z\frac{\partial w}{\partial z}+\frac{z^2}{2!}\frac{\partial^2 w}{\partial z^2}+\cdots+(-1)^r\frac{z^r}{r!}\frac{\partial^r w}{\partial z^r}\right)$$

for $0\leq r\leq k$ when $z>0$.

For t fixed, $w(t, z)$ has a Taylor expansion of the form

$$w(t, \zeta)=w(t, z)+\frac{\partial w}{\partial z}(t, z)(\zeta-z)+\cdots+\frac{\partial^r w}{\partial z^r}(t, z)\frac{(\zeta-z)^r}{r!}$$
$$+\frac{\partial^{r+1}w}{\partial z^{r+1}}(t, \xi)\frac{(\zeta-z)^{r+1}}{(r+1)!}$$

for $0\leq r\leq k$, where $0\leq\zeta<\xi(t, z, \zeta, r)<z$ so that, when $\zeta=0$,

$$0=w(t, 0)=w(t, z)-z\frac{\partial w}{\partial z}(t, z)+\cdots+\frac{(-1)^r}{r!}z^r\frac{\partial^r w}{\partial z^r}(t, z)$$
$$+\frac{(-1)^{r+1}z^{r+1}}{(r+1)!}\frac{\partial^{r+1}w}{\partial z^{r+1}}(t, \xi).$$

Hence

$$\frac{\partial^r}{\partial z^r}\left(\frac{w}{z}\right)=\frac{1}{r+1}\frac{\partial^{r+1}}{\partial z^{r+1}}w(t, \xi),$$

which has a limit as the point (t, z) approaches Q_0^{**}.

We have thus shown that the partial derivatives of w/z, with respect to z alone, of order $\leq k$ have limits at Q_0^{**}.

We next show that the partial derivatives of w/z with respect to t alone have limits at Q_0^{**}. First note that the derivatives of w with respect to t alone vanish at $z=0$. For, $w(t, z)=v^*(s, h)$ so that

$$\frac{\partial v}{\partial s}=\frac{\partial v^*}{\partial s}=\frac{\partial w}{\partial t}+\frac{\partial w}{\partial z}\frac{\partial z}{\partial s}=\frac{\partial w}{\partial t}+\frac{\partial w}{\partial z}\frac{\partial G}{\partial s}$$

and, as we have seen, $\partial v/\partial s$ and $\partial G/\partial s$ vanish at $z=0$. Thus $\partial w/\partial t=0$ at $z=0$. Similarly, successive differentiation shows $\partial^r w/\partial t^r=0$ on γ_0^{**}, $0\leq r\leq k$.

We apply Taylor's theorem to obtain

$$\frac{\partial^r}{\partial t^r}\left(\frac{w}{z}\right)=\frac{1}{z}\frac{\partial^r w(t, z)}{\partial t^r}=\frac{1}{z}\left\{z\frac{\partial}{\partial z}\left[\frac{\partial^r w(t, \xi)}{\partial t^r}\right]\right\}=\frac{\partial}{\partial z}\frac{\partial^r w(t, \xi)}{\partial t^r},$$

$$0<\xi(z, r)<z$$

and conclude that $(\partial^r/\partial t^r)(w/z)$ has a limit at Q_0^{**} for $0 \leq r \leq k$,

Finally, any mixed derivative may be written as

$$\frac{\partial^{n+m}}{\partial z^n \partial t^m}\left(\frac{w}{z}\right), \qquad n+m=r\leq k$$

and this may be written as $\frac{\partial^n}{\partial z^n}\left\{\frac{1}{z}\frac{\partial^m w}{\partial t^m}\right\}$

where $\partial^m w/\partial t^m$ vanishes on γ_0^{**} and is of class c^{k-m} on $N^{**}(Q_0)$ and of class c^{k-m+1} on $(N^{**}(Q_0)-\gamma_0^{**})+Q_0^{**}$. By the first results for derivatives with respect to z, the mixed derivatives have the desired property.

THEOREM 1. *Let R be a bounded region whose boundary Γ consists of a finite number of non-intersecting simply closed regular arcs of class c^k, $(k\geq 2)$. Let $G(x, y)$ be a function of class c^k on $R+\Gamma$, vanishing on Γ, positive in R, with $\partial G/\partial \nu \geq \delta > 0$ on Γ.*

Let H be the Hilbert space formed by completing the linear vector space V of variations—functions of class c^k on $\bar{R}$ and vanishing on Γ—, using the functional

$$D[\xi]=\iint(a\xi_x^2+b\xi_y^2+c\xi^2)dx\,dy$$

for $\xi \in V$ as the square of the norm, where a, b, c are bounded and integrable, $a>0$, $b>0$, $c\geq 0$ in $R+\Gamma$.

Then the set of functions $G\tau$, where τ is a polynomial in x and y, is dense in H. The set $\{f_i\}$ obtained by orthonormalizing the set $\{Gx^i y^j\}$ is complete in H.

If $g(x, y)$ is a function of class c^k on $\bar{R}$ and Ψ is the set of functions ψ of class c^k on $\bar{R}$, assuming the values of $g(x, y)$ on Γ, and if for any $\psi \in \Psi$ we define $b_i = D[\psi - g, f_i]$, then

$$\left\|\psi-g-\sum_{i=1}^{n} b_i f_i\right\|^2 = D\left[\psi-g-\sum_{i=1}^{n} b_i f_i\right] \leq \frac{\theta(n)}{n^{k-2}}$$

where $\lim_{n\to\infty}\theta(n)=0$, *$\theta$ depending on $\psi-g$.*

In particular, if f is integrable,

$$I[\psi]=\iint(a\psi_x^2+b\psi_y^2+c\psi^2+2f\psi)dx\,dy,$$

and there exists an admissible function ψ_0 which minimizes $I[\psi]$ for $\psi \in \Psi$, and we define

$$a_i = -\iint f f_i \, dx\, dy - D[g, f_i], \quad \phi_n = g + \sum_{i=1}^{n} a_i f_i ,$$

then

$$\| \psi_0 - \phi_n \|^2 = D[\psi_0 - \phi_n] \leq \frac{\theta(n)}{n^{k-2}}$$

where $\lim_{n\to\infty} \theta(n) = 0$.

Proof. If v is a variation, we show there is a sequence Q_j of polynomials such that

$$\lim_{j\to\infty} \| v - G\, Q_j \|^2 = \lim_{j\to\infty} \iint [a(v-G\,Q_j)_x^2 + b(v-G\,Q_j)_y^2 + c(v-G\,Q_j)^2]\, dx\, dy = 0.$$

In view of Lemma 4, v/G is of class c^{k-1} on $\overline{R}$ and it is thus possible to extend the definition of v/G over the entire plane so that it is of class c^{k-1} over the entire plane. Let $\Omega(\delta)$ be the uniform modulus of continuity of the $(k-1)$st partial derivatives of v/G over a rectangle with sides parallel to the axes containing R in its interior.

By Lemma 2, with $s=2$, $j=1$, $m_1=m_2=j$ there is a sequence Q_j of polynomials of degree $2j$ in x and y such that, for (x, y) in $\overline{R}$,

$$\left|\frac{v}{G} - Q_j\right|, \quad \left|\left(\frac{v}{G}\right)_x - Q_{j_x}\right|, \text{ and } \left|\left(\frac{v}{G}\right)_y - Q_{j_y}\right| \text{ are all } O\left(\frac{1}{j^{k-2}}\,\Omega\left(\frac{1}{j}\right)\right).$$

Hence

$$(v-G\,Q_j)_x^2 = \left[\left(\frac{v}{G} - Q_j\right) G\right]_x^2 = \left[\left(\frac{v}{G} - Q_j\right)_x G + \left(\frac{v}{G} - Q_j\right) G_x\right]^2$$

$$\leq \left(\frac{v}{G} - Q_j\right)_x^2 G^2 + G_x^2\left(\frac{v}{G} - Q_j\right)^2 + 2G|G_x|\left|\left(\frac{v}{G} - Q_j\right)_x\right|\left|\frac{v}{G} - Q_j\right|$$

$$= O\left(\frac{1}{j^{2(k-2)}}\left[\Omega\left(\frac{1}{j}\right)\right]^2\right).$$

A similar result is true for $(v-G\,Q_j)_y^2$ and $(v-G\,Q_j)^2$. Thus $\lim_{j\to\infty} D[v - G\,Q_j] = 0$ for $k \geq 2$.

It has thus been proved that the linear manifold formed by $\{Gx^iy^j\}$ is dense in V and thus in H. By the previous discussion the set $\{f_i\}$ is complete in H.

Now let v in the above be the particular variation $\psi - g$ and let $[N]$ represent the largest integer $\leq N$. For fixed n, let $j = [(\sqrt{n}/2) - 1]$ and $\tau_n(x, y) = Q_j(x, y)$. Thus there is a sequence τ_n of degree at most

$$2j \leq \left[2\left(\frac{\sqrt{n}}{2} - 1\right)\right] = [\sqrt{2n} - 2]$$

such that

$$D[\psi-g-G\tau_n]=O\left(\frac{1}{n^{k-2}}\Omega^2\left(\frac{8}{\sqrt{n}}\right)\right).$$

Now $\sum_{i=1}^{n} b_i f_i = G\mu_n$ where μ_n is a polynomial of degree greater than $\sqrt{2n}-2$, and it is known that

$$\left\| \psi-g-\sum_{i=1}^{n} c_i f_i \right\|$$

is a minimum when $c_i=(\psi-g,\ f_i)=b_i$. Thus

$$D\left[\psi-g-\sum_{i=1}^{n} b_i f_i\right]=O\left(\frac{1}{n^{k-2}}\theta\right), \quad \lim_{n\to\infty}\theta=0.$$

In particular, if ψ_0 minimizes $I[\psi]$, then we have seen that

$$D[\psi_0-g,\ f_i]=-\iint f f_i\,dx\,dy-D[g,\ f_i].$$

Thus, in this case, the Fourier coefficients depend only on known quantities.

COROLLARY $$b_n=O\left(\sqrt{\frac{\theta(n)}{n^{k-2}}}\right).$$

Proof. $$\begin{aligned} b_n^2=D[b_n f_n]&=D\left[\left(\psi-g-\sum_{i=1}^{n-1} b_i f_i\right)-\left(\psi-g-\sum_{i=1}^{n} b_i f_i\right)\right] \\ &\le D\left[\psi-g-\sum_{i=1}^{n-1} b_i f_i\right]+2\left(D\left[\psi-g-\sum_{i=1}^{n-1} b_i f_i\right]\right. \\ &\quad\left.\cdot D\left[\psi-g-\sum_{i=1}^{n} b_i f_i\right]\right)^{1/2}+D\left[\psi-g-\sum_{i=1}^{n} b_i f_i\right]=O\left(\frac{\theta(n)}{n^{k-2}}\right). \end{aligned}$$

§ 3. **Expansion Theorems.** We use the notations in Theorem 1 and seek conditions which insure that convergence in H yields uniform convergence in $\bar{R}$.

THEOREM 2. *Let R be a bounded region with boundary Γ. Let ψ, ψ_n be continuous on $\bar{R}$, absolutely continuous on each line in $\bar{R}$ and all taking on the same values on Γ. Let $D[\psi]<\infty$, $D[\psi_n]<\infty$. If $\lim_{n\to\infty} D[\psi-\psi_n]=0$, then a necessary and sufficient condition that $\lim_{n\to\infty}\psi_n=\psi$ uniformly on $\bar{R}$ is that ψ_n be equicontinuous on $\bar{R}$. If $\lim_{n,m\to\infty} D[\psi_n-\psi_m]=0$ then a necessary and sufficient condition that $\lim_{n\to\infty}\psi_n$ exists uniformly on $\bar{R}$ is that ψ_n be equicontinuous on $\bar{R}$.*

Proof. The necessity is clear since a sequence of continuous functions which converge uniformly are equicontinuous.

Let $u(x, y)$ be a function with the continuity properties of $\psi(x, y)$ and vanishing on Γ. Let P_0 be a point interior to R. Place polar coordinates at P_0. If a ray from P_0 meets the circle S_ρ of radius $\rho \leq d$, d being the diameter of R, with P_0 as center, before it meets Γ, label P_1 the first intersection point with S_ρ and Q the first intersection with Γ. Otherwise both P_1 and Q will refer to the first intersection point of the ray and Γ.

$$\frac{1}{2\pi}\left\{\int_0^{2\pi}|u(P_1)|\,d\theta\right\}^2 \leq \int_0^{2\pi} u^2(P_1)d\theta = \int_0^{2\pi}\left[\int_{P_1}^{Q}\frac{\partial u}{\partial r}dr\right]^2 d\theta$$

$$= \int_0^{2\pi}\left[\int_{P_1}^{Q}\frac{1}{\sqrt{r}}\sqrt{r}\,\frac{\partial u}{\partial r}dr\right]^2 d\theta \leq \int_0^{2\pi}\log\frac{d}{\rho}\int_{P_1}^{Q} r\left(\frac{\partial u}{\partial r}\right)^2 dr\, d\theta$$

$$\leq \log\frac{d}{\rho}\iint (u_x^2+u_y^2)dx\,dy \leq \alpha\log\frac{d}{\rho}D[u]$$

where $\alpha = 1/\min\limits_{R+\Gamma}(a, b)$, since

$$\iint (au_x^2+bu_y^2+cu^2)dx\,dy \geq \min(a,\ b)\iint (u_x^2+u_y^2)dx\,dy\ .$$

Apply this result to the functions $u_n = \psi - \psi_n$ (or to $u_{nm} = \psi_n - \psi_m$) which are equicontinuous on $R+\Gamma$ and thus have a uniform modulus of continuity $\omega(\delta)$, which approaches 0 with δ.

Since P_1 is on or interior to the circle of radius ρ, we have $|u_n(P_1) - u_n(P_0)| \leq \omega(\rho)$, whence $|u_n(P_1)| \geq |u_n(P_0)| - \omega(\rho)$ and

$$2\pi[|u_n(P_0)| - \omega(\rho)] \leq \sqrt{2\pi\alpha\log d/\rho\, D[u_n]}\ .$$

Thus

$$|u_n(P_0)| \leq \sqrt{\frac{\alpha}{2\pi}D[u_n]\log\frac{d}{\rho}} + \omega(\rho)\ ,$$

which is true even if P_0 is on Γ.

Now, for $\varepsilon > 0$, choose $\rho = \rho_1$ so small that $\omega(\rho_1) < \varepsilon/2$ and then choose N so large that

$$\frac{\alpha}{2\pi}D[u_n]\log\frac{d}{\rho_1} < \frac{\varepsilon^2}{4}$$

for $n > N$. Hence

$$\varepsilon > 0 \cdot\supset\cdot \exists\, N(\varepsilon) \ni n > N \cdot\supset\cdot |\psi(P_0) - \psi_n(P_0)| < \varepsilon\ .$$

LEMMA 5. *Let R be a bounded region with boundary Γ and diameter d.*

Let $u(x, y)$ *be continuous on* $R+\Gamma$, *absolutely continuous on each line in* $R+\Gamma$, *and vanish on* Γ, *and let* $0<D[u]<\infty$. *Let* $\alpha=1/\min_{R+\Gamma}(a, b)$. *Let* $P_0 \in R+\Gamma$. *If there exists* $\delta>0$, $K\geq 0$ *and*

$$|u(P)-u(P_0)|\leq K\,\|P-P_0\|^\delta$$

for all points P *such that the ray* P_0P *is in* $R+\Gamma$, *then*

$$|u(P_0)|\leq\sqrt{\frac{\alpha D[u]}{2\pi\delta}\log^+\frac{d^\delta K}{\Delta D[u]}}+\Delta D[u]$$

where Δ *is any number* >0, *and*

$$\log^+x=\begin{cases}\log x & \textit{if} \quad x>1\\ 0 & \textit{if} \quad x\leq 1\,.\end{cases}$$

Proof. If P_0 is interior to R, and $\rho\leq d$, then as in Theorem 2

$$\frac{1}{2\pi}\left\{\int_0^{2\pi}|u(P_1)|\,d\theta\right\}^2\leq \alpha\log d/\rho\, D[u],$$

where P_1 is a point which is the first intersection of a ray from P_0 with either Γ or the circle of radius $\rho\leq d$ about P_0 as center.

Since P_1 is on or interior to the circle of radius ρ, we have

$$|u(P_1)-u(P_0)|\leq K\rho^\delta,\quad |u(P_1)|\geq|u(P_0)|-K\rho^\delta,$$

$$2\pi[u(P_0)-K\rho^\delta]\leq\sqrt{2\pi\alpha\log d/\rho\, D[u]},$$

$$|u(P_0)|\leq\sqrt{\frac{\alpha}{2\pi}D[u]\log\frac{d}{\rho}}+K\rho^\delta,$$

which holds even if P_0 is on Γ.

Let $\Delta>0$. If

$$\left(\frac{\Delta D[u]}{K}\right)^{1/\delta}<d,$$

choose

$$\rho=\left(\frac{\Delta D[u]}{K}\right)^{1/\delta}$$

to obtain

$$|u(P_0)|\leq\sqrt{\frac{\alpha}{2\pi\delta}D[u]\log\frac{d^\delta K}{\Delta\, D[u]}}+\Delta\, D[u]\,.$$

Otherwise,

$$\left(\frac{\Delta\, D[u]}{K}\right)^{1/\delta}\geq d,\qquad K\leq\frac{\Delta\, D[u]}{d^\delta}$$

and we may replace K to obtain

$$|u(P_0)| \leq \sqrt{\frac{\alpha}{2\pi} D[u] \log \frac{d}{\rho}} + \frac{\Delta D[u]}{d^\delta} \rho^\delta .$$

Choose $\rho = d$ to obtain $|u(P_0)| \leq \Delta D[u]$.

COROLLARY 1. *A sufficient condition that a sequence u_n, absolutely continuous on each line in $\bar{R}$, vanishing on Γ, continuous on $\bar{R}$, and having $\lim_{n\to\infty} D[u_n]=0$, converge to 0 at P_0 is that $\exists\, \delta > 0$ and a sequence K_n, with $\lim_{n\to\infty} D[u_n] \log K_n = 0$ such that*

$$|u_n(p) - u_n(P_0)| \leq K_n \| P - P_0 \|^\delta$$

for all P with ray P_0P in $\bar{R}$. If δ, K_n are independent of P_0, the convergence is uniform. In any case,

$$|u_n(P_0)| \leq \sqrt{\frac{\alpha}{2\pi\delta} D[u_n] \log^+ \frac{d^\delta K_n}{\Delta_n D[u_n]}} + \Delta_n D[u_n]$$

for any $\Delta_n > 0$.

LEMMA 6. *Let R be a bounded domain with boundary Γ. Let $P_0 \in \bar{R}$ and suppose there is a circle of radius ε lying in $\bar{R}$ and containing P_0. Place polar coordinates (r, θ) at P_0. Let $u(x, y)$ be of class c^1 in $\bar{R}$ and suppose that there exist $\lambda > 0$, $\sigma \geq 0$ such that*

$$|u_r(P) - u_r(P_0)| \leq \sigma \| P - P_0 \|^\lambda$$

for all points P such that the ray P_0P is in $\bar{R}$.
Then

$$|\nabla u(P_0)|^2 \leq (\sigma^2 D[u]^\lambda)^{1/(\lambda+1)} \left(\frac{\alpha}{\lambda\pi}\right)^{\lambda/(\lambda+1)} 2^{(5\lambda+3)/(\lambda+1)} (\lambda+1) + \frac{8\alpha D[u]}{\pi\varepsilon^2} \left(\frac{\lambda+1}{\lambda}\right) .$$

Proof. $$|u_r(P_0)| \leq |u_r(P)| + \sigma r^\lambda$$

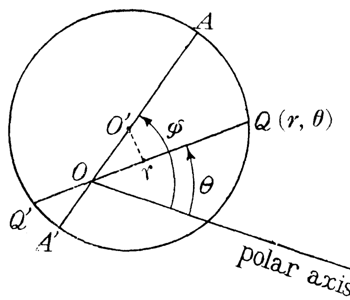
Integrating over a circle S_ρ of radius $\rho \leq \varepsilon$ which contains P_0, $S_\rho \subset S_\varepsilon$, we obtain

$$\iint_{S_\rho} |u_r(P_0)|^2 r\, dr\, d\theta \leq 2 \iint_{S_\rho} u_r(P)^2 r\, dr\, d\theta + 2 \iint_{S_\rho} \sigma^2 r^{2\lambda} r\, dr\, d\theta .$$

We may assume that the polar axis lies in the direction of $\nabla u(P_0)$. Hence $u_r(P_0) = |\nabla u(P_0)| \cos\theta$ and

$$\iint_{S_\rho} |\nabla u(P_0)|^2 (\cos^2\theta) r\, dr\, d\theta < 2\alpha D[u] + 2\sigma^2 (2\rho)^{2\lambda} \cdot \pi\rho^2 .$$

We will show that the minimum value of $\iint_{S_\rho}(\cos^2\theta)r\,dr\,d\theta$ is $\pi\rho^2/4$. Suppose first that the pole O is interior to S_ρ. Let $r(\theta)$ be the equation of the circle relative to the pole O. Let Q be the point $(r(\theta), \theta)$ and Q' the point $(r(\theta+\pi), \theta+\pi)$. Q and Q' are thus the intersections of a ray through O with the circle. Let O' be the center of the circle and suppose the coordinates of O' relative to O are (c, ϕ). Then the angle between OQ and OO' is $\phi-\theta$. Drop a perpendicular from O' to QOQ' hitting the latter at T, the length of OT being $|c\cos(\phi-\theta)|$. Thus one of the lengths $\|OQ\|$, $\|OQ'\|$ is $m+|c\cos(\phi-\theta)|$ and the other is $m-|c\cos(\phi-\theta)|$ where $2m$ is the length of QQ', and the product $\|OQ'\|\cdot\|OQ\|=m^2-c^2\cos^2(\phi-\theta)$. Also, if OO' meets the circle in points A, A' it is easily seen that $\|OA'\|\,\|OA\|=\|OQ\|\,\|OQ'\|$ so that $(\rho+c)(\rho-c)=m^2-c^2\cos^2(\phi-\theta)$ and $m^2=\rho^2-c^2+c^2\cos^2(\phi-\theta)$. Hence



$$\|OQ\|^2+\|OQ'\|^2=[m+|c\cos(\phi-\theta)|]^2+[m-|c\cos(\phi-\theta)|]^2$$
$$=2m^2+2c^2\cos^2(\phi-\theta)=2\rho^2-2c^2+4c^2\cos^2(\phi-\theta).$$

We note that

$$\iint_{S_\rho}(\cos^2\theta)r\,dr\,d\theta=\frac{1}{2}\int_0^{2\pi}r^2(\theta)\cos^2\theta\,d\theta=\frac{1}{2}\int_0^{\pi}(\|OQ\|^2+\|OQ'\|^2)\cos^2\theta\,d\theta$$
$$=\frac{1}{2}\int_0^{\pi}[2\rho^2-2c^2+4c^2\cos^2(\phi-\theta)]\cos^2\theta\,d\theta\ .$$

Moreover this formula holds even if O is a point on the circumference for in this case

$$\iint_{S_\rho}(\cos^2\theta)\,r\,dr\,d\theta=1/2\int_\gamma^{\gamma+\pi}r^2(\theta)\cos^2\theta\,d\theta$$

where γ is the angle between the polar axis and the tangent to the circle at O in that direction which has the area to the left of the tangent line. Here $r^2=[2\rho\cos(\phi-\theta)]^2$ and since the square of the cosine has period π, the integral reduces to

$$\frac{1}{2}\int_0^{\pi}4\rho^2\cos^2(\phi-\theta)\cos^2\theta\,d\theta\ .$$

Thus, in any case,

$$\iint_{S_\rho} (\cos^2\theta) r\,dr\,d\theta = \frac{1}{2}\int_0^\pi [2\rho^2 - 2c^2 + 4c^2\cos^2(\phi-\theta)]\cos^2\theta\,d\theta$$

$$= \frac{1}{2}\pi\rho^2 - \frac{1}{2}\pi c^2 + \frac{\pi c^2}{4}[1 + 2\cos^2\phi]\,.$$

For fixed c, the minimum is obtained when $\phi = \frac{\pi}{2}$, and is $\frac{\pi\rho^2}{2} - \frac{1}{4}\pi c^2$. The absolute minimum is obtained when $c=\rho$ and is $\pi\rho^2/4$.

It follows from this result, that

$$\frac{\pi\rho^2}{4}|\nabla u(P_0)|^2 \leq 2\alpha D[u] + 2\sigma^2(2\rho)^{2\lambda}\pi\rho^2\,,$$

$$|\nabla u(P_0)|^2 \leq \frac{8\alpha D[u]}{\pi\rho^2} + 2^{2\lambda+3}\sigma^2\rho^{2\lambda}\,.$$

Consider the function $y = A/\rho^2 + B\rho^{2\lambda}$ where $A = 8\alpha D[u]/\pi$, $B = 2^{2\lambda+3}\sigma^2$. The minimum value is

$$y_{\min} = A^{\lambda/(\lambda+1)}B^{1/(\lambda+1)}(\lambda+1)\lambda^{-\lambda/(\lambda+1)}$$
$$= (\sigma^2 D[u]^\lambda)^{1/(\lambda+1)}\left(\frac{\alpha}{\lambda\pi}\right)^{\lambda/(\lambda+1)}(\lambda+1)\,2^{(5\lambda+3)/(\lambda+1)}$$

obtained when

$$\rho = \left(\frac{A}{B\lambda}\right)^{1/(2\lambda+2)} = \left(\frac{\alpha D[u]}{\lambda\pi\sigma^2 2^{2\lambda}}\right)^{1/(2\lambda+2)}.$$

If

$$\left(\frac{\alpha D[u]}{\lambda\pi\sigma^2 2^{2\lambda}}\right)^{1/(2\lambda+2)} \leq \varepsilon\,.$$

choose

$$\rho = \left(\frac{\alpha D[u]}{\lambda\pi\sigma^2 2^{2\lambda}}\right)^{1/(2\lambda+2)}$$

and have

$$|\nabla u(P_0)|^2 \leq (\sigma^2 D[u]^\lambda)^{1/(\lambda+1)}\left(\frac{\alpha}{\pi\lambda}\right)^{\lambda/(\lambda+1)} 2^{(5\lambda+3)/(\lambda+1)}(\lambda+1).$$

However, if

$$\left(\frac{\alpha D[u]}{\lambda\pi\sigma^2 2^{2\lambda}}\right)^{1/(2\lambda+2)} > \varepsilon\,,$$

we have

$$\sigma^2 \leq \frac{\alpha D[u]}{2^{2\lambda}\lambda\pi\varepsilon^{2\lambda+2}}$$

and integrating over S_ε, as in the beginning of this proof, we find that

$$\iint_{S_\varepsilon} |u_r(P_0)|^2 r\,dr\,d\theta \leq 2\iint_{S_\varepsilon} u_r(P)^2 r\,dr\,d\theta + 2\frac{\alpha\,D[u]}{2^{2\lambda}\lambda\pi\varepsilon^{2\lambda+2}}(2\varepsilon)^{2\lambda}\pi\varepsilon^2,$$

$$\frac{\pi\varepsilon^2}{4}|\nabla u(P_0)|^2 \leq 2\alpha D[u] + \frac{2\alpha\,D[u]}{\pi\lambda\varepsilon^2}\cdot\pi\varepsilon^2,$$

$$|\nabla u(P_0)|^2 \leq \frac{8\alpha\,D[u]}{\pi\varepsilon^2} + \frac{8\alpha\,D[u]}{\lambda\pi\varepsilon^2} = \left(1+\frac{1}{\lambda}\right)\frac{8\alpha\,D[u]}{\pi\varepsilon^2}.$$

Thus, in any case,

$$|\nabla u(P_0)|^2 \leq (\sigma^2 D[u]^\lambda)^{1/(\lambda+1)}\left(\frac{\alpha}{\lambda\pi}\right)^{\lambda/(\lambda+1)} 2^{(5\lambda+3)/(\lambda+1)}(\lambda+1) + \frac{8\alpha\,D[u]}{\pi\varepsilon^2}\left(\frac{\lambda+1}{\lambda}\right).$$

LEMMA 7. *Let R be a bounded region with boundary Γ and diameter d and let R have the property that there exists an $\varepsilon > 0$ such that every point of $R+\Gamma$ is within some circle of radius ε lying in $R+\Gamma$.*

Let $u(x, y) = G\tau + H$ where τ is a polynomial of degree m, G and H are of class c^1 on $R+\Gamma$ and vanish on Γ, $G > 0$ in R, $|\nabla G| \geq \delta > 0$ on Γ. Let $|G| \leq G_1$, $|H| \leq H_1$, $|\nabla G| \leq G_2$, $|\nabla H| \leq H_2$ for constants G_1, G_2, H_1, H_2.

Suppose also that

$$|G_x(P) - G_x(P_0)| \leq G_0 \| P - P_0 \|, \quad |G_y(P) - G_y(P_0)| \leq G_0 \| P - P_0 \|,$$

$|H_x(P) - H_x(P_0)| \leq H_0 \| P - P_0 \|$, $|H_y(P) - H_y(P_0)| \leq H_0 \| P - P_0 \|$ *for constants* G_0, H_0, *whenever* P, P_0 *are points in* $\bar{R}$ *such that the line* P_0P *is in* $\bar{R}$. *Let A be an upper bound for $D[u]$ and $D[u]\log m$.*

Then there exists a constant B, depending only on α, A, G_0, G_1, G_2, H_0, H_1, H_2, δ, ε, d, G but not on m or τ, such that for $P_0 \in \bar{R}$.

$$|u(P_0)| \leq \sqrt{\frac{2\alpha}{\pi} D[u] \log^+ \frac{m}{\Delta\, D[u]}} + B(\Delta\, D[u])^4 (D[u])^{1/4}$$

for any $\Delta > 0$. (m to be replaced by 1 if it is 0).

Proof. We may assume $D[u] > 0$ for otherwise $u = 0$ in $\bar{R}$.

Let $L = \max_{\bar{R}} |\tau|$. By a theorem of Kellogg [3], $|\nabla\tau(P)| \leq Lm^2/\varepsilon$ for $P \in \bar{R}$.

If P and P_0 are on a straight line in $\bar{R}$, then

$$|\tau(P) - \tau(P_0)| = \left|\int_{P_0}^{P} \frac{\partial\tau}{\partial r} dr\right| \leq \frac{Lm^2}{\varepsilon} \| P - P_0 \|.$$

$$|H(P) - H(P_0)| \leq H_2 \| P - P_0 \|, \quad |G(P) - G(P_0)| \leq G_2 \| P - P_0 \|,$$

$$|u(P) - u(P_0)| \leq |G(P)\tau(P) - G(P)\tau(P_0)| + |G(P)\tau(P_0) - G(P_0)\tau(P_0) + |H(P) - H(P_0)|$$

$$\leq\left(G_1\frac{Lm^2}{\varepsilon}+LG_2+H_2\right)\|P-P_0\|=K\|P-P_0\|.$$

By Lemma 5, with $\varDelta=D[u]^{-1/2}$,

$$(2)\qquad |u(P_0)|\leq\sqrt{\frac{\alpha}{2\pi}D[u]\log^+\frac{dK}{\sqrt{D[u]}}}+\sqrt{D[u]}\,.$$

Also, τ_x, τ_y are polynomials of degree m and absolute value less than or equal to Lm^2/ε, so that $|\nabla\tau_x|\leq(Lm^2/\varepsilon)(m^2/\varepsilon,)$ and

$$|\tau_x(P)-\tau_x(P_0)|\leq\int_{P_0}^{P}|\nabla\tau_x|dr\leq\frac{Lm^4}{\varepsilon^2}\|P-P_0\|.$$

Thus

$$|\nabla\tau(P)-\nabla\tau(P_0)|\leq\frac{2Lm^4}{\varepsilon^2}\|P-P_0\|.$$

Then

$$\begin{aligned}
&|\nabla u(P)-\nabla u(P_0)|\\
&\quad\leq|G(P)\nabla\tau(P)-G(P_0)\nabla\tau(P_0)|+|\tau(P)\nabla G(P)-\tau(P_0)\nabla G(P_0)|+|\nabla H(P)-\nabla H(P_0)|\\
&\quad\leq|G(P)\nabla\tau(P)-G(P)\nabla\tau(P_0)|+|G(P)\nabla\tau(P_0)-G(P_0)\nabla\tau(P_0)|\\
&\qquad+|\tau(P)\nabla G(P)-\tau(P)\nabla G(P_0)|+|\tau(P)\nabla G(P_0)-\tau(P_0)\nabla G(P_0)|\\
&\qquad+|H_x(P)-H_x(P_0)|+|H_y(P)-H_y(P_0)|\\
&\quad\leq\left(G_1\frac{2Lm^4}{\varepsilon^2}+\frac{Lm^2}{\varepsilon}G_2+L\,2G_0+G_2\frac{Lm^2}{\varepsilon}+2H_0\right)\|P-P_0\|=\sigma\|P-P_0\|.
\end{aligned}$$

Whence Lemma 6 yields

$$(3)\qquad |\nabla u(P_0)|\leq\sqrt{(\sigma^2D[u])^{1/2}\left(\frac{\alpha}{\pi}\right)^{1/2}32+\frac{16\alpha\,D[u]}{\pi\varepsilon^2}}\,.$$

By use of inequalities (2) and (3) we now find a bound for L.

Either $L\leq1$ or else there exist constants c_1, c_2 such that $K< c_1Lm^2$, $\sigma<c_2Lm^4$ where the factor m is to be omitted if it is zero, and c_1, c_2 depend only on ε, G_1, G_2, H_2, H_0, G_0.

Assume $L>1$. Since $|\nabla G|\neq0$ on Γ, there exists a continuous curve (or curves) γ dividing $\bar{R}$ into two closed sets $\bar{R}_1$ and $\bar{R}_2$, such that $\bar{R}_1\bar{R}_2=\gamma$, $\bar{R}_1$ being a boundary set where $|\nabla G|\geq\delta/2>0$, and $\bar{R}_2$ the set separated from Γ by γ. There is a constant c_3 such that $G(P)\geq c_3>0$ for $P\in\bar{R}_2$.

Suppose first that $|\tau|$ assumes its maximum L at a point $P_0\in\bar{R}_2$. Then, by (2),

$$|G(P_0)\tau(P_0)+H(P_0)|\leq\sqrt{\frac{\alpha}{2\pi}D[u]\log^+\frac{d\,c_1Lm^2}{\sqrt{D[u]}}}+\sqrt{D[u]}$$

or,

$$(4) \qquad L \leq \frac{1}{c_3}\left[H_1 + \sqrt{\frac{\alpha}{2\pi} D[u] \log^+ \frac{d\, c_1 L m^2}{\sqrt{D[u]}}} + \sqrt{D[u]}\right].$$

Since $D[u] \log m$ and $D[u]$ are bounded by A, equation (4) implies the existence of a constant c_4 depending on c_3, c_1, d, A, α, H_1 such that $L < c_4$.

On the other hand, if $|\tau|$ assumes its maximum L at a point $P_0 \in \bar{R}_1$, write

$$\nabla u = G \nabla \tau + \tau \nabla G + \nabla H \quad \tau^2 \nabla G = \tau \nabla u - u \nabla \tau + H \nabla \tau - \tau \nabla H,$$

$$\tau^2 |\nabla G| \leq |\tau|\ |\nabla u| + |u|\ |\nabla \tau| + |H|\ |\nabla \tau| + |\tau|\ |\nabla H|$$

$$\leq L\sqrt{(c_2^2 L^2 m^8 D[u])^{1/2}\left(\frac{\alpha}{\pi}\right)^{1/2} 32 + \frac{16\alpha\, D[u]}{\pi\varepsilon^2}}$$

$$+ \frac{Lm^2}{\varepsilon}\left(\sqrt{\frac{\alpha}{2\pi} D[u] \log^+ \frac{d\, c_1 L m^2}{\sqrt{D[u]}}} + \sqrt{D[u]}\right) + H_1 \frac{Lm^2}{\varepsilon} + LH_2 .$$

Therefore,

$$(5) \qquad L \leq \frac{2}{\delta}\left\{\sqrt{32\,(c_2^2 L^2 m^8 D[u])^{1/2}\left(\frac{\alpha}{\pi}\right)^{1/2} + \frac{16\alpha\, D[u]}{\pi\varepsilon^2}}\right.$$

$$\left. + \frac{m^2}{\varepsilon}\left(\sqrt{\frac{\alpha\, D[u]}{2\pi} \log^+ \frac{d\, c_1 L m^2}{\sqrt{D[u]}}} + \sqrt{D[u]}\right) + \frac{H_1 m^2}{\varepsilon} + H_2\right\}.$$

This inequality, which is of the form

$$L \leq K_1 + K_2 m^2 + K_3 m^2 \sqrt{\log L} + K_4 m^2 \sqrt{L},$$

shows that

$$\sqrt{L} \leq \frac{K_1}{\sqrt{L}} + \frac{K_2 m^2}{\sqrt{L}} + K_3 m^2 \sqrt{\frac{\log L}{L}} + K_4 m^2 \leq K_1 + K_2 m^2 + K_3 m^2 + K_4 m^2,$$

since $L > 1$, whence $L \leq c_5 m^4$ for some constant c_5.

Thus, in any case, there is a constant c_6 such that $L < c_6 m^4$, where the factor m is to be omitted if it is zero. From this one can conclude that $K < c_1 c_6 m^6$. However, we may obtain a better estimate by noticing that K merely serves as a number such that $|u(P) - u(P_0)| \leq K \| P - P_0 \|$ whenever P and P_0 are on a straight line in $\bar{R}$. Hence K may be replaced by $\sup_{\bar{R}} |\nabla u|$.

The inequality $\sigma < c_2 L m^4 < c_2 c_6 m^8$ and formula (3) yield

$$|\nabla u(P_0)| \leq \sqrt{c_2 c_6 m^8 D[u]^{1/2}\left(\frac{\alpha}{\pi}\right)^{1/2} \cdot 32 + \frac{16\alpha D[u]^{1/2} D[u]^{1/2}}{\pi\varepsilon^2}} = c_7 D[u]^{1/4} m^4 ,$$

since $D[u]\leq A$. Thus we may replace K by $c_7 D[u]^{1/4}m^4$ and substitute in Lemma 5 to obtain

$$|u(P_0)|\leq\sqrt{\frac{\alpha}{2\pi}D[u]\log^+\frac{dc_7D[u]^{1/4}m^4}{\Delta D[u]}}+\Delta D[u]\ .$$

Let $\Delta=dc_7\Delta_1^4D[u]^{13/4}$ and $B=dc_7$ to obtain the conclusion.

LEMMA 8. *Let R, G, have the properties in Lemma 7 and let $u=G\tau$ where τ is a polynomial of degree m.*

Then $D[u]\geq c_{12}|u(P_0)|^2/\log m$ where $c_{12}>0$ is a constant depending only on G_0, G_1, G_2, d, ε, α, δ, G. The factor $\log m$ is to be omitted if $m=0$ or 1.

Proof. Whether $L\leq 1$ or not, the formulas for K, σ show that $K<c_1Lm^2$, $\sigma<c_2Lm^4$. Moreover, either formula (4) or (5) holds, with $H_1=0$, $H_2=0$. If (4) holds, we have

$$\frac{L}{\sqrt{D[u]}}\leq\frac{1}{c_3}\left[\sqrt{\frac{\alpha}{2\pi}\log^+\frac{dc_1Lm^2}{\sqrt{D[u]}}}+1\right].$$

Let $w=L/\sqrt{D[u]}$. The above inequality is then of the form $w\leq K_1\sqrt{\log wm^2}+K_2$ whence $L/\sqrt{D[u]}\leq c_8\log m$ for some constant c_8, depending on α, c_3, d, c_1. Here the factor $\log m$ is to be omitted if $m=0$ or 1. On the other hand, if (5) holds, we have

$$\frac{L}{\sqrt{D[u]}}\leq\frac{2}{\delta}\left\{\sqrt{c_2m^4\frac{L}{\sqrt{D[u]}}\left(\frac{\alpha}{\pi}\right)^{1/2}\cdot 32}+\sqrt{\frac{16\alpha}{\pi\varepsilon^2}}\right.$$
$$\left.+\frac{m^2}{\varepsilon}\left(\sqrt{\frac{\alpha}{2\pi}\log^+dc_1m^2\frac{L}{\sqrt{D[u]}}}+1\right)\right\},$$

from which we conclude $L/\sqrt{D[u]}\leq c_9m^4\log m$ (m and $\log m$ to be omitted if $m=0$ or 1).

Thus, in any case, there is a constant c_{10} such that $L/\sqrt{D[u]}\leq c_{10}m^5$. Therefore

$$\frac{K}{\sqrt{D[u]}}\leq\frac{c_1Lm^2}{\sqrt{D[u]}}\leq c_1c_{10}m^7\ .$$

Substituting in equation (2), we have

$$|u(P_0)|\leq\sqrt{\frac{\alpha}{2\pi}D[u]\log^+dc_1c_{10}m^7}+\sqrt{D[u]}\leq c_{11}\sqrt{\log m}\sqrt{D[u]}\ ,$$

m to be omitted if it is 0 or 1.

THEOREM 3. *Let R be a bounded region whose boundary Γ consists of a finite number of simply closed regular arcs of class c^k, $k \geq 3$. Let $G(x, y)$ be a function of class c^k on $R+\Gamma$, vanishing on Γ, positive in R, with $\partial G/\partial\nu \geq \delta > 0$ on Γ. Let f_i be the set obtained by orthonormalizing the set $\{Gx^i y^j\}$ using the functional*

$$D[\xi]=\iint_R (a\xi_x^2+b\xi_y^2+c\xi^2)dx\,dy$$

as the square of the norm, where a, b, c are bounded and integrable, $a>0$, $b>0$, $c \geq 0$ on $R+\Gamma$. Let $g(x, y)$ be any function of class c^k on $R+\Gamma$. Let $\psi(x, y)$ be any function of class c^k on $R+\Gamma$ assuming the values of $g(x, y)$ on Γ. Define $b_i=D[\psi-g, f_i]$.

Then

$$\left|\psi-g-\sum_{i=1}^{n} b_i f_i\right|=O\left(\sqrt{\frac{\sqrt{\theta(n)}}{n^{k-2}}\log\frac{n}{(\log n)^N}}\right),$$

where

$$D\left[\psi-g-\sum_{i=1}^{n} b_i f_i\right] \leq \frac{\theta(n)}{n^{k-2}},$$

with $\lim_{n\to\infty}\theta(n)=0$, θ depending on $\psi-g$, and where N is any fixed constant >0. Moreover, if $k \geq 10$, then

$$\left|\nabla\psi-\nabla\left(g+\sum_{i=1}^{n} b_i f_i\right)\right|=O\left(\left[\frac{\theta(n)}{n^{k-10}}\right]^{1/4}\right).$$

Finally, if S is any closed domain in R, $k \geq 7$, then for points P in S,

$$\left|\nabla\psi-\nabla\left(g+\sum_{i=1}^{n} b_i f_i\right)\right|=O\left(\left[\frac{\theta(n)\log n}{n^{k-6}}\right]^{1/4}\right).$$

Proof. Let $u_n=\psi-g-\sum_{i=1}^{n} b_i f_i$. Then u_n is of the form $G\tau_n+H$ where the degree m_n of τ_n is less than $\sqrt{2n}-2$ and greater than $\sqrt{2n}-2$. By Theorem 1, $D[u_n] \leq \theta(n)/n^{k-2}$, $k \geq 3$, where $\lim_{n\to\infty}\theta(n)=0$ so that $D[u_n]\log m_n \leq A$ for some constant A independent of n. By Lemma 7,

$$|u_n(P_0)| \leq \sqrt{\frac{2\alpha}{\pi} D[u_n]\log^+\frac{m_n}{\Delta_n D[u_n]}}+B(\Delta_n D[u_n])^4 D[u_n]^{1/4}$$

for any $\Delta_n>0$.

$$|u_n| \leq \sqrt{\frac{2\alpha}{\pi} D[u_n]\log^+\frac{\sqrt{2n}}{\Delta_n D[u_n]}}+B(\Delta_n D[u_n])^4 D[u_n]^{1/4}.$$

There is a constant E, depending on N, such that $1/n < E/(\log n)^N e$, $n \geq 3$. Then

$$D[u_n] \leq \frac{\theta(n)}{n^{k-2}} \leq \frac{\theta(n)E^{k-2}}{e^{k-2}(\log n)^{N(k-2)}} = \frac{\sqrt{2n}}{e^{k-2}\Delta_n} \leq \frac{\sqrt{2n}}{e\Delta_n}$$

if

$$\Delta_n = \frac{\sqrt{2n}(\log n)^{N(k-2)}}{\theta(n)E^{k-2}} .$$

The function $x \log(\sqrt{2n}/\Delta x)$ is monotone increasing for $0 \leq x \leq \sqrt{2n}/e\Delta$ so that we may replace $D[u_n]$ by $\theta(n)/n^{k-2}$ to obtain

$$\begin{aligned} |u_n| \leq & \sqrt{\frac{2\alpha}{\pi} \frac{\theta(n)}{n^{k-2}} \log^+ \frac{\theta(n)E^{k-2}}{(\log n)^{N(k-2)}\theta(n)/n^{k-2}}} \\ & + B\left(\frac{\sqrt{2n}(\log n)^{N(k-2)}}{\theta(n)E^{k-2}} \frac{\theta(n)}{n^{k-2}}\right)^4 \left(\frac{\theta(n)}{n^{k-2}}\right)^{1/4} \\ = & O\left(\sqrt{\frac{\sqrt{\theta(n)}}{n^{k-2}} \log \frac{n}{(\log n)^N}}\right) . \end{aligned}$$

In the proof of Lemma 7, we saw that $L < c_6 m^4$ and $\sigma < c_2 L M^4 < c_2 c_6 m^8$. Hence by equation (3) of Lemma 7,

$$|\nabla u_n| \leq \sqrt{c_2 c_6 m_n^8 \left(\frac{\theta(n)}{n^{k-2}}\right)^{1/2} \left(\frac{\alpha}{\pi}\right)^{1/2} \cdot 32 + \frac{16\alpha}{\pi\varepsilon^2} \frac{\theta(n)}{n^{k-2}}} .$$

Since $m_n < \sqrt{2n}$, we obtain the statement of the theorem regarding uniform convergence in $\bar{R}$ of $|\nabla u_n|$ for $k \geq 10$.

Next, let S be any closed domain in R. We may suppose the boundary Γ' of S is sufficiently smooth so that a circle of radius ε may be rolled around Γ' while lying in S. Let $L_n' = \sup_S |\tau_n|$ and $P_0^{(n)}$ be the point in S where $L_n' = |\tau_n(P_0^{(n)})|$. As in the proof of Lemma 7,

$$|\nabla u_n(P_0)| \leq \sqrt{(\bar{\sigma}_n^2 D[u_n])^{1/2} \left(\frac{\alpha}{\pi}\right)^{1/2} \cdot 32 + \frac{16\alpha D[u_n]}{\pi\varepsilon^2}} \qquad \text{for } P_0 \in S$$

where

$$\bar{\sigma}_n = G_1 \frac{2L_n' m_n^4}{\varepsilon^2} + \frac{L_n' m_n^2}{\varepsilon} G_2 + L_n' 2G_0 + \frac{G_2 L_n' m_n^2}{\varepsilon} + 2H_0 .$$

Using $G\tau_n$ as the function u of Lemma 8 defined over $\bar{R}$ and remembering that $G\tau_n = -\sum_{i=1}^{n} b_i f_i$, we obtain

$$|G(P_0^{(n)})L_n'| \leq \sqrt{D\left[\sum_{i=1}^{n} b_i f_i\right] \log m_n} \Big/ c_{12} \ .$$

In S, $|G(P)| \geq c_{13} > 0$. Also

$$D\left[\sum_{i=1}^{n} b_i f_i\right] = \sum_{i=1}^{n} b_i^2 \leq \sum_{i=1}^{\infty} b_i^2 = D[\psi - g] \ .$$

Therefore $L_n' < c_{14}\sqrt{\log n}$, $\bar{\sigma}_n < c_{15} n^2 \sqrt{\log n}$, and

$$|\nabla u_n(P_0)| \leq \sqrt{c_{15} n^2 \sqrt{\log n} \left(\frac{\theta(n)}{n^{k-2}}\right)^{1/2} \left(\frac{\alpha}{\pi}\right)^{1/2} \cdot 32 + \frac{16\alpha\theta(n)}{\pi\varepsilon^2 n^{k-2}}}$$

$$= O\left(\left[\frac{\theta(n)\log n}{n^{k-6}}\right]^{1/4}\right) .$$

THEOREM 4. *Let R, Γ, G, f_i be defined as in Theorem 3. Then there is a constant c_{17} such that whenever $P_0 \in \bar{R}$, then*

$$\sum_{K=1}^{n} f_K^2(P_0) \leq c_{17} \log n \ .$$

The theorem is true if P_0 is a point where $f_1, \cdots, f_n$ all vanish, in particular on Γ. Let P_0 be a point in R where not all f_K, $K=1, \cdots, n$ vanish. Consider the problem of minimizing $D[u]$, where u is of the form $u = \sum_{K=1}^{n} c_K f_K$, under the condition $u(P_0) = T \neq 0$. Now

$$D[u] = D\left[\sum_{K=1}^{n} c_K f_K\right] = \sum_{K=1}^{n} c_K^2 \ ,$$

so that we must minimize $\sum_{K=1}^{n} c_K^2$ under the condition $\sum_{K=1}^{n} c_K f_K(P_0) = T$. By Lagrange multipliers we find a necessary condition for a minimum to be

$$c_K = \bar{c}_K = \frac{T f_K(P_0)}{\sum_{j=1}^{n} f_j^2(P_0)} \ ,$$

and the function $\bar{u} = \sum_{K=1}^{n} \bar{c}_K f_K$ satisfies

$$D[\bar{u}] = T^2 \Big/ \sum_{K=1}^{n} f_K^2(P_0) \ .$$

This is actually a minimum value, for, if $u = \sum_{K=1}^{n} c_K f_K$, then

$$T^2 = \left[\sum_{K=1}^{n} c_K f_K(P_0)\right]^2 \leq \sum_{K=1}^{n} c_K^2 \sum_{K=1}^{n} f_K^2(P_0)$$

so

$$\frac{T^2}{\sum_{K=1}^{n} f_K^2(P_0)} \leq \sum_{K=1}^{n} c_K^2 = D[u] .$$

Now $\bar{u}$ is of the form $G\,\tau_n$ where τ_n has the degree of f_n and this degree is less than $\sqrt{2n-2}$.

By Lemma 8, we have $D[\bar{u}] \geq c_{12} T^2 / \log \sqrt{2n-2}$.
Hence

$$\frac{T^2}{\sum_{K=1}^{n} f_K^2(P_0)} = D[\bar{u}] \geq \frac{c_{12} T^2}{\log \sqrt{2n-2}} ,$$

$$\sum_{K=1}^{n} f_K^2(P_0) \leq \frac{1}{c_{12}} \log \sqrt{2n-2} .$$

4. An Associated Problem. As in the previous sections, let R be a bounded region whose boundary Γ consists of a finite number of simply closed regular arcs of class c^k, $k \geq 3$; $G(x, y)$ be a function of class c^k on $R+\Gamma$, vanishing on Γ, positive in R, with $\partial G/\partial \nu \geq \delta > 0$ on Γ; $g(x, y)$ be any function of class c^k on $R+\Gamma$; a variation be a function of class c^k on $R+\Gamma$ vanishing on Γ.

Let

$$D^p[\xi] = \iint_R (a\xi_x^2 + b\xi_y^2 + c\xi^2)^p dx\, dy ,$$

where $a > 0$, $b > 0$, $c \geq 0$ on $R+\Gamma$; a, b, and c are bounded and integrable on $\bar{R}$; p is a real number greater than or equal to 1.

Assuming the existence of a function ψ_0, yielding minimum value to $D^p[\psi]$ in the set of admissible functions of class c^k on $R+\Gamma$, which take the value of g on Γ, can we obtain ψ_0 by the Rayleigh Ritz method? This question is answered in the affirmative and an estimate is obtained for the rate of convergence.

Let $\|\xi\| = (D^p[\xi])^{1/2p}$, for ξ in the set of functions of class c^k on $R+\Gamma$. This functional has the properties $\|\xi\| \geq 0$, $\|a\xi\| = |a|\, \|\xi\|$ for real a, $\|\xi+\eta\| \leq \|\xi\| + \|\eta\|$.

The functional $\|\xi\|$ is a true norm in the linear space V of variations. Let H be the Banach space formed by completing V with respect to this norm. As in the proof of Theorem 1, we see that the set of functions $G\tau$, where τ is a polynomial in x and y, is dense in H. Moreover, if ψ is admissible, there exists a sequence of polynomials Q_j of degree at most j such that

$$\|\psi-g-GQ_j\|=D^p[\psi-g-GQ_j]^{1/2p}=O\left(\frac{\theta(j)}{j^{k-2}}\right),$$

where θ depends on $\psi-g$ and $\lim_{j\to\infty}\theta(j)=0$.

There exists $\inf D^p[\psi]\geq 0$ for admissible ψ. Let τ_j be a polynominal of degree at most j which makes $D^p[g+G\tau_j]\leq D^p[g+GQ_j]$ for all polynomials Q_j of degree at most j.

That such a polynomial τ_j exists can be seen as follows. The class of all functions GQ_j where Q_j is a polynominal of degree at most j is also the linear manifold determined by $f_i=GT_i$, the orthonormal sequence of Theorem 1, whose polynomial factor T_i is of degree at most j. As stated in the introduction, $1\leq i\leq j\binom{j+1}{2}+j+1=\sigma$ so that we may write $GQ_j=\sum_{i=1}^{\sigma}c_if_i$. Now let Q_j' be any fixed Q_j. We may restrict ourselves to those Q_j such that $D^p[g+GQ_j]\leq D^p[G+GQ_j']$. For such Q_j we have

$$\|g\|+\|g+GQ_j'\|\geq\|g\|+\|g+GQ_j\|\geq\|GQ_j\|.$$

Since $D[\xi]\leq D^p[\xi]^{1/p}|R|^{1/q}$ where $(1/p)+(1/q)=1$, $|R|=$area of R, we find that

$$|R|^{1/q}[\|g\|+\|g+GQ_j'\|]^2\geq D[GQ_j]=D\left[\sum_{i=1}^{\sigma}c_if_i\right]=\sum_{i=1}^{\sigma}c_i^2.$$

Thus the permissible c_i lie in a bounded closed set S in σ-dimensional space. Since

$$D^p[g+GQ_j]=D^p\left[g+\sum_{i=1}^{\sigma}c_if_i\right]$$

is a continuous function of c_i in S, it attains its minimum in S.

Since $D^p[g+G\tau_j]$ is a decreasing function of j, we have

$$\lim_{j\to\infty}\|g+G\tau_j\|\leq\liminf_{j\to\infty}\|g+GQ_j\|.$$

Let ψ be admissible and choose Q_j so that $\lim_{j\to\infty}D^p[\psi-g-GQ_j]=0$. Then $\|g+GQ_j\|\leq\|\psi\|+\|\psi-g-GQ_j\|$ implies that $\liminf_{j\to\infty}\|g+GQ_j\|\leq\|\psi\|$. It follows that $\lim_{j\to\infty}\|g+G\tau_j\|\leq\|\psi\|$ for every admissible ψ and thus $g+G\tau_j$ is a minimizing sequence.

If $c>0$ in a set of positive measure in R, the functional $\|\xi\|$ is a true norm in the linear space (c^k) of functions of class c^k on $R+\Gamma$. If $c=0$, a.e. in R, this is still true provided we identify functions differing by a constant. In either case we will complete the space (c^k) to form a Banach space B.

A set S in a normed linear space is uniformly convex if there exists a continuous monotone increasing function $g(\varepsilon)$, $0\leq\varepsilon<1$, with $\lim_{\varepsilon\to\infty} g(\varepsilon)=0$, such that whenever ξ, η are in S and $\|\xi\|=\|\eta\|=1$, $\|(\xi+\eta)/2\|\geq 1-\varepsilon$, then $\|\xi-\eta\|\leq g(\varepsilon)$.

We shall show (c^k) is uniformly convex. It is easily verified that if α, β are ≥ 0, and $p\geq 1$ then

$$3\alpha^p+\beta^p\leq 2|\alpha+\beta|^p+|\alpha-\beta|^p.$$

Apply the inequality to the integrand below, where we assume ϕ and ψ are in (c^k), $\|\phi\|=\|\psi\|=1$, $\|(\phi+\psi)/2\|\geq 1-\varepsilon$.

$$\iint 3\left[a\left(\frac{\phi+\psi}{2}\right)_x^2+b\left(\frac{\phi+\psi}{2}\right)_y^2+c\left(\frac{\phi+\psi}{2}\right)^2\right]^p$$

$$+\left[a\left(\frac{\phi-\psi}{2}\right)_x^2+b\left(\frac{\phi-\psi}{2}\right)_y^2+c\left(\frac{\phi-\psi}{2}\right)^2\right]^p dx\,dy$$

$$\leq\frac{2}{2^p}\iint[(a\phi_x^2+b\phi_y^2+c\phi^2)+(a\psi_x^2+b\psi_y^2+c\psi^2)]^p dx\,dy$$

$$+\iint|a\phi_x\psi_x+b_y\phi_y\psi_y+c\phi\psi|^p dx\,dy$$

$$\leq\frac{2}{2^p}\left\{\left(\iint(a\phi_x^2+b\phi_y^2+c\phi^2)^p dxdy\right)^{1/p}+\left(\iint_R(a\psi_x^2+b\psi_y^2+\psi c^2)^p dx\,dy\right)^{1/p}\right\}^p$$

$$+\iint(\sqrt{a\phi_x^2+b\phi_y^2+c\phi^2}\sqrt{a\psi_x^2+b\psi_y^2+c\psi^2})^p dx\,dy$$

$$\leq 2+\sqrt{\iint(a\phi_x^2+b\phi_y^2+c\phi^2)^p dx\,dy}\sqrt{\iint(a\psi_x^2+b\psi_y^2+c\psi^2)^p dx\,dy}=3$$

Hence

$$\iint\left(a\left(\frac{\phi-\psi}{2}\right)_x^2+b\left(\frac{\phi-\psi}{2}\right)_y^2+c\left(\frac{\phi-\psi}{2}\right)^2\right)^p dx\,dy\leq 3-3(1-\varepsilon)^{2p},$$

and

$$\|\phi-\psi\|=\left[\iint\left(a(\phi-\psi)_x^2+b(\phi-\psi)_y+c(\phi-\psi)^2\right)^p dx\,dy\right]^{1/2p}$$

$$\leq 2(3[1-(1-\varepsilon)^{2p}])^{1/2p}\leq 2(3^{1/2p})(2p\varepsilon)^{1/2p}=g(\varepsilon)$$

for $\varepsilon<1$, since the function $y=[1-(1-x)^{2p}]-2px$ vanishes at 0 and is a decreasing function of x for $0\leq x<1$.

LEMMA 9. *Let B be a Banach space, Y a set in B with the property that if y_1, y_2 are in Y, then so is $(y_1+y_2)/2$. Let the linear manifold spanned by Y be a uniformly convex set in B. Let*

$$\rho = \inf_{y \in Y} \| y \| > 0,$$

let y_n *be a sequence in* Y *with*

$$\lim_{n\to\infty} \| y_n \| = \rho, \quad \rho_n = \| y_n \| .$$

Then there exists a unique x *in* B *such that* $\| x \| = \rho$ *and we have*

$$\| x - y_n \| \leq \rho g\left(\frac{\rho_n - \rho}{2\rho}\right) + \rho_n - \rho ,$$

where $g(\varepsilon)$ *is the function in the definition of uniform convexity. If* $\rho = \inf_{y \in Y} \| y \| = 0$, *and* $\lim_{n\to\infty} \| y_n \| = \rho$, *then there is a unique* x *in* B *such that* $\| x \| = \rho$, *and we have* $\| x - y_n \| = \rho_n - \rho$.

Proof. Let $z_n = y_n / \rho_n$ so that $\| z_n \| = 1$. Write

$$\frac{z_n + z_m}{2} - \frac{1}{\rho}\left(\frac{y_n + y_m}{2}\right) = \frac{y_n}{2}\left(\frac{1}{\rho_n} - \frac{1}{\rho}\right) + \frac{y_m}{2}\left(\frac{1}{\rho_m} - \frac{1}{\rho}\right).$$

$$\left\| \frac{z_n + z_m}{2} \right\| \geq \frac{1}{\rho} \left\| \frac{y_n + y_m}{2} \right\| - \frac{\| y_n \|}{2}\left(\frac{1}{\rho} - \frac{1}{\rho_n}\right) - \frac{\| y_m \|}{2}\left(\frac{1}{\rho} - \frac{1}{\rho_m}\right)$$

$$\geq \frac{1}{\rho} \cdot \rho - \frac{\rho_n}{2}\left(\frac{1}{\rho} - \frac{1}{\rho_n}\right) - \frac{\rho_m}{2}\left(\frac{1}{\rho} - \frac{1}{\rho_m}\right) = 1 - \frac{(\rho_n - \rho) + (\rho_m - \rho)}{2\rho} .$$

Hence

$$\| z_n - z_m \| \leq g\left(\frac{(\rho_n - \rho) + (\rho_m - \rho)}{2\rho}\right) \quad \text{for} \quad \frac{\rho_n - \rho + \rho_m - \rho}{2\rho} < 1 .$$

Thus there exists $z = \lim_{n\to\infty} z_n$ in B. Let $x = \rho z = \lim_{n\to\infty} \rho z_n = \lim_{n\to\infty} \rho_n z_n = \lim_{n\to\infty} y_n$. Then $\| x \| = \lim_{n\to\infty} \| y_n \| = \rho$. Also $\| z_n - z \| \leq g((\rho_n - \rho)/2\rho)$ implies

$$\| x - y_n \| = \| \rho z - \rho_n z_n \| \leq \| \rho z - \rho z_n \| + \| \rho z_n - \rho_n z_n \| \leq \rho g\left(\frac{\rho_n - \rho}{2\rho}\right) + \rho_n - \rho .$$

To show x is unique, suppose also $y'_n \in Y$, $\lim_{n\to\infty} \| y'_n \| = \rho$, $x' \in B$, $\| x' \| = \rho$, $x' = \lim_{n\to\infty} y'_n$. Then form the sequence $\{y''_n\} = y_1, y'_1, y_2, y'_2$, etc. of elements of Y with $\lim_{n\to\infty} y''_n = \rho$. As above, $\exists x'' \in B$ with $x'' = \lim_{n\to\infty} y''_n = \lim_{n\to\infty} y'_n = \lim_{n\to\infty} y_n$. The last part of the lemma is obvious, since only $\| 0 \| = 0$.

To apply the lemma, let B be the completion of (c^k), Y the set of admissible functions,

$$y_n = g + G\tau_n, \quad \rho = \inf D^p[\psi]^{1/2p} ,$$

for admissible ψ. By the lemma, there is a unique x such that $\| x \| =$

ρ. Assuming that $x=\psi_0$ is in Y, we can choose polynomials Q_j of degree at most j such that

$$\|\psi_0-g-GQ_j\|=O\left(\frac{\theta(j)}{j^{k-2}}\right).$$

Then

$$\rho_j-\rho=\|g+G\tau_j\|-\|\psi_0\|\leq\|g+GQ_j\|-\|\psi_0\|$$
$$=\|g+GQ_j-\psi_0+\psi_0\|-\|\psi_0\|\leq\|\psi_0-g-GQ_j\|=O\left(\frac{\theta(j)}{j^{k-2}}\right).$$

By the lemma,

$$\|\psi_0-g-G\tau_j\|\leq 2\rho(6p)^{1/2p}\left(\frac{\rho_j-\rho}{2\rho}\right)^{1/2p}+\rho_j-\rho=O\left(\frac{\theta(j)}{j^{k-2}}\right)^{1/2p},$$

a better result, $O(\theta(j)/j^{k-2})$, is obtained in the case $\rho=0$.

Since

$$D[u]<(D^p[u])^{1/p}|R|^{1/q},$$

where $|R|$ is the area of R and $(1/p)+(1/q)=1$, we find

$$D[u_j]\leq\left(\frac{\theta(j)}{j^{k-2}}\right)^{1/p},$$

where $\lim_{j\to\infty}\theta(j)=0$, when we take $u_j=\psi_0-g-G\tau_j$. A proof similar to that of Theorem 3 can now be constructed for the following result.

THEOREM 5. *Let R be a bounded region whose boundary Γ consists of a finite number of simply closed regular arcs of class c^k, $k\geq 3$. Let $G(x, y)$ be a function of class c^k on $R+\Gamma$, vanishing on Γ, positive in R,* with *$\partial G/\partial\nu\geq\delta>0$ on Γ. Let a, b, c be bounded and integrable on $\bar{R}$, and $a>0$, $b>0$, $c\geq 0$ on $\bar{R}$. Let $g(x, y)$ be any function of class c^k on $R+\Gamma$. Choose polynomials τ_j minimizing $D^p[g+GQ_j]$ in the set of all polynomials Q_j of degree at most j. Then, if ψ_0 yields minimum value to $D^p[\psi]$ for ψ in the set of functions of class c^k on $R+\Gamma$ assuming the values of g on Γ, we have*

$$|\psi_0-g-G\tau_j|=O\left(\sqrt{\left(\frac{\sqrt{\theta(j)}}{j^{k-2}}\right)^{1/p}\log\frac{j}{(\log j)^N}}\right),$$

where N is any fixed positive constant, $\theta(j)$ depends on ψ_0-g and $\lim_{j\to\infty}\theta(j)=0$.

If $k \geq 16p+2$, *then*

$$|\nabla\psi_0 - \nabla(g+G\tau_j)| = O\left(\left[\frac{\theta(j)}{j^{k-2-16p}}\right]^{1/4p}\right).$$

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UNIVERSITY OF MINNESOTA

THE FLEXURE OF A NON-UNIFORM BEAM

E. E. Jones

Summary. The flexure of a beam of non-uniform flexural rigidity and non-uniform loading is deduced by the use of the method of the Laplace transform, the results being in the form of a single equation involving integrals which are in a suitable form for evaluation, either numerically or otherwise. Two examples of practical importance are introduced to illustrate the method, and the results are also applied to determine the equation to the elastica of a beam supported by many rigid supports.

1. Introduction. The method of solution of linear differential equations by means of the Laplace transform was used by Jaegar [6] to deduce the deflection of a beam with concentrated loads along its length, the beam having uniform flexural rigidity and variable loading. These results were extended considerably by Thomson [10], who indicated that the Laplace transformation method led to the simplest approach to the beam problem. These results were obtained in the form of a single equation in terms of certain end conditions, and eliminated the necessity of determining the equations between points of discontinuity of load, and then connecting them at these points, [9], [1]. Thomson's results apply to problems concerning beams of uniform flexural rigidity, and in order to extend them to problems involving beams of varying and discontinuous cross-sectional inertia it was necessary to reduce these latter problems to the former by the introduction of an artificial modified loading of the beam, [4], [11]. This present paper indicates how the problem of the beam with non-uniform loading and flexural rigidity can be solved directly by the use of standard operational methods, an appeal being made only to well-known results in the calculus, [3, p. 257], [7, pp. 71, 82], [12].

It is assumed in this paper, that if $\mathscr{L}y(x)$ is the Laplace transform of $y(x)$, then

$$\mathscr{L}y(x)=\int_0^\infty e^{-px}y(x)dx,$$

and in conjunction with this the following theorem is also required:

$$\mathscr{L}y_1(x)\mathscr{L}y_2(x)=\mathscr{L}\int_0^x y_1(u)y_2(x-u)du,$$

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if these integrals exist.

The results of the subsequent analysis can also be put into more convenient forms by the introduction of the unit step-function, defined by

$$H(x-a)\begin{cases} =0\ , & x<a\ , \\ =1\ , & x>a\ . \end{cases}$$

2. **The beam** under consideration is assumed to have s sections, separated by the points x_n, $(n=1, 2, \cdots, s-1)$, the origin of coordinates being at one end of the beam, and the x-axis directed along the undistorted position of the beam. The y-axis is then taken in the direction vertically downwards, i.e. in the direction in which the gravitational forces act. The weight per unit length of the beam in the section $x_{n-1}<x<x_n$ is $w_n(x)$, and in order to simplify the notation, the flexural rigidity in this section is defined as $B_n^{-1}(x)$. The beam is subjected to m concentrated loads P_n, acting at the points X_n, $(n=1, 2, \cdots, m)$.

In order to avoid assumptions regarding the distribution of the concentrated loads along an element of the beam at the positions where they act, it is more convenient to deduce an expression for the shear force acting on a right section of the beam in terms of the forces acting on the beam. If z measures the bending moment at a point of the beam distant x from the origin, then $-dz/dx$ measures the shear force at this point. Assuming that z_1 is the value of dz/dx at the origin, then the shear force at a distance x from this origin is given by the differential equation

$$\frac{dz}{dx}=z_1+\phi(x)\ , \tag{2.1}$$

where

$$\phi(x)=\int_0^x w(u)du+\sum_{X_n<x} P(X_n)\ . \tag{2.2}$$

Here $\phi(x)$ is equal to an integral plus a step-function, and $P(X_n)\equiv P_n$. Any distributive loads can be included in w, which is a simply discontinuous function of x of the form

$$w(x)=\sum_{n=0}^{s-1}(w_{n+1}-w_n)H(x-x_n)\ ,$$

where $w_0=0$.
Equation (2.1) can easily be deduced by resolving all the forces acting on the length of beam between the origin and the point distant x

from this origin normally to the beam in the direction of the y-axis.

The Laplace transform of equation (2.1) is

$$(2.3) \qquad p\mathscr{L}z - z_0 = z_1/p + \int_0^\infty \phi(x)e^{-px}dx\ ,$$

it being sufficient to assume that $p>0$, since ϕ is bounded, and possesses a finite number of finite discontinuities in the range of integration. On rearranging equation (2.3),

$$\mathscr{L}z = \frac{z_0}{p} + \frac{z_1}{p^2} + \frac{1}{p}\int_0^\infty \phi(x)e^{-px}dx.$$

The inverse of this equation is determined by using the convolution integral, giving

$$z(x) = z_0 + z_1x + \int_0^x \phi(u)du\ .$$

On integrating by parts, this leads to

$$(2.4) \qquad z = z_0 + z_1x + [u\phi(u)]_0^x - \int_0^x ud\phi(u) = z_0 + z_1x + \int_0^x (x-u)d\phi(u)\ .$$

This equation expresses the bending moment z at a point of the beam in terms of a Stieltjes integral, [13, chap I], and thus can be interpreted in a series form.

From equation (2.2), by substituting for $\phi(x)$ into the integral involved in equation (2.4),

$$(2.5) \qquad \int_0^x (x-u)d\phi(u) = \int_0^x (x-u)w(u)du + \sum_{X_n<x} (x-X_n)P_n\ ,$$

since contributions to the integral from the step-function only occur when u passes through a point of discontinuity. Hence finally equation (2.4) takes the form

$$(2.6) \qquad z = z_0 + z_1x + \int_0^x (x-u)w(u)du + \sum_{n=1}^{m} P_n(x-X_n)H(x-X_n)\ ,$$

where the last term in equation (2.5) has been modified by the use of the unit step-function.

The deflection y at the point x of the beam is given by the differential equation

$$\frac{d^2y}{dx^2} = z(x)B(x)\ ,$$

where

$$B(x)=\sum_{n=0}^{s-1}(B_{n+1}-B_n)H(x-x_n)\ ,$$

with $B_0=0$. Here $B(x)$ is a simply discontinuous function of x, and $z(x)$ is defined by equation (2.6)

If $y_0=(y)_{x=0}$, and $y_1=(dy/dx)_{x=0}$, then by repeating the above process

$$\mathscr{L}y=\frac{y_0}{p}+\frac{y_1}{p^2}+\frac{1}{p^2}\mathscr{L}(zB)\ ,$$

whence

$$y=y_0+y_1x+\int_0^x(x-u)z(u)B(u)du\ , \tag{2.7}$$

using again the property of the convolution integral.

By combining equations (2.6) and (2.7), the deflection of the beam can be written in the more convenient form

$$\begin{aligned} y=y_0+y_1x+\int_0^x(x-u)(z_0+z_1u)Bdu+\int_0^x(x-v)Bdv\int_0^v(v-u)wdu \\ +\sum_{n=1}^{m}P_nH(x-X_n)\int_{X_n}^x(x-u)(u-X_n)Bdu\ . \end{aligned} \tag{2.8}$$

The integrals involved in this expression are all interpreted in the same manner, the range of integration is subdivided into intervals corresponding to the subdivisions of the functions B and w, thus

$$\int_0^x(x-u)Bdu=\sum_{n=0}^{r-1}\int_{x_n}^{x_{n+1}}(x-u)\,B_{n+1}du+\int_{x_r}^{x}(x-u)B_{r+1}du\ ,$$

when $x_r<x<x_{r+1}$, $(0\leq r\leq s-1)$. This integral may also be interpreted in the form

$$\sum_{n=0}^{s-1}H(x-x_n)\int_{x_n}^{x}(x-u)(B_{n+1}-B_n)du.$$

Similar expressions occur for the remaining integrals although greater care must be taken over the subdivision of the last two integrals of equation (2.8).

It follows from equation (2.7) that

$$\begin{aligned} \frac{dy}{dx}=y_1+\int_0^x(z_0+z_1u)Bdu+\int_0^x Bdv\int_0^v(v-u)wdu \\ +\sum_{n=1}^{m}P_nH(x-X_n)\int_{X_n}^x(u-X_n)Bdu\ . \end{aligned} \tag{2.9}$$

In any practical problem the values of the constants y_0, y_1, z_0, and z_1 can be deduced from the given end conditions, it being noticed that the equations apply along the whole length of the beam.

3. **The first example** illustrates the effect on the flexure of a beam of a variation in the flexural rigidity of the beam. The beam is assumed to have uniform loading w, and is freely supported at the same level at the ends $x=0$, l. The beam is subdivided and stepped in cross-section at the points x_n, $(n=1, 2, \cdots, 2s)$, so that $x_{2s+1}=l$, and these points are symmetrically placed with respect to the mid-point of the beam, such that

$$\left.\begin{array}{l} x_{2s-n+1}+x_n=l \ , \\ x_{2s-n+1}-x_n=l_n \ . \end{array}\right\}(n=0, 1, \cdots, s). \tag{3.1}$$

The flexural rigidity of the stepped beam is constant in each section, and is also symmetrically distributed, such that, in the usual notation, $B_{2s-n+1}=B_{n+1}$, $(n=0, 1, \cdots, s)$.

The deflection of the beam at a point distant x from one end, given by equation (2.8), is

$$\begin{aligned} y=y_1x+z_1\sum_{n=0}^{2s}(B_{n+1}-B_n)H(x-x_n)\int_{x_n}^{x} u(x-u)du \\ +\frac{w}{2}\sum_{n=0}^{2s}(B_{n+1}-B_n)H(x-x_n)\int_{x_n}^{x} v^2(x-v)dv \ , \end{aligned} \tag{3.2}$$

since $y_0=z_0=0$ at $x=0$, where $y=d^2y/dx^2=0$. Also $y=d^2y/dx^2=0$ at $x=l$, hence from equation (2.6), $z_1=-wl/2$, and from equation (3.2), after some reduction,

$$y_1=\frac{w}{24l}\sum_{n=0}^{2s}(B_{n+1}-B_n)(l+3x_n)(l-x_n)^3 \ .$$

The integrals of equation (3.2) are easily evaluated, and after substituting for y_1 and z_1, rearrangement leads to the final expression for the deflection

$$\begin{aligned} y=\frac{1}{24}wxB_1(l^3+x^3-2lx^2)+\frac{1}{48}wx\sum_{n=1}^{s}(B_{n+1}-B_n)(3l^2-l_n^2)l_n \\ -\frac{w}{384}\sum_{n=1}^{s}H(x-x_n)(B_{n+1}-B_n)(5l^2+4lx-4x^2-2l\,l_n+4xl_n-3l_n^{\,2})(2x-l+l_n)^2 \\ +\frac{w}{384}\sum_{n=1}^{s}H(x-x_{2s-n+1})(B_{n+1}-B_n)(5l^2+4lx-4x^2+2l\,l_n-4xl_n-3l_n^2)(2x-l-l_n)^2 . \end{aligned}$$

When $x=l/2$ this relation reduces to the result deduced by Hetényi, [5], using another method.

4. **The second example** refers to a cantilever beam clamped horizontally at the end $x=0$, free at $x=l$, and loaded linearly according to the relation $w=mx$, where m is a constant. The beam is subdivided and stepped in cross-section at x_n, $(n=1, 2, \cdots, s-1)$, in such a way that B_n is constant in each section, but increases in magnitude as n increases. A concentrated load acts at the mid-point X of the end section $x_{s-1} \leq x \leq x_s$.

The equation governing the deflection of the beam reduces to

$$y=\sum_{n=0}^{s-1} H(x-x_n)(B_{n+1}-B_n)\int_{x_n}^{x}(x-u)(z_0+z_1u)du$$

$$+m\sum_{n=0}^{s-1} H(x-x_n)(B_{n+1}-B_n)\int_{x_n}^{x}(x-v)dv\int_0^v(v-u)udu$$

$$+PH(x-X)\sum_{n=0}^{s-1}(B_{n+1}-B_n)\int_X^x(x-u)(u-X)du ,$$

since $y_0=y_1=0$ at $x=0$, where $y=dy/dx=0$.

When $x=x_s=l$, then $z=dz/dx=0$, hence from equations (2.6) and (2.1),

$$z_1l+z_0+P(l-X)+ml^3/6=0 ,$$
$$z_1+P+ml^2/2=0 .$$

Thus

$$z_0=PX+ml^3/3 , \quad \text{and} \quad z_1=-(P+ml^2/2) .$$

The deflection at any point x of the beam then becomes

$$y=\frac{1}{12}\sum_{n=0}^{s-1} H(x-x_n)(B_{n+1}-B_n)(x-x_n)^2\{2P(3X-x-2x_n)+ml^2(2l-x-2x_n)\}$$

$$+\frac{m}{120}\sum_{n=0}^{s-1} H(x-x_n)(B_{n+1}-B_n)(x^5-5xx_n^4+4x_n^5)$$

$$+\frac{P}{6}H(x-X)\sum_{n=0}^{s-1}(B_{n+1}-B_n)(x-X)^3 .$$

5. **When a beam is constrained** at various points along its length by means of rigid supports, the reactions at these points will occur in the equations for the flexture of the beam. It is thus necessary to eliminate, or at least to determine these reactions. A particular example will suffice to indicate the procedure. It is required to determine the form of the elastica of a beam of varying section clamped at each end, and supported at several points along its length, one of

these supports being a distance d out of alignment with the remainder.

There are m supports, one at each of the points X_r, $(r=1, 2, \cdots, m)$, the beam being divided into s sections at points x_r, $(r=1, 2, \cdots, s-1)$. The beam is clamped horizontally at $x=0$ and at $x=x_s=X_s$, and there $y=dy/dx=0$.

The following notation is introduced:

$$a_r=\int_0^{X_r}(X_r-u)Bdu \ , \qquad b_r=\int_0^{X_r}(X_r-u)uBdu \ ,$$

$$c_r=\int_0^{X_r}(X_r-v)Bdv\int_0^{v}(v-u)wdu, \quad d_{nr}=\int_{X_n}^{X_r}(X_r-u)(u-X_n)Bdu \ ,$$

the integrals being interpreted as in § 2.

If P_n, $(n=1, 2, \cdots, m)$, are the reactions at the supports, then from equations (2.8) and (2.9), at $x=0$, $y_0=y_1=0$, and at $x=x_s=X_s$, then

$$\begin{aligned} &z_0a_s+z_1b_s+c_s+\sum_{n=1}^{m}P_nd_{ns}=0 \ , \\ &z_0a'_s+z_1b'_s+c'_s+\sum_{n=1}^{m}P_nd'_{ns}=0 \ , \end{aligned} \tag{5.1}$$

where $a'_s=\int_0^{X_s}Bdu$, etc., i.e. the partial derivatives of the integrals with respect to x at $x=X_s$.

Solving equation (5.1) for z_0 and z_1 we obtain

$$z_0=f_s+\sum_{n=1}^{m}P_nF_{ns} \ , \qquad z_1=g_s+\sum_{n=1}^{m}P_nG_{ns} \ ,$$

where

$$f_s=(c_sb'_s-c'_sb_s)/(a'_sb_s-a_sb'_s) \ , \qquad g_s=(a_sc'_s-a'_sc_s)/(a'_sb_s-a_sb'_s) \ ,$$
$$F_{ns}=(b'_sd_{ns}-b_sd'_{ns})/(a'_sb_s-a_sb'_s) \ , \qquad G_{ns}=(a_sd'_{ns}-a'_sd_{ns})/(a'_sb_s-a_sb'_s) \ .$$

It is assumed that the supports are in line along $y=0$, with the exception of the support at the point (x_t, d). If $\delta_{rt}=1$ when $r=t$, and is zero when $r\neq t$, then $y_r=d\delta_{rt}$, and from equation (2.8),

$$-d\delta_{rt}-a_rf_s-b_rg_s-c_r=\sum_{n=1}^{m}\{a_rF_{ns}+b_rG_{ns}+d_{nr}H(n-r)\}P_n \ .$$

This equation can be written in the matrix form

$$p_{rn}P_n=q_r \ , \qquad (r, n=1, 2, \cdots, m), \tag{5.2}$$

where

$$p_{rn} = a_r F_{ns} + b_r G_{ns} + d_{nr} H(n-r) \ ,$$
$$q_r = -(d\delta_{rt} + a_r f_s + b_r g_s + c_r) \ .$$

The matrix equation can be solved for P_n by any of the standard methods [2, pp. 96–155], i.e. by an iterative process, or by forming a triangular matrix by premultiplying both sides of equation (5.2) by a suitable matrix and solving the resulting equations either directly or by considering the reciprocal matrix solution.

The elastica is determined by inserting the values of P_n in equation (2.8), since $y_0 = y_1 = 0$, and z_0 and z_1 are already known. The procedure is similar for other end conditions. When the reactions at the supports are known, it is also possible to determine the slope, the bending moment, and the shear stress at any point of the beam. All the integrals can be evaluated numerically, [8], or directly if the variation of B and w is in a simple form, and a tabular process can be readily set up.

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UNIVERSITY OF NOTTINGHAM, ENGLAND.

NOTE ON NONCOOPERATIVE CONVEX GAMES

HUKUKANE NIKAIDÔ AND KAZUO ISODA

1. Introduction. Nash's equilibrium-point theorem for many-person games can be approached by two methods: first, the Kakutani-type fixed-point theorem[1] is very useful for this game problem; second, in case of finite-dimensional multilinear payoffs, J. Nash himself has given an elegant procedure [7] which is directly based on Brouwer's fixed-point theorem. In a previous paper [10] one of us proved a general minimax theorem in making use of a procedure analogous to that of Nash. The present note is a continuation of this paper, and its main purpose is to offer further improvements of Nash's method so as to treat noncooperative many-person games played over infinite-dimensional convex sets, based on a generalization of von Neumann's symmetrization method[2] of game matrices. The results thus obtained contain further weakening of (especially topological) assumptions of the equilibrium-point theorem.

Next we shall discuss the equilibrium-point problem of some general noncooperative games by reducing them to suitable convex games. This will clarify the relevance of convex games to general games.

2. Definitions and notations. We mean by a *convex game* [3] a noncooperative n-person game with the following conditions:

a) The ith player's strategy space is a compact convex set X_i of a topological linear space E_i.

b) The ith player's payoff $K_i(x_1, \cdots, x_i, \cdots, x_n)$ is concave with respect to his own strategy variable $x_i \in X_i$.

c) The sum of payoffs $\sum_{i=1}^{n} K_i(x_1, \cdots, x_i, \cdots, x_n)$ is continuous over the cartesian product space $X_1 \otimes X_2 \otimes \cdots \otimes X_n$.

d) For each fixed x_i, $K_i(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)$ is a continuous

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[1] See [6], [4], [5], or [9]. A supplementary note to [9] will be published shortly.

[2] See G. W. Brown and J. von Neumann, *Solutions of games by differential equations* in [1], and D. Gale, H. W. Kuhn and A. W. Tucker, *On symmetric games* in [1].

function of the $(n-1)$-tuple $[x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n] \in X_1 \otimes \cdots \otimes X_{i-1} \otimes X_{i+1} \otimes \cdots \otimes X_n$ respectively.

REMARK In view of the usual classification of games in terms of total gains, c) may be of interest. Indeed, in case of constant-sum games, c) is automatically fulfilled. If all the payoffs are continuous over $X_1 \otimes \cdots \otimes X_n$, c) and d) are also fulfilled.

A point $[\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n] \in X_1 \otimes X_2 \otimes \cdots \otimes X_n$ is said to be an *equilibrium* point if the x_i-function $K_i(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_{i-1}, x_i, \hat{x}_{i+1}, \cdots, \hat{x}_n)$ assumes its maximum at $x_i = \hat{x}_i$ $(i=1, 2, \cdots, n)$.

REMARK The notion of equilibrium points first appeared in the celebrated work of Augustin Cournot (see [2]) and was investigated by him by means of differential calculus. But the contemporary concern about it is to see the existence of these points in the global sense by topological methods. The equilibrium-point problem under conditions a)–d) cannot, however, be treated by the Kakutani fixed-point theorem, since the required upper semi-continuity is not always assured in these cases. Thus, the proof in the following section may deserve some general attention.

3. Generalization of von Neumann's symmetrization and proof of the equilibrium-point theorem. To see the existence of equilibrium points for a convex game, we introduce an auxiliary function. To begin with, denote by

$$x = [x_1, x_2, \cdots, x_n], \qquad y = [y_1, y_2, \cdots, y_n]$$

two mutually independent variables with the same domain

$$X = X_1 \otimes X_2 \otimes \cdots \otimes X_n,$$

which is again compact and convex.

Next put

$$\Phi(x, y) = \sum_{i=1}^{n} K_i(y_1, y_2, \cdots, y_{i-1}, x_i, y_{i+1}, \cdots, y_n). \tag{1}$$

It is noted that $\Phi(x, y)$ is also concave with respect to $x \in X$. The importance of this function is clarified by:

LEMMA 3. 1. *A point*

$$\hat{y} = [\hat{y}_1, \hat{y}_2, \cdots, \hat{y}_n] \in X$$

is an equilibrium point for the given game, if and only if $\Phi(x, \hat{y})$ assumes *its maximum at* $x=\hat{y}$.

Proof. The necessity is obvious. If, conversely,

$$\Phi(\hat{y}, \hat{y}) \geqq \Phi(x, \hat{y})$$

for any $x \in X$, setting

$$x=[\hat{y}_1, \hat{y}_2, \cdots, \hat{y}_{i-1}, x_i, \hat{y}_{i+1}, \cdots, \hat{y}_n]$$

gives

$$K_i(\hat{y}_1, \hat{y}_2, \cdots, \hat{y}_{i-1}, \hat{y}_i, \hat{y}_{i+1}, \cdots, \hat{y}_n) \geqq K_i(\hat{y}_1, \cdots, \hat{y}_{i-1}, x_i, \hat{y}_{i+1}, \cdots, \hat{y}_n)$$

for any $x_i \in X_i$.

REMARK For a zero-sum two-person game, we have

$$\Phi(x, y)=K(x_1, y_2)-K(y_1, x_2)\ , \qquad \Phi(y, y)=0\ ,$$

where $K(x_1, x_2)$ is the payoff from player 2 to player 1. This implies the functional form of von Neumann's symmetrization procedure[3]. We shall later present an interpretation of this function with regard to player's behavior.

With this setup, we next prove:

THEOREM 3. 1. *A convex game always has at least one equilibrium point.*

Proof. By Lemma 3. 1., we have only to see the existence of a point $\hat{y} \in X$ such that $\Phi(\hat{y}, \hat{y}) \geqq \Phi(x, \hat{y})$ for any $x \in X$. Suppose the contrary were valid. Then, to each $y \in X$, there exists some $x \in X$ such that

$$(2) \qquad \Phi(y, y)<\Phi(x, y)\ .$$

Put $G_x=\{y; \Phi(y, y)<\Phi(x, y)\}$ then G_x is open by conditions c) and d), and

$$X \subset \bigcup_{x \in X} G_x$$

by (2). Hence, in view of the compactness of X, we can find a finite

[3] It is noted that $\Phi(x, y)$ does not provide a *real* generalization of von Neumann's symmetrization, since x_i's refer, in special cases, to mixed strategies. We can also construct, however, the function Φ in terms of pure strategies, and this will give a real generalization of von Neumann's method symmetrizing game matrices; instead of the cartesian product of mixed strategy spaces we must, then, consider the mixed strategies over the cartesian product of pure strategy spaces. But in either cases the *formal* procedures in constructing Φ are exactly the same.

set $A=\{a_1, a_2, \cdots, a_s\}\subset X$ such that

$$X\subset \bigcup_{j=1}^{s} G_{a_j} .$$

This implies $\Phi(y, y)<\max_j \Phi(a_j, y)$ for any $y\in X$. Now, put

$$f_j(y)=\max[\Phi(a_j, y)-\Phi(y, y), 0] \qquad (j=1, 2, \cdots, s) .$$

These s functions are all continuous by conditions c) and d), and satisfy $f_j(y)\geqq 0$, $\sum_{j=1}^{s} f_j(y)>0$ for any $y\in X$.

The continuous mapping

$$y\to \sum_{j=1}^{s} f_j(y)a_j\Big/\sum_{j=1}^{s} f_j(y) \tag{3}$$

maps X into the convex hull $C(A)$ of A and therefore in particular $C(A)$ into $C(A)$. Since $C(A)$ is homeomorphic to a compact convex set in a Euclidean space, there exists a fixed point by Brouwer's fixed-point theorem.

Denote by $\hat{y}$ one such point. We have then

$$\hat{y}=\sum_{j=1}^{s} f_j(\hat{y})a_j\Big/\sum_{j=1}^{s} f_j(\hat{y})\in C(A)\subset X .$$

But for such a j that $f_j(y)>0$, we have, by definition, $\Phi(a_j, \hat{y})>\Phi(\hat{y}, \hat{y})$. Since $\Phi(x, y)$ is x-concave, this implies $\Phi(\hat{y}, \hat{y})>\Phi(\hat{y}, \hat{y})$, which is a contradiction.

REMARK The foregoing proof is essentially a repetition of the argument in [10]; the application of this argument to many-person cases is made possible by the use of $\Phi(x, y)$. It should be noticed, however, that despite the generality of Theorem 3. 1, it does not contain the result of [10]. The main reason for this fact is: the quasi-concavity (see [10]) of the original payoff may be lost in constructing $\Phi(x, y)$. So the theorem in [10] needs separate discussion.

4. An interpretation of $\Phi(x, y)$. Lemma 3. 1 can be rewritten as follows: *An n-person game has an equilibrium point if and only if*

$$\min_{y\in X}\max_{x\in X}[\Phi(x, y)-\Phi(y, y)]=0 . \tag{4}$$

Now (4) may be interpreted in the following way: Suppose there are n persons $P_1, P_2, \cdots, P_n$. We consider the cases where all the persons $P_2, \cdots, P_n$ except P_1 cooperate. Denote the coalition consisting of only P_1 by Q_1 and that consisting of $P_2, P_3, \cdots, P_n$ by Q_2. Q_1 and Q_2 play n original games simultaneously, conforming to the following new rules: We denote these n games by $G_1, G_2, \cdots, G_n$, respectively. In G_i ($i=1$,

$2, \cdots, n)$, Q_1 participates in the n simultaneous games as the ith player, while Q_2 occupies all the other positions. Then

$$x=[x_1, x_2, \cdots, x_n]\in X$$

indicates the strategies of Q_1, and

$$y=[y_1, y_2, \cdots, y_n]\in X$$

indicates those of Q_2. If Q_1 chooses x and Q_2 chooses y, Q_2 pays to Q_1 the amount

$$K_i(y_1, \cdots, y_{i-1}, x_i, y_{i+1}, \cdots, y_n)$$

as the outcome of G_i. On the other hand, Q_1 pays to Q_2 the amount

$$\sum_{i=1}^{n} K_i(y_1, y_2, \cdots, y_n)$$

as the rent for gambling, after the game is over. Thus $\Phi(x, y)-\Phi(y, y)$ indicates the total gain of Q_1, while $\Phi(y, y)-\Phi(x, y)$ indicates that of Q_2. With the notion of this new zero-sum two-person game, (4) gives a criterion for the existence of equilibrium points for the original n-person game. If the given n-person game is constant sum, (4) is reduced to the more natural formula:

$$\min_{y\in X}\max_{x\in X}\Phi(x, y)=\pi ,$$

where π denotes the corresponding constant sum.

5. Reduction to convex games. In this section we assume E_i is a normed linear space. We further assume regarding the payoffs $H_i(x_1, x_2, \cdots, x_n)$ the following conditions:

(i) The x_i-function $H_i(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)$ is upper semi-continuous for each fixed $(n-1)$-tuple $[x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n]$.

(ii) The x_i-set

$$\{x_i\,;\ \max_{x_i\in X_i} H_i(x_1, \cdots, x_i, \cdots, x_n)=H_i(x_1, \cdots, x_i, \cdots, x_n)\}$$

is convex for each fixed $(n-1)$-tuple $[x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n]$.

(iii) The family $\{H_i(x_1, \cdots, x_i, \cdots, x_n)\,;\ x_i\in X_i\}$ is a uniformly equi-continuous family of functions on $X_1\otimes\cdots\otimes X_{i-1}\otimes X_{i+1}\otimes\cdots\otimes X_n$.

These games are usually treated by means of Kakutani's fixed-point theorem. We shall next, however, prove the following:

THEOREM 5. 1. *To each game of foregoing type there exists a convex game with the same strategy spaces whose equilibrium points are exactly those of the original game.*

As a direct application of Theorems 3.1 and 5.1 we can see the existence of equilibrium points for games of the foregoing type without Kakutani's theorem.

We now proceed to prove some lemmas.

Let R and S be normed linear spaces. We denote by $||x||$ the norm of a point $x \in R$. A continuous function $f(x)$ over R will be called linear if

$$f(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

for $x_1, x_2 \in R$, $\alpha_1 + \alpha_2 = 1$. We define the norm of f as usual:

$$||f|| = \sup_{||x|| \leqq 1} |f(x) - f(0)| .$$

Now, let $H(x, y)$ be a function on $X \otimes Y$, where X and Y are compact convex sets in R and S, respectively, and suppose that the family of functions $\{H(x, y) ; x \in X\}$ is uniformly equi-continuous.

Let further F_y be the totality of linear functions f over R such that (I) $||f|| \leqq 1$ and (II) $f(x) \geqq H(x, y)$ for any $x \in X$.

Putting

$$K(x, y) = \inf_{f \in F_y} f(x) ,$$

we obtain an x-concave function on $X \otimes Y$. We call $K(x, y)$ the x-concave envelope of $H(x, y)$. We shall show the continuity of this function by proving the following lemmas.

LEMMA 5. 1. $\{K(x, y) ; x \in X\}$ is a uniformly equi-continuous family of functions on Y.

Proof. Since $\{H(x, y) ; x \in X\}$ is uniformly equi-continuous, we can find for $\varepsilon > 0$ a $\delta > 0$ such that $||y_1 - y_2|| \leqq \delta$ implies $|H(x, y_1) - H(x, y_2)| \leqq \varepsilon$ for any $x \in X$. We shall show that, for this same δ, $||y_1 - y_2|| \leqq \delta$ implies $|K(x, y_1) - K(x, y_2)| \leqq \varepsilon$ for any $x \in X$.

Indeed, if $f \in F_{y_1}$, then

$$f(x) \geqq H(x, y_1) \geqq H(x, y_2) - \varepsilon$$

for all $x \in X$, and $||f + \varepsilon|| = ||f||$; namely, we have $f + \varepsilon \in F_{y_2}$.

In the same way, we have $g + \varepsilon \in F_{y_1}$ for $g \in F_{y_2}$.

Hence, if $||y_1 - y_2|| \leqq \delta$, we obtain

$$K(x, y_1) + \varepsilon = \inf_{f \in F_{y_1}} f(x) + \varepsilon = \inf_{f \in F_{y_1}} [f(x) + \varepsilon] \geqq \inf_{g \in F_{y_2}} g(x) = K(x, y_2) ,$$

and similarly $K(x, y_2)+\varepsilon \geqq K(x, y_1)$ for any $x \in X$. This means that $|K(x, y_1)-K(x, y_2)| \leqq \varepsilon$ for $y_1, y_2 \in Y$, $||y_1-y_2|| \leqq \delta$, and all $x \in X$.

LEMMA 5. 2. *$K(x, y)$ is continuous on X for each fixed $y \in Y$.*

Proof. Let y be an arbitrary fixed point in Y. If $||x-\hat{x}|| \leqq \varepsilon$, then $f \in F_y$ implies

$$|f(x)-f(\hat{x})| \leqq ||f|| \, ||x-\hat{x}|| \leqq \varepsilon .$$

It follows that

$$\left| \inf_{f \in F_y} f(x) - \inf_{f \in F_y} f(\hat{x}) \right| \leqq \varepsilon ,$$

proving the desired continuity.

LEMMA 5. 3. *$K(x, y)$ is continuous on $X \otimes Y$.*

Proof. We have this lemma immediately by taking Lemmas 5. 1 and 5. 2 together into consideration.

LEMMA 5. 4. *Suppose $H(x, y)$ is upper semi-continuous in x for each fixed $y \in Y$, and the x-set*

$$\Gamma_y = \{x \, ; \max_{x \in X} H(x, y) = H(x, y)\}$$

is a convex subset of X for each fixed $y \in Y$. Put

$$\Delta_y = \{x \, ; \max_{x \in X} K(x, y) = K(x, y)\} .$$

Then we have $\Gamma_y = \Delta_y$ for each fixed $y \in Y$.

Proof. Let y be any fixed point $\in Y$, and put

$$\omega_y = \max_{x \in X} H(x, y) .$$

Then the linear function $g(x) \equiv \omega_y$ belongs to F_y. Hence we have

$$H(x, y) \leqq K(x, y) = \inf_{f \in F_y} f(x) \leqq \omega_y$$

for all $x \in X$, which implies $\Gamma_y \subset \Delta_y$.

Conversely, by the above formula, it is obvious that if $\hat{x} \in \Delta_y$ then $K(\hat{x}, y) = \omega_y$. Thus, to see that $\Delta_y \subset \Gamma_y$, it suffices to show that $K(\hat{x}, y) < \omega_y$ for $\hat{x} \notin \Gamma_v$.

Let $\hat{x}$ be any point not belonging to Γ_y. Then

$$\text{dist}\,(\hat{x}, \Gamma_y)=2\alpha>0 .$$

Putting

$$M=\{x\,;\,\text{dist}\,(x, \Gamma_y)<\alpha\} ,$$

we obtain an open convex set M and a closed convex set $\overline{M}$ (the closure of M) in R. Moreover, it is clear that $\hat{x}\notin\overline{M}$.

Let $e(x)$ be such a linear function that $e(x)\geqq 0$ on $\overline{M}$ and $e(\hat{x})=-1$; its existence is a well-known fact (known as Mazur's theorem) in the theory of convex sets. Denote by N the complement of M within X; N is compact and, in view of the definition of M, we have

$$\omega_y-\max_{x\in N} H(x, y)=\gamma>0\,; \qquad \min_{x\in N} e(x)=\eta<0 .$$

Put

$$f(x)=\omega_y+\frac{\delta e(x)}{|\eta|} ,$$

where $\delta>0$ is so small that $\delta\leqq\gamma$ and $\delta\|e\|\leqq|\eta|$. Then $\|f\|\leqq 1$, $f(x)\geqq\omega_y$ on M, and

$$f(x)=\omega_y+\frac{\delta e(x)}{|\eta|}\geqq\omega_y+\frac{\gamma\eta}{|\eta|}=\omega_y-\gamma\geqq H(x, y)$$

for any $x\in N$.

Hence $f\in F_y$. Moreover,

$$f(\hat{x})=\omega_y+\frac{\delta e(\hat{x})}{|\eta|}=\omega_y-\frac{\delta}{|\eta|}<\omega_y ,$$

which means $K(\hat{x}, y)<\omega_y$, proving the lemma.

The proof of Theorem 5. 1. is now immediate. Indeed, let us construct the x_i-concave envelope $K_i(x_1, \cdots, x_i, \cdots, x_n)$ of

$$H_i(x_1, \cdots, x_i, \cdots, x_n) \qquad (i=1, 2, \cdots, n) .$$

Then $K_i(x_1, x_2, \cdots, x_n)$ is clearly x_i-concave, and is continuous by Lemma 5. 3. Thus, we obtain a convex game. Moreover, the set of equilibrium points of this game coincides with that of the original game, by Lemma 5. 4.

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TOKYO COLLEGE OF SCIENCE, TOKYO, JAPAN
TOKYO INSTITUTE OF TECHNOLOGY, TOKYO, JAPAN.

ON THE CONVERGENCE OF ASYMPTOTIC SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

R. M. REDHEFFER and W. WASOW

1. Introduction. In the differential equation

$$L[y, \varepsilon] \equiv y^{(n)} + \sum_{j=1}^{n} A_j(x, \varepsilon) y^{(n-j)} = 0 \tag{1}$$

let the coefficients $A_j(x, \varepsilon)$ be analytic functions of x and ε. For all values of x and ε for which these coefficients are holomorphic in both variables the differential equation admits a fundamental system of solutions with the same property. But if some coefficients of (1) have poles, as functions of ε, for a certain value of ε, say for $\varepsilon=0$, then the solutions of the differential equation will in general have singularities, as functions of ε, at $\varepsilon=0$. The purpose of this paper is to collect some observations on the question of when solutions holomorphic at $\varepsilon=0$ exist even in this case.

The theory of asymptotic integration of such differential equations [6], [8], [3], [9], [10] teaches that in this case there exist fundamental solutions which are asymptotically represented by generally divergent expansions of the form

$$e^{P(x,\varepsilon)} \sum_{\nu=0}^{\infty} y_\nu(x) \varepsilon^{\nu/r}, \tag{2}$$

where r is a positive integer and $P(x, \varepsilon)$ is a polynomial in $\varepsilon^{-1/r}$. Our problem might naturally be generalized to include the question of the convergence of any, or all, of these asymptotic series, whether $P(x, \varepsilon)$ be identically zero or not. But this will not be done here.

The analogous problem for differential equations without a parameter, at a point where the coefficients have a singularity has been quite thoroughly investigated (cf. [1, 486–489]). By contrast, there seem to exist no studies of corresponding questions for the dependence on a parameter, nor does it seem possible to transfer the results obtained for one problem to the other by an easy analogy. In view of this situation the results of this paper may be of some interest.

2. Necessary conditions. Let us assume that $A_j(x, \varepsilon)$ are of the form

$$A_j(x, \varepsilon) = \varepsilon^{-h} \sum_{k=0}^{\infty} A_{jk}(x) \varepsilon^k, \qquad (j=1, \cdots, n) \tag{3}$$

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where at least one $A_{j0}(x)$ is not identically zero and h is a positive integer. The series are supposed to converge when x is in a fixed region X of the x-plane, and for ε in a circle E: $|\varepsilon|\leq\varepsilon_0$, ε_0 being independent of x. The functions $A_{jk}(x)$ are to be holomorphic in X. In order to shorten the terminology the self-explanatory expressions x-holomorphic and ε-holomorphic will sometimes be used. A function that is holomorphic in both variables may be called (x, ε)-holomorphic.

The differential equation (1) can be rewritten in the from

$$\varepsilon^h L[y, \varepsilon]\equiv\varepsilon^h N[y, \varepsilon]+M[y, \varepsilon]=0 . \tag{4}$$

Here,

$$N[y, \varepsilon]=y^{(n)}+\sum_{\nu=1}^{n} a_\nu(x, \varepsilon)y^{(n-\nu)} \tag{4a}$$

$$M[y, \varepsilon]=\sum_{\mu=0}^{m} b_\mu(x, \varepsilon)y^{(m-\mu)} , \tag{4b}$$

and the $a_\nu(x, \varepsilon)$ and $b_\mu(x, \varepsilon)$ are (x, ε)-holomorphic in the product space of X and E. The $b_\mu(x, \varepsilon)$ are polynomials in ε of degree less than h. The coefficient $b_0(x, 0)$ is not identically zero. Furthermore, $0\leq m<n$. By formal substitution of a power series $\sum_{j=0}^{\infty} y_j(x)\varepsilon^j$ into (4) it is seen that nontrivial *formal* power series solutions can be constructed if, and only if,

$$m>0 . \tag{5}$$

If (5) is satisfied, then the function $y_0(x)$ may be any solution of the "*reduced*" *differential equation*

$$M[y, 0]=\sum_{\mu=0}^{m} b_\mu(x, 0)y^{(m-\mu)}=0 , \tag{6}$$

and the functions $y_j(x)$, $j\geq1$ can be successively calculated, in infinitely many ways, as solutions of a sequence of nonhomogeneous differential equations whose homogeneous part is $M[y_j, 0]$.

Let us call a solution which is ε-holomorphic at $\varepsilon=0$, a *regular* solution. Unless it is important in the context, we shall not specify the x-domain for which such a solution is regular. A set of regular solutions will be simply called *independent*, if the solutions are linearly independent at $\varepsilon=0$, and hence in some neighborhood of $\varepsilon=0$. From the preceding discussion it follows that *the differential equation* (1) *cannot have more than m independent regular solutions.*

It is easy to construct examples for which the number of independent regular solutions is equal to m. Let, for instance, $Y_j(x, \varepsilon)$, $(j=1, \cdots, m)$ be m linearly independent functions that are (x, ε)-holomorphic in the product space of X and E, and denote by $M[y, \varepsilon]=0$

the linear differential equation of order m with leading coefficient one that is satisfied by these functions. If D designates the operation of differentiation with respect to x, then

$$\varepsilon^{n-m}D^{n-m}M[y, \varepsilon]+M[y, \varepsilon]=0 , \qquad n>m \tag{7}$$

is an nth order differential equation with m regular solutions. The standard asymptotic theory (see e.g. [9]) shows that the functions $Y_j(x, \varepsilon)$ are part of a fundamental system of (7) whose $n-m$ remaining solutions have asymptotic representations of the form (2) with $P(x, \varepsilon)$ equal to the $n-m$ determinations of $(-1)^{1/(n-m)}\varepsilon^{-1}x$.

In spite of this, the occurrence of any regular solution must be regarded as exceptional. In order to show this we prove the following lemma, which generalizes a result of Horn [2].

LEMMA 1. *Let the coefficients* $\alpha_{j\mu}(x, \varepsilon)$ *of the system of differential equations*

$$\frac{du_j}{dx}=\varepsilon^{-h}\sum_{\mu=1}^{n}\alpha_{j\mu}(x, \varepsilon)u_\mu , \qquad (j=1, \cdots, n) \tag{8}$$

be (x, ε)*-holomorphic for* x *in* X *and for* $|\varepsilon|\leq\varepsilon_0$. *Let the solution* $u_j=U_j(x, \varepsilon)$ *of* (8) *be characterized by the initial values*

$$U_j(a, \varepsilon)=p_j(\varepsilon) \qquad (j=1, \cdots, n) \tag{9}$$

at a point a *of* X, *where the functions* $p_j(\varepsilon)$ *are holomorphic for* $|\varepsilon|\leq\varepsilon_0$, *except possibly for a pole at* $\varepsilon=0$. *Then*

$$U_j(x, \varepsilon)=U_j^*(x, \varepsilon)+U_j^{**}\left(x, \frac{1}{\varepsilon}\right)$$

where U_j^*, U_j^{**} *are* x*-holomorphic in* X, *and* ε*-holomorphic for* $|\varepsilon|\leq\varepsilon_0$ *and* $|\varepsilon|>0$, *respectively.*

Proof. Define the functions $U_{jr}(x, \varepsilon)$ by the relations

$$U_{jr}(x, \varepsilon)=\begin{cases} p_j(\varepsilon), & r=0 \\ \varepsilon^{-h}\displaystyle\int_{\Gamma_{ax}}\sum_{\mu=1}^{n}\alpha_{j\mu}(t, \varepsilon)U_{j,r-1}(t, \varepsilon)dt, & r>0 \end{cases} \tag{10}$$

where Γ_{ax} is a path connecting a and x in X. By the standard argument of Picard's iteration method it follows that for $0<\varepsilon_1\leq|\varepsilon|\leq\varepsilon_0$, and for x in any closed and bounded subdomain of X,

$$U_j(x, \varepsilon)=\sum_{r=0}^{\infty}U_{jr}(x, \varepsilon) , \tag{11}$$

where the series, as well as the series of its termwise derivatives with respect to x, converge uniformly and absolutely in the indicated domain. If k is the highest order of the poles of the functions $p_j(\varepsilon)$, the formulas (10) show that the iterants are of the form

$$U_{jr}(x, \varepsilon)=\varepsilon^{-(rh+k)} V_{jr}(x, \varepsilon) \tag{12}$$

where the $V_{jr}(x, \varepsilon)$ are (x, ε)-holomorphic for x in X and $|\varepsilon|\leq\varepsilon_0$. From (11) and (12) we conclude, by means of Weierstrass's theorem on interchange of summations in double series that $U_j(x, \varepsilon)$ admits a convergent representation of the form

$$U_j(x, \varepsilon)=\sum_{\nu=-\infty}^{\infty} F_{j\nu}(x)\varepsilon^{\nu} ,$$

which is uniformly valid for x in every closed subdomain of X, and for $0<\varepsilon_1\leq|\varepsilon|\leq\varepsilon_0$, where ε_1 is arbitrary, and the $F_{j\nu}(x)$ are holomorphic in X. This proves the lemma.

Suppose, now, that the differential equation (1) admits a regular solution $Y(x, \varepsilon)$ in some subdomain X^* of X. If a is any point of X^*, then $Y(x, \varepsilon)$ can be uniquely characterized by the values of $Y^{(s)}(a, \varepsilon)$, $(s=0, \cdots, n-1)$, which are ε-holomorphic for $|\varepsilon|\leq\varepsilon_0$. Since the differential equation (1) is equivalent to a system of the form (8), it follows from the lemma just proved that

$$Y(x, \varepsilon)=\phi_1(x, \varepsilon)+\phi_2(x, \varepsilon) , \qquad x\in X,$$

where ϕ_1, ϕ_2 are ε-holomorphic in $|\varepsilon|\leq\varepsilon_0$, and $|\varepsilon|>0$, respectively, and x-holomorphic in X. But since $Y(x, \varepsilon)$ is ε-holomorphic for $|\varepsilon|\leq\varepsilon_0$ and x in X^*, the uniqueness theorem for Laurent's expansion leads to the conclusion that $\phi_2(x, \varepsilon)\equiv 0$ for x in X^* and all ε. Being x-holomorphic in X by Lemma 1, $\phi_2(x, \varepsilon)$ vanishes therefore identically in the whole domain X. This implies, in particular, that $Y(x, 0)$ is x-holomorphic in X. On the other hand, $Y(x, 0)$ is a solution of the reduced equation $M[y, 0]=0$, and we have proved the following theorem.

THEOREM 1. *If the full differential equation* (4) *possesses a regular solution* $Y(x, \varepsilon)$, *then the corresponding solution* $Y(x, 0)$ *of the reduced equation* $M[y, 0]=0$ *must be x-holomorphic in every domain X where the coefficients of the full equation are x-holomorphic.*

This is a rather strong restriction on the coefficients of $M[y, 0]=0$, in particular on $b_0(x)$. For the equation $M[y, 0]=0$ has, in general, singularities at all zeros of $b_0(x)$, and there will rarely exist a solution of $M[y, 0]=0$ that is holomorphic at all the zeros of $b_0(x)$ which lie in X.

Theorem 1 sheds some light on Theorem 9.2 of [11]. That paper was concerned with the special case in which the expression $N[y, \varepsilon]\equiv N[y]$ of (4) was of order four and independent of ε, and $M[y, \varepsilon]$ was of the form

$$M[y]\equiv b_0(x)y''+b_2(x)y ,$$

where $b_0(x)$ had a first-order zero at $x=0$. In Theorem 9.2 it was proved that the full equation

$$\varepsilon N[y]+M[y]=0 \tag{13}$$

possesses in this case a solution $y=V(x, \varepsilon)$ that approaches, uniformly in a full neighborhood X^* of $x=0$, the x-holomorphic solution $v(x)$ of $M[y]=0$, as $\varepsilon\to 0$ along a given ray of the ε-plane. It would seem plausible to conjecture that $V(x, \varepsilon)$ is ε-holomorphic at $\varepsilon=0$. But Theorem 1 shows that this is, in general, not the case, at least, if $b_0(x)$ possesses other zeros besides $x=0$.

3. Some remarks on sufficient conditions for convergence. The problem of finding sufficient conditions for the convergence of an asymptotic series in ε seems to be much more difficult than the topic discussed in the preceding section but some special classes of differential equations admitting regular solution can be constructed.

a) *Constant coefficients.* If the coefficients of the differential equation (4) are independent of x it possesses a solution of the form $y=e^{\lambda(\varepsilon)x}$ corresponding to every distinct root $\lambda(\varepsilon)$ of the polynomial equation

$$H(\lambda, \varepsilon)\equiv\varepsilon^h\left\{\lambda^n+\sum_{\nu=1}^{n} a_\nu(\varepsilon)\lambda^{n-\nu}\right\}+\sum_{\mu=0}^{m} b_\mu(\varepsilon)\lambda^{m-\mu}=0 .$$

Let $\lambda=\lambda_0$ be a root of the equation

$$H(\lambda, 0)\equiv\sum_{\mu=0}^{m} b_\mu(0)\lambda^{m-\mu}=0 ,$$

then by classical implicit function theorems $H(\lambda, \varepsilon)$ possesses an ε-holomorphic root for which $\lambda(0)=\lambda_0$, provided $\partial H/\partial\lambda$ does not vanish for $\varepsilon=0$, $\lambda=\lambda_0$, that is, provided λ_0 is a simple root of $H(\lambda, 0)$. If all roots of $H(\lambda, 0)$ are multiple, $H(\lambda, \varepsilon)=0$ may or may not define an ε-holomorphic function, as can be seen from the example

$$H(\lambda, \varepsilon)\equiv\varepsilon^h\lambda^3+\lambda^2-2\lambda+1=0 ,$$

which possesses an ε-holomorphic solution for $h=2$, but not for $h=1$.

b) *Linear coefficients.* In formulas (4a) and (4b) let

$$\begin{aligned} a_\nu(x, \varepsilon)&=a_{0\nu}(\varepsilon)+a_{1\nu}(\varepsilon)x \\ b_\mu(x, \varepsilon)&=b_{0\mu}(\varepsilon)+b_{1\mu}(\varepsilon)x . \end{aligned} \tag{14}$$

For many differential equations of this type regular solutions can be found by means of complex Laplace transformation. If we introduce the polynomials

$$A_j = A_j(t, \varepsilon) = \sum_{\nu=1}^{n} a_{j\nu}(\varepsilon) t^{n-\nu} ,$$

(15) $j = 0, 1,$

$$B_j = B_j(t, \varepsilon) = \sum_{\mu=0}^{m} b_{j\mu}(\varepsilon) t^{m-\mu} ,$$

then the differential equation (4) with coefficients of the form (14) admits [4, §§ 8, 18] solutions of the form

$$y(x, \varepsilon) = \int_C v(t, \varepsilon) e^{tx} dt \tag{16}$$

where $v(t, \varepsilon)$ is a solution of the differential equation

$$(\varepsilon^h t^n + \varepsilon^h A_0 + B_0) v - \frac{d}{dt} [(\varepsilon^h A_1 + B_1) v] = 0 \tag{17}$$

and C a suitable contour.

If a closed contour C on the Riemann surface of $v(t, \varepsilon)$ as a function of t can be found such that C is independent of ε and the integral (16) exists for all small ε, then the integral will be either zero or furnish a regular solution, since C can then be chosen so as to avoid the points where $v(t, \varepsilon)$ is not ε-holomorphic. It is possible, but not very illuminating, to formulate more explicit sufficient conditions on the coefficients under which the preceding condition can be satisfied. Some special differential equations of this type were treated in [12] and [13]. The equations

(18)
$$\varepsilon y^{(4)} + x y'' + y = 0$$
$$\varepsilon y^{(4)} + x(y'' + y) = 0$$

do possess regular solutions. The differential equation

$$\varepsilon y'' + k x y' + y = 0$$

turns out to have a regular solution when the constant $1/k$ is a negative integer. For other values of k the solution of the reduced equation has a singular point at $x=0$ and the sole regular solution is $y \equiv 0$, in consequence of Theorem 1.

4. The differential equation $\varepsilon y'' + a(x) y' + b(x) y = 0$.

a) *Polynomial initial conditions.* The theorem of § 2 suggests the conjecture that regular solutions exist if the coefficients of the differential equation are entire functions without zeros. But the example $\varepsilon y''' + y'' - 2y' + y = 0$ mentioned in the preceding section shows that this conjecture is certainly not true in full generality. In this section some sufficient conditions are established for regularity, attention being confined to the equation $\varepsilon y'' + a(x) y' + b(x) y = 0$. In agreement with the foregoing results, the conditions on $a(x)$ and $b(x)$ take account of the behavior in the large. For example, a solution x-holomorphic for $|\varepsilon|<\varepsilon_0$, $|x|<x_0$ must be in fact x-holomorphic for $\varepsilon=0$, whenever $x\in X$, the domain of regularity of a and b. A hypothesis ensuring regularity must therefore ensure, at least implicitly, that the reduced equation $ay'+by=0$ has a solution of the indicated type.

We consider the following statements:

STATEMENT (A). The equation $\varepsilon y''+a(x)y'+b(x)y=0$ has a solution

$$y(x,\varepsilon)=\sum_{n=0}^{\infty}y_n(x)\varepsilon^n \not\equiv 0 \tag{19}$$

convergent for $|\varepsilon|<\varepsilon_0$, $|x|<x$, and satisfying $y_n(0)=P(n)$, $y_n'(0)=Q(n)$, where $P(n)$ and $Q(n)$ are polynomials of degree $\leq k$.

STATEMENT (B). The equation admits a regular solution $y(x,\varepsilon)$ such that $y_x(0,\varepsilon)/y(0,\varepsilon)$ is a rational function of ε, whose numerator and denominator have degree $\leq k$.

STATEMENT (C). The equation admits a solution of the form

$$\sum_{i=0}^{k}h_i(x)\varepsilon^i\not\equiv 0\ ,$$

where the h's are holomorphic near $x=0$.

STATEMENT (D). We have $H^k[f(x)]\equiv 0$ for a linear function $f(x)\not\equiv 0$, where the operator H is defined by

$$H(p)=p+\int_0^x ap\,dx+\int_0^x\int_0^{x_1}(b-a')p\,dx\,dx_1\ .$$

It will be shown, now, that these statements are closely related, $a(x)$ and $b(x)$ being holomorphic near $x=0$:

THEOREM 2. *Statements* (A) *and* (C) *are equivalent*; (B) *is equivalent to them provided* $a(0)\not\equiv 0$; *and* (D) *implies all three.*

To establish the theorem, suppose Statement (A) given, and equate coefficients to find

$$-y''_{n-1}=ay'_n+by_n\ ,\quad \text{for}\quad n\geq 1\ ,\quad 0=ay'_0+by_0\ , \tag{20}$$

which becomes

$$\begin{aligned} y'_{n-1}-Q(n-1)&=\int_0^x(ay'_n+by_n)dx=-ay_n\Big|_0^x+\int_0^x y_n(b-a')dx\\ &=-ay_n+a(0)P(n)+\int_0^x y_n(b-a')dx\ . \end{aligned} \tag{21}$$

Further integration yields

$$y_{n-1}-P(n-1)-xQ(n-1)=\int_0^x\int_0^{x_1}y_n(b-a')dx\,dx_1-\int_0^x ay_n dx+a(0)P(n)x\ . \tag{22}$$

Let $Y_n = \Delta^{k+1} y_n$, the $(k+1)$th difference. We have, by (22),

$$Y_{n-1} = \int_0^x \int_0^{x_1} Y_n(b-a')\,dx\,dx_1 - \int_0^x aY_n\,dx\ . \tag{23}$$

Regularity of a and b, convergence of $\sum y_n \varepsilon^n$, ensure that for $|x| \leq x^* < x_0$ we have

$$|Y_n| \leq BA^n\ , \quad |a| \leq M\ , \quad |a'| \leq M\ , \quad |b| \leq M\ ,$$

where A, B, and M are suitable positive constants (that may depend on x^*). Thus, we have by (23) in every circle $|x| \leq \delta < 1$, with $\delta \leq x^*$,

$$\max|Y_{n-1}| \leq 3\delta M(\max|Y_n|) \leq (2\delta M)^{m+1}(\max|Y_{n+m}|)\ , \quad m=0, 1, \cdots$$

the latter relation following by iteration. Choose δ so small that $4\delta M < 1/A$. Then

$$|Y_{n-1}| < B(1/2A)^{m+1} A^{n+m} = BA^{n-1}/2^{m+1}\ .$$

Letting $m \to \infty$ shows that $|Y_n| = 0$, and hence

$$y_n = g_0(x) + g_1(x)n + g_2(x)n^2 + \cdots + g_k(x)n^k\ .$$

It follows that $y(x, \varepsilon)$ has the form

$$y(x, \varepsilon) = \sum_{i=0}^{k} \frac{f_i(x)}{(1-\varepsilon)^{i+1}} \tag{24}$$

as we see by using factorial polynomials in place of powers of n. Multiplying through by $(1-\varepsilon)^{k+1}$ shows that (A) implies (C).[1]

To see that (C) implies (A), express the given polynomial as a new polynomial in $1-\varepsilon$ and divide by $(1-\varepsilon)^{k+1}$. We are led to a solution of the form (24), and expansion of $(1-\varepsilon)^{-i-1}$ gives the initial conditions described in (A). We have incidentally established the rather curious fact that $y_n(x)$ and $y_n'(x)$ are polynomials in n for every small fixed x, if for the single value $x=0$.

Suppose now that (C) is given. We may assume (24). With $s = 1/(1-\varepsilon)$, equating powers of s in $\varepsilon y'' + ay' + by = 0$ gives

$$\left\{\begin{array}{l} f_0'' = 0 \\ f_1'' = L(f_0) \\ f_2'' = L(f_1) \\ \cdots\cdots\cdots \\ f_k'' = L(f_{k-1}) \\ 0 = L(f_k'') \end{array}\right\}, \quad L(y) = y'' + ay' + by\ , \tag{25}$$

and conversely, the system (25) for some $f_i \not\equiv 0$ ensures a solution of

[1] A simpler proof has been given by Robert Steinberg, starting with the observation that $(1-\varepsilon)^k y(x, \varepsilon) = \sum Y_n(x)\varepsilon^n$ has $Y_n(0) = Y_n'(0) = 0$.

the form (24), hence of the form described in Statement (C).

We have $\frac{d^2}{dx^2}H(p)=L(p)$, and hence, when the constants of integration are taken as zero in (25), this system is equivalent to

$$(26)\qquad \begin{cases} f_0''=0 \\ f_1=H(f_0) \\ f_2=H(f_1) \\ \cdots\cdots \\ f_k=H(f_{k-1}) \\ 0=H(f_k)\ . \end{cases}$$

Hence $H^k f_0=0$ is sufficient to ensure a solution of (25), and indeed with $f_i'(0)=f_i(0)=0$ for $i\geq 1$. Thus, (D) implies (C), and hence (D) implies (A). The converse is false; but if we define

$$H(p, q)f\equiv Hf+px+q\ .$$

Statement (A) or (C) is equivalent to

$$(27)\qquad \prod_{i=1}^{k} H(p_i, q_i)f\equiv 0\ , \qquad f=cx+d\not\equiv 0\ ,$$

for some constants p_i, q_i. Here f is the first function f_i in (25) which is not identically zero.

If (B) is given, suppose $y(0, \varepsilon)$ has a zero of order $h>0$ at $\varepsilon=0$. Then $y_0(0)=y_1(0)=\cdots=y_{h-1}(0)=0$. The system (20) gives $y_0=c\exp\left[-\int_0^x (b/a)dx\right]$ where c is constant. If $a(0)\not\equiv 0$, it follows that $y_0(x)\equiv 0$ for small x and hence for all $x\in X$. Similarly, $y_1, \cdots, y_{h-1}\equiv 0$ for small x. Hence the function $\varepsilon^{-h}y(x, \varepsilon)$ is ε-holomorphic for $\varepsilon=0$ and small x.

If $y_x(0, \varepsilon)/y(0, \varepsilon)=P(\varepsilon)/Q(\varepsilon)$, where P and Q have degree $\leq k$, then the function $\varepsilon^{-h}y(x, \varepsilon)$ satisfies the same condition. Combining this observation with the preceding, we see that one may suppose $y(0, 0)\not\equiv 0$ in Statement (B), provided $a(0)\not\equiv 0$.

Putting $t=1-\varepsilon$, dividing numerator and denominator by t^{k+1}, and relabeling coefficients, transforms the given condition into

$$\frac{y_x(0, \varepsilon)}{y(0, \varepsilon)}=\frac{A(\varepsilon)}{B(\varepsilon)}$$

where

$$A(\varepsilon)=a_0(1-\varepsilon)^{-1}+a_1(1-\varepsilon)^{-2}+\cdots+a_k(1-\varepsilon)^{-k-1}\ ,$$

and similarly for $B(\varepsilon)$. The function

$$Y(x, \varepsilon)=\frac{y(x, \varepsilon)}{y(0, \varepsilon)}B(\varepsilon)$$

is regular near $\varepsilon=0$; it satisfies the given differential equation; and also $Y(0, \varepsilon)=B(\varepsilon)$, $Y_x(0, \varepsilon)=A(\varepsilon)$. Hence $Y(x, \varepsilon)$ satisfies the requirements of Statement (A).

Finally, it is clear that (B) follows from (C) if $y(0, \varepsilon)\not\equiv 0$ in (C). If $y(0, \varepsilon)\equiv 0$, however, we have $y_n(0)=0$ in (20), which implies $y_n(x)\equiv 0$ for $a(0)\not\equiv 0$ as above. Hence $y(x, \varepsilon)\equiv 0$ contrary to the assumption in Statement (C).

The condition on the operator H admits a simple interpretation. If I is the identity operator, then formally

$$(I-Hz)^{-1}p=\sum_{i=0}^{\infty} z^i H^i p\,, \qquad H^0=I\,.$$

Now, when $H^i(cx+d)=0$ for $i\geq k$, as in Statement (D), then the above expression is a polynomial in z for $p=cx+d$. Suppose, more generally, that

$$(I-Hz)^{-1}(cx+d)=\phi(z, x)\,,$$

a function holomorphic in z at $z=1$. Then $cx+d=(I-Hz)\phi$ or, by differentiating,

$$0=(1-z)\phi''-za\phi'-zb\phi\,.$$

With $\varepsilon=1-1/z$ this yields

$$\varepsilon\phi''+a\phi'+b\phi=0$$

where ϕ is ε-holomorphic near $\varepsilon=0$.

The above treatment is purely formal. If

$$\overline{\lim}|H^k p|^{1/k}\leq\theta<1$$

for $p=cx+d$ and $|x|<\delta$, however, then the formal equalities become true equalities. We define $(I-Hz)^{-1}$ by the foregoing series, which converges uniformly in x near $z=1$. The function $H^k p$ being analytic for each k, we may differentiate the series to find that $\phi(z, x)$ is in fact a solution holomorphic in z for $|z|<1/\theta$. The corresponding domain of ε is $|1-\varepsilon|>\theta$. Hence a sufficient condition that the equation have a solution (ε, x)-holomorphic for $|1-\varepsilon|>\theta$ and $|x|<\delta$ is that $\overline{\lim}|H^k p|^{1/k}\leq\theta$ for $p=cx+d\not\equiv 0$. An extension can be given after the manner of (27).

b) *Examples and discussion.* The preceding result enables us to construct equations admitting regular solutions. If the polynomials in Statement (A) are constant, so that $k=0$, $P(n)=p$, $Q(n)=q$, then Statement (C) yields $y(x, \varepsilon)=h(x)$. The differential equation shows that

$h(x)$ is linear, whence $h(x)=p+qx$ by the initial conditions. Such a function is a solution if and only if $aq+b(p+qx)=0$. The differential equation then takes the form

$$\varepsilon y''+b(x)[(c-x)y'+y]=0 \tag{28}$$

where c is constant. For every choice of $b(x)$ there is obviously a regular solution; namely, $y=x-c$.

The case $k=0$ just considered can be regarded in a different light. Let $a(x)$ and $b(x)$ be integrable and satisfy $|a|\leq M$, $|b|\leq M$ for a domain of the (possibly complex) variable x. The Picard iteration procedure shows then that

$$\varepsilon y''+ay'+by=0$$

has a unique solution $y(x, \varepsilon)$ subject to $y(0, \varepsilon)=c$, $y'(0, \varepsilon)=d$, where c and d are indepenent of ε. Moreover, this solution is an entire function in $1/\varepsilon$, of exponential type M at most. If we require a solution $Y(x, \varepsilon)$ ε-holomorphic near $\varepsilon=0$ and satisfying the same initial condition, it is necessary that $y\equiv Y$. This shows that both y and Y are ε-holomorphic in the extended ε-plane, hence independent of ε. Thus we are led to the situation found otherwise above. This discussion resembles that used previously for the more general equation (4).

Turning now to the case $k=1$ in Statement (A), we find

$$\begin{cases} f_0''=0 \\ f_1''=af_0'+bf_0 \\ \ \ 0=f_1''+af_1'+bf_1 \end{cases} \tag{29}$$

by (25). Adding the three equations, or considering $y(x, 0)$, we see that $s=f_0+f_1$ satisfies the reduced equation $as'+bs=0$. Hence, with c_0 constant,

$$f_0+f_1=c_0R(x)\,, \qquad R(x)=e^{-\int_0^x(b/a)dx} \tag{30}$$

where R must be regular since f_0 and f_1 are. If $c_0=0$ one easily shows that the problem reduces to the case $k=0$ just considered. Without loss of generality, therefore, we may take $c_0=1$. In terms of R, the original differential equation is

$$\varepsilon y''+Ra(x)(y/R)'=0 \tag{31}$$

and the system (29) is equivalent to the three conditions (30), $f_0=cx+d$ with c, d constant, and

$$R''=[(cx+d)/R]'aR\,. \tag{32}$$

The differential equation, then, is

$$\varepsilon y'' + \frac{R''}{[(cx+d)/R]'}\left(\frac{y}{R}\right)' = 0 \tag{33}$$

and the solution is given by Statement (C) as

$$y(x, \varepsilon) = R - (cx+d)\varepsilon \ . \tag{34}$$

That (34) is in fact a solution is easily verified by actual substitution. In summary, there is a regular solution, with $y_n(0)$ and $y_n'(0)$ linear functions of n, if and only if the equation can be put in the form (33); and the sole such solution is then a constant multiple of (34), divided by $(1-\varepsilon)^2$.

The case $k=2$ is more complicated. It is found that $a(x)$ must satisfy a certain first-order nonlinear differential equation, R being given, and the case corresponding to $c_0=0$ in (30) reduces to the case $k=1$. It would be desirable to find an explicit form of the equation for $k\geq 2$, but we have not been able to do this.

Although the foregoing considerations restrict the behavior of a and b in the large (by virtue of analytic continuation) the analyticity of a and b plays no very essential role. Indeed a corresponding real-variable result might be given, with hypothesis on the local behavior only. It seems difficult to give criteria in which the complex-variable character of the problem is more fully used. This difficulty is illustrated by the following two examples.

Let $a=b$, in the discussion leading to (33) and (34), so that $R=e^{-x}$. If $c=0$, $d=1$ the differential equation is

$$\varepsilon y'' + y'e^{-x} + ye^{-x} = 0 \tag{35}$$

with solution $y=e^{-x}-\varepsilon$. It is seen that R, a, b, and $1/a$ are entire functions of exponential type, as is the solution y.

Consider, next, the equation

$$\varepsilon y'' + y'e^{x} + ye^{x} = 0 \ . \tag{36}$$

Despite the resemblance to (35), there is no regular solution, as we now show; and thus the conditions just described, stringent though they be, are yet insufficient.

Suppose there is a regular solution of (36). Since $a(0)=1$ we may assume $y(0, 0)\neq 0$, as in the above discussion. The function $y(x, \varepsilon)/y(0, \varepsilon)$ therefore is regular and has $y_0(0)=1$, $y_n(0)=0$, $(n\geq 1)$ in the series representation. The system (20) gives

$$y_n = e^{-x}\{c_n + y_{n-1}'\} \ , \qquad y_0 = e^{-x}$$

where the c_n are constants. By induction we see that

$$y_n = n!\, e^{-(n+1)x} + (n+1)!\, c_1 e^{-nx} + (n-2)!\, c_2 e^{-(n-1)x} + \cdots + c_{n-1} e^{-2x} - c_n e^{-x}$$

and, in view of the initial conditions, that $c_i > 0$. We have therefore

$$e^x y_n(x) = e^x y_n(x) - y_n(0) = n!\,(e^{-nx} - 1) + \cdots , \qquad (n \geq 1) ,$$

where the terms not written are of the same sign as the leading term, for real x, since c_n drops out. Thus it is that $|y_n(x)| > n!\, e^{-x} |e^{-nx} - 1|$, and the series diverges for $x \neq 0$.

c) *Related partial differential equations.* Consider the following problems, with regular a, b:

Problem (A). To find a solution $y(x, \varepsilon) \not\equiv 0$ of $\varepsilon y'' + ay' + by = 0$ which is holomorphic in ε for $|\varepsilon| < r$, in a given set S of x values.

Problem (B). To find a solution $Y(x, \varepsilon) \not\equiv 0$ of $Y_{xx} + aY_{x\varepsilon} + bY_\varepsilon = 0$ which is an entire function of type $1/r$ in ε for a given set S of x values, and satisfies

$$Y(x, 0) = e^{-\int_0^x (b/a)dx}$$

or $Y(x, 0) = 0$.

It will be shown, now, that these problems are completely equivalent. If

$$y(x, \varepsilon) = \sum_0^\infty y_n(x) \varepsilon^n$$

is a solution of Problem (A) then

$$Y(x, \varepsilon) = \sum_0^\infty y_n(x) \varepsilon^n / n!$$

satisfies the differential equation $Y_{xx} + aY_{x\varepsilon} + bY_\varepsilon = 0$. This can be verified by termwise differentiation, insertion into the partial differential equation and use of equations (20). Since $Y(x, 0) = y_0(x)$, the first equation in (20) shows that the initial condition of Problem (B) is also statisfied. Finally, it is easy to prove and doubtless well known that Y is an entire function of ε of type $1/r$, if and only if $y(x, \varepsilon)$ is ε-holomorphic for $|\varepsilon| < r$.

To show, conversely, that a solution $Y(x, \varepsilon) = \sum z_n(x) \varepsilon^n$ of Problem (B), leads to a solution of (A) we observe that, by virtue of the statement in the last sentence, the series

$$y(x, \varepsilon) = \sum z_n(x) n!\, \varepsilon^n$$

converges. The functions $y_n(x) = z_n(x) n!$ are then seen to satisfy the recursion formulas (20) for $n \geq 1$. That they also hold for $n = 0$ follows from the initial condition imposed on $y(x, \varepsilon)$. This completes the proof.

We remark in passing that y and Y are transforms of each other:

$$\int_0^\infty e^{-pt} Y(x, \varepsilon) d\varepsilon = y(x, p) .$$

The change of variable

$$s = \varepsilon - \int_0^x a(x) dx , \qquad t = \varepsilon$$

is suggested by the characteristics, and reduces the partial differential equation to canonical form

$$u_{st} = \frac{b - a'}{a^2} u_s + \frac{b}{a^2} u_t$$

where $u(s, t) = Y(x, \varepsilon)$ and the coefficients are evaluated at x. The initial condition is

$$u\left(-\int_0^x a(x) dx, 0 \right) = e^{-\int_0^x (b/a) dx} .$$

With $z = e^{-x} - 1$ this becomes $u(z, 0) = z + 1$ when $a = b = e^{-x}$; but $u(z, 0) = 1/(1-z)$ when $a = b = e^x$. Thus the initial values have a pole at $x = 0$, in the second case. We have seen already that the solution is regular in the first case but not in the second.

A related partial differential equation arises in another way if we seek a solution $y(x, \varepsilon)$ which is an entire function of type k and such that

$$\int_{-\infty}^\infty |y(x, i\sigma)|^2 d\sigma < \infty .$$

Such functions are equivalent with those representable in the form

$$y(x, \varepsilon) = \int_{-k}^k e^{\varepsilon t} f(x, t) dt , \qquad \int_{-k}^k |f(x, t)|^2 dt < \infty .$$

One obtains, formally,

$$\int_{-k}^k (\varepsilon f_{xx} + a f_x + b f) e^{\varepsilon t} dt = 0 .$$

Integration by parts yields

$$\int_{-k}^k \varepsilon (F_{txx} - a F_x - b F) dt + e^{\varepsilon t} [a F_x(x, k) + b F(x, k)] = 0 ,$$

where

$$F(x, t) = \int_{-k}^t f(x, t) dt .$$

Letting $\varepsilon \to 0$ shows that the integrated part vanishes. Hence the original problem leads to a two-point boundary-value problem

$$F_{txx} = aF_x + bF ,$$

$$F(x, k) = c \exp\left(-\int_0^x (b/a)dx\right) ,$$

$$F(x, -k) = 0 .$$

Conversely, from such an F we can construct, at least formally, a solution to the question first proposed.

Many of the foregoing considerations apply with only slight change to the equation

$$\varepsilon y'' + a(x)y' + b(x)y = c(x) . \tag{37}$$

The condition $f_0'' = 0$ in (26) is replaced by $f_0'' = c(x)$, and we are led to consider $H^k p$ with $p = f_0$. Similarly, the condition $0 = ay_0' + by_0$ in (20) becomes

$$h(x) = ay_0' + by_0$$

with corresponding change in the boundary condition for the associated partial differential equation. (The equation itself does not change.)

That there is always a solution regular in ε, for *some* $c(x)$, is evident when we take $y = 1$, $c = b(x)$. Actually, one can find a $c(x)$ such that the regular solution depends on ε. For example, let f satisfy

$$f'' + af' + bf = 0 , \qquad f \not\equiv 0 ,$$

and let $c(x) = -f''(x)$. Then $y = f(x)/(1 - \varepsilon)$ is a solution of (37).

5. A hydrodynamic application. Differential equations of the type (4) with

$$M[y, \varepsilon] \equiv xM^*[y, \varepsilon] \tag{38}$$

where the leading coefficient $b_0^*(x, 0)$ of $M^*[y, 0]$ does not vanish at $x = 0$ occur in the theory of hydrodynamic stability. This application will be explained below. We shall be concerned here with necessary conditions on a differential equation (4), for which (38) is satisfied, in order that it possess a full contingent of m solutions that converge to solutions of $M^*[y, 0] = 0$, as $\varepsilon \to 0$, uniformly in a full neighborhood of $x = 0$.

Before stating our theorem concerning this case we recall ([4], p. 126) that for linear differential expressions there exists division algorithm involving only rational operations and differentiations, by means of which $N[y, \varepsilon]$ can be represented in a unique fashion in the form

$$N[y, \varepsilon] = Q[M^*[y, \varepsilon], \varepsilon] + R[y, \varepsilon] \,. \tag{39}$$

Here $Q[u, \varepsilon]$ and $R[y, \varepsilon]$ are linear differential expressions with (x, ε)-holomorphic coefficients. The order of $R[y, \varepsilon]$ is at most $m-1$. Let us call $R[y, \varepsilon]$ the remainder of $N[y, \varepsilon]$ with respect to $M^*[y, \varepsilon]$.

THEOREM 3. *Assume that the differential equation*

$$\varepsilon N[y, \varepsilon] + xM^*[y, \varepsilon] = 0 \tag{40}$$

possesses m solutions of the form

$$Y_j(x, \varepsilon) = y_{j0}(x) + \varepsilon v_j(x, \varepsilon) \,, \qquad (j=1, \cdots, m) \tag{41}$$

where the $y_{j0}(x)$ form a fundamental system of the reduced equation

$$M^*[y, 0] = 0 \tag{42}$$

and the $v_j(x, \varepsilon)$ are bounded, together with their first n derivatives with respect to x, at $x=0$, and for ε in some point set E^* having $\varepsilon=0$ as an accumulation point. Then the remainder $R[y, \varepsilon]$ of $N[y, \varepsilon]$ with respect to $M^*[y, \varepsilon]$ vanishes for $x=\varepsilon=0$, identically for all $y(x)$.

The conditions on $Y_j(x, \varepsilon)$ in this theorem are much weaker than regularity. The meaning of Theorem 3 is essentially that even these weaker conditions will only exceptionally be satisfied, since for arbitrary $N[y, \varepsilon]$ and $M^*[y, \varepsilon]$ the remainder will, in general, not vanish identically in y, for $x=\varepsilon=0$.

Proof of Theorem 3. Without loss of generality we may assume that

$$y_{j0}^{(k-1)}(0) = \delta_{jk} \,, \qquad (j, k=1, \cdots, m) \tag{43}$$

If (39) is inserted in (40) and y is replaced by $Y_j(x, \varepsilon)$, then use of (41) leads to a relation of the form

$$\varepsilon^2 \phi_j(x, \varepsilon) + \varepsilon x \psi_j(x, \varepsilon) + \varepsilon R[y_{j0}, 0] = 0 \,, \qquad (j=1, \cdots, m) \tag{44}$$

where $\phi_j(0, \varepsilon)$ and $\psi_j(0, \varepsilon)$ remain bounded as $\varepsilon \to 0$ in E^*. Setting $x=0$ and letting $\varepsilon \to 0$ in E^*, this yields

$$R[y_{j0}, 0] = 0 \,, \qquad \text{for} \quad x=0 \qquad (j=1, \cdots, m) \,.$$

Because of (43) we conclude that every coefficient of $R[y, 0]$ vanishes at $x=0$. This proves the theorem.

Application. By a simple change of variables the Orr-Sommerfeld equation in the theory of hydrodynamic stability, [5],

$$\frac{i}{\alpha R}\left[\frac{d^4\phi}{dz^4}-2\alpha^2\frac{d^2\phi}{dz^2}+\alpha^4\phi\right]+(w(z)-c)\left(\frac{d^2\phi}{dz^2}-\alpha^2\phi\right)-w''(z)\phi=0$$

can be written in the form

$$\varepsilon(y^{(4)}-2\alpha^2y''+\alpha^4y)+b_0(x)(y''-\alpha^2y)-b_0''(x)y=0 \tag{45}$$

with $b_0(0)=0$. (The dependence of $b_0(x)$ on the complex parameter c is not set in evidence in our notation. The letter α denotes a positive constant.) The special case that $b_0''(x)$ also vanishes at $x=0$ is of some interest in hydrodynamics. If c is real, for instance, and $b_0''(0)=0$, one has the case of a periodic disturbance of the flow such that the critical layer where the disturbance and the main flow travel with equal velocities, coincides with an inflection point of the main flow profile $w(z)$, [7]. In the present case

$$b_0(x)=\beta_1x+\beta_3x^3+\beta_4x^4+\cdots,$$

and the remainder $R[y,\varepsilon]$ in Theorem 3 is independent of ε. A straightforward calculation, not reproduced here, shows that this remainder vanishes for $x=0$, if and only if

$$\beta_4=0\ , \qquad 3\beta_3^2-5\beta_5\beta_1=0\ . \tag{46}$$

Since the coefficients β_j depend on c these conditions can be satisfied for very exceptional profiles and very special disturbances only. Now, it is known, [14], that corresponding to every solution of the reduced equation there exist solutions of the full equation (45) having the form (41), with $v_j(x,\varepsilon)$ and its derivatives bounded in some region S of the x-plane. As we have just seen, S will not include the origin, at least not for *all* such solutions, unless very exceptional conditions are satisfied. From this it can easily be deduced that S cannot be a doubly connected domain surrounding the origin completely, i.e., some solutions which converge in certain regions to a solution of the reduced equation, must diverge in certain other regions. It follows from this fact (cf. [14], [5]) that the damped disturbances of the corresponding hydrodynamic flow possess a so-called "inner friction layer," i.e., a layer in which the effects of viscosity cannot be neglected no matter how small the viscosity coefficient.

Thus Theorem 3 leads to the result that *even if $w(z)-c$ and $w''(z)$ vanish at the same point for a certain damped disturbance, an inner friction layer will be present unless the disturbance and the velocity profile are of an extremely exceptional type.*

It can be shown that the vanishing of $R[y,\varepsilon]$ at $x=\varepsilon=0$ is only one of infinitely many conditions necessary for the existence of m regular solutions. It is therefore very likely, but not yet proved, that $b_0(x)=x$ (Couette flow) is the only flow for which inner friction layers are ever absent. In the Couette case the remainder $R[y, \varepsilon]$ is, of course, identically zero for all x and ε, and every solution of the reduced equation is trivially a regular solution of the full equation.

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University of California, Los Angeles
Institute for Numerical Analysis, Los Angeles

ON A THEOREM OF L. LICHTENSTEIN

S. E. WARSCHAWSKI

1. Introduction. The object of this note is the proof of the following:

THEOREM. *Let C be a closed Jordan curve in the z-plane which possesses a corner of opening $\pi\alpha$, $0<\alpha\leq 2$ at $z=0$. Suppose that this corner is formed by two regular analytic arcs γ_a and γ_b:*

$$\gamma_a: z=A(t)=\sum_{\nu=1}^{\infty} a_\nu t^\nu; \quad \gamma_b: z=B(t)=\sum_{\nu=1}^{\infty} b_\nu t^\nu, \qquad 0\leq t\leq 1,\ a_1\neq 0,\ b_1\neq 0.$$

If $\zeta=f(z)$ maps the interior Δ of C conformally onto the half-plane $\mathscr{I}[\zeta]>0$ so that $f(0)=0$, then, for every integer n,

$$\lim_{z\to 0}\left\{z^{n-1/\alpha}\frac{d^n f(z)}{dz^n}\right\}=c\,\frac{1}{\alpha}\left(\frac{1}{\alpha}-1\right)\cdots\left(\frac{1}{\alpha}-n+1\right), \tag{1}$$

for unrestricted approach, where $c=\lim_{z\to 0}[f(z)z^{-1/\alpha}]$.

This theorem was stated by L. Lichtenstein [2] and [3], but proved only for the case that α is *irrational*. He remarks, however, that it is most likely true for all α, $0<\alpha\leq 2$, but that his proof does not yield this result. In the following a simple proof based on a different approach is given for the complete theorem[1].

2. Lemmas. In the proof of theorem we shall make use of the following two lemmas.

LEMMA 1. *Suppose Γ is a closed Jordan curve with a corner at $z=0$ of opening $\pi\alpha$, $0<\alpha\leq 2$, and that each of the two arcs forming the corner has bounded curvature in the neighborhood of $z=0$. If $w=g(z)$ maps the interior D of Γ conformally onto the angle $0<\arg w<\pi\alpha$, so that $g(0)=0$, then for non-tangential approach,*

$$\lim_{z\to 0}\frac{g(z)}{z}=\mu \quad \textit{exists and} \quad \mu\neq 0\ . \tag{2}$$

This is just a weaker statement of a well known result [4, 5]; (2) holds under more general assumptions on the arcs which form the corner

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[1] This note is the result of an inquiry from Dr. George Forsythe of the Institute of Numerical Analysis regarding the validity of Lichtenstein's theorem for all α. Dr. Forsythe applies this result in his preceding paper on "Asymptotic lower bounds for the fundamental frequency of convex membranes".

and for unrestricted approach [5, p. 427]. However, for the sake of completeness we give an elementary proof of this lemma in § 4.

LEMMA 2. *Suppose that* $F(w)$ *is analytic in an angle* A: $\alpha < \arg w < \beta$, $\beta - \alpha \leqq 2\pi$, *and that in every sub-angle* B *of* A *with the vertex at* $w=0$, $\lim_{w\to 0} \frac{F(w)}{w} = \mu$. *Then for any integer* $n \geqq 1$, *as* $w \to 0$ *in any sub-angle* B *of* A

$$(3) \qquad \lim_{w\to 0}[w^{n-1}F^{(n)}(w)] = \begin{cases} \mu & \text{when } n=1 \\ 0 & \text{when } n>1 \,. \end{cases}$$

Proof. Let B be the angle $\alpha+\delta < \arg w < \beta-\delta$, $0 < 2\delta < \beta-\alpha$. About $w \in B$ we describe a circle c of radius r which is contained in and tangent to a side of the angle $\alpha + \frac{\delta}{2} \leqq \arg w \leqq \arg \beta - \frac{\delta}{2}$. Clearly, $\frac{r}{|w|} \geqq \sin\frac{\delta}{2}$. We set $G(w) = F(w) - \mu w$. Then

$$w^{n-1}G^{(n)}(w) = \frac{n!}{2\pi i}\int_c \frac{G(t)w^{n-1}}{(t-w)^{n+1}}dt = \frac{n!}{2\pi i}\int_c \frac{G(t)}{t}\frac{tw^{n-1}}{(t-w)^{n+1}}dt \,.$$

Since $|t| \leqq |t-w| + |w|$ and $|t-w| = r$ for t on c, we have

$$|w^{n-1}G^{(n)}(w)| \leqq \frac{n!}{2\pi}\int_c \left|\frac{G(t)}{t}\right| \frac{|w|^{n-1}(r+|w|)}{r^{n+1}}|dt|$$

$$\leqq n! \frac{2}{\sin^n(\delta/2)} \max_{t\in c}\left|\frac{G(t)}{t}\right|$$

and the last expression approaches 0 as $w \to 0$ in B. This proves (3).

3. **Proof of the theorem.** (i) We may and shall assume in the following that C consists of two regular analytic arcs $\widehat{OA}$ and $\widehat{OB}$ and a circular arc γ about O through A and B. (The size of the radius r of this arc will be restricted below). For, if D is a subregion of Δ bounded by the just described curves, and if $f_1(z)$ maps D onto the upper half plane such that $f_1(0)=0$, then $f(z) = h[f_1(z)]$ for $z \in D$, where $h(\zeta)$ is an analytic function in a neighborhood of $\zeta = 0$ and $h'(0) \neq 0$. The result (1) on $f^{(n)}(z)$ follows then from that on $f_1^{(n)}(z)$.

The theorem will be proved by the following statement: *if* $w = g(z)$ *maps* Δ *onto the angle* $0 < \arg w < \pi\alpha$ *such that* $z=0$ *corresponds to* $w=0$, *then, for unrestricted approach,*

$$(4) \qquad \lim_{z\to 0} g'(z) = \lambda,\ 0 < |\lambda| < \infty, \text{ and } \lim_{z\to 0}[z^{n-1}g^{(n)}(z)] = 0, \text{ for } n>1.$$

The result (1) of the theorem is then obtained from (4) by use of the fact that $f(z)=[g(z)]^{1/\alpha}$.

For the proof of (4) we may presuppose that $0<\alpha<1$; for if $1\leqq\alpha\leqq 2$ we apply first the auxiliary transformation $z'=z^{1/4}$. For $|t|\leqq\delta$, where $\delta>0$ is sufficiently small, $\overset{\frown}{OA}$ and $\overset{\frown}{OB}$ are transformed into regular analytic arcs in $\tau=t^{1/4}$. We assume r so small that $\overset{\frown}{OA}$ and $\overset{\frown}{OB}$ are obtained for values of the parameter $t\leqq\delta$.

We now impose a further restriction on δ and thus on r. There exists a $\rho>0$ such that $z=A(t)$ and $z=B(t)$ have analytic and univalent inverse functions $t=a(z)$ and $t=b(z)$ in $|z|\leqq\rho$. We take δ so small that $\overset{\frown}{OA}$ and $\overset{\frown}{OB}$ are contained in $|z|<\rho$. Thus, $r<\rho$.

(ii) Consider the maps of $\varDelta$ by means of $t=a(z)$: $\overset{\frown}{OA}$ is transformed into a segment $\overset{\frown}{O_1A_1}$ of the real t-axis and $\overset{\frown}{OB}$ into an arc $\overset{\frown}{O_1B_1}$ which makes an angle of opening $\pi\alpha$ with O_1A_1. The circular arc γ: $\overset{\frown}{AB}$ is mapped onto an arc $\overset{\frown}{A_1B_1}$. If r is sufficiently small, the arcs $\overset{\frown}{O_1B_1}$ and $\overset{\frown}{A_1B_1}$ will lie in the upper half of the t-plane[2]. We assume that r has been so chosen (third and final restriction on r). Let $\varDelta_1$ denote the image of $\varDelta$ in the t-plane.

Suppose that $w=\phi(t)$ maps $\varDelta_1$ onto the angle $0<\arg w<\pi\alpha$ such that $t=0$ corresponds to $w=0$ and A_1 to $w=\infty$. The segment O_1A_1 is then transformed into the positive real axis of the w-plane. We reflect the arc $O_1B_1A_1$ with respect to the positive real axis and denote the image of B_1 by B_1'. By Schwarz's reflection principle the function $w=\phi(t)$ maps the region bounded by the Jordan curve Γ: $O_1B_1A_1B_1'O_1$ conformally onto the angle $-\pi\alpha<\arg w<\pi\alpha$.

We apply now Lemma 1 to the curve Γ, which has a corner of opening $2\alpha\pi$, $0<2\alpha<2$, at $t=0$, formed by the regular analytic arcs $\overset{\frown}{O_1B_1}$ and $\overset{\frown}{O_1B_1'}$. Hence, for non-tangential approach,

$$\lim_{t\to 0}\frac{\phi(t)}{t}=\frac{1}{\mu}$$

exists and $0<|\mu|<\infty$. Next, observing that the mapping $w=\phi(t)$ preserves angles at $t=0$ and applying Lemma 2 to the inverse $F(w)$ of $\phi(t)$ we find that in any angle $-\pi\alpha+\varepsilon<\arg w<\pi\alpha-\varepsilon$ $(0<\varepsilon<\pi\alpha)$:

$$\lim_{w\to 0}F'(w)=\mu\,, \qquad \lim_{w\to 0}[w^{n-1}F^{(n-1)}(w)]=0\,, \quad \text{for} \quad n>1\,.$$

Hence, in any sector $|\arg t|\leqq\pi\beta$, $|t|\leqq\eta$, where $0<\beta<\alpha$ and η is sufficiently small,

$$\lim_{t\to 0}\phi'(t)=\frac{1}{\mu}\,, \qquad \lim_{t\to 0}[t^{n-1}\phi^{(n)}(t)]=0\,, \qquad \text{for } n>1. \tag{5}$$

[2] We assume here that O,A,B follow in counter-clockwise order along C.

Since $\phi[a(z)]=g(z)$, it follows from (5) that, for $\lambda=\frac{a'(0)}{\mu}=\frac{1}{\mu\alpha_1}$,

$$(6) \qquad \lim_{z\to 0} g'(z)=\lambda \quad \text{and,} \quad \lim_{z\to 0}[z^{n-1}g^{(n-1)}(z)]=0\,, \qquad \text{for } n>1,$$

in any curvilinear angle in $C+\Delta$ *formed by* $\widehat{OA}$ *and any Jordan arc* j *in* Δ *which has a tangent at* O *making the angle* $\pi\beta$ *with the tangent to* $\widehat{OA}$ *at* O.

(iii) By applying the same argument in which the arc $\widehat{OB}$ takes the role of $\widehat{OA}$ we find that (6) holds in any curvilinear angle formed by $\widehat{OB}$ and any Jordan arc j' in Δ which has a tangent at O making an angle $\pi\beta$ with the tangent to OB at O. Since β may be taken so that the two curvilinear angles overlap, we obtain (4), and this completes the proof.

4. Proof of Lemma 1. We can construct a Jordan curve Γ_i contained in $D+\Gamma$ and one Γ_e exterior to D, each consisting of two circular arcs intersecting at the angle $\pi\alpha$ at $z=0$ (and at another point). The interion $I(\Gamma_i)$ of Γ_i is in D, and we may assume that the exterior $E(\Gamma_e)$ contains D. If $h_i(z)$ and $h_e(z)$ are the bilinear transformations which map $I(\Gamma_i)$ and $E(\Gamma_e)$, respectively, onto the angle $0<\arg w<\pi\alpha$, such that $h_i(0)=h_e(0)=0$, then clearly

$$\lim_{z\to 0}\frac{h_i(z)}{z}=\lambda_i \quad \text{and} \quad \lim_{z\to 0}\frac{h_e(z)}{z}=\lambda_e$$

exist for unrestricted approach, $0<|\lambda_i|,\ |\lambda_e|<\infty$. The function $\zeta=h_e^{1/\alpha}(z)$ maps $E(\Gamma_e)$ onto $\mathscr{I}[\zeta]>0$, Γ and Γ_i onto closed curves Γ^* and Γ_i^*, respectively, which lie in $\mathscr{I}[\zeta]\geqq 0$ and are tangent to the real axis at $\zeta=0$. Let $\phi(\zeta)$ and $\phi_i(\zeta)$ map the interiors of Γ^* and Γ_i^*, respectively, onto the upper half plane, so that $\phi(0)=\phi_i(0)=0$ and, for a point ζ_0 interior to Γ_i^*, $\phi(\zeta_0)=\phi_i(\zeta_0)$. An application of the Wolff-Carathéodory-Landau-Valiron lemma [1, 5] shows that

$$\lim_{\zeta\to 0}\frac{\phi(\zeta)}{\zeta}=l\,, \qquad 0\leqq l<\infty,$$

exists for non-tangential approach. Since

$$\phi_i(\zeta)=h_i^{1/\alpha}[h_e^{-1}(\zeta^\alpha)]\,,$$

where h_e^{-1} denotes the inverse of h_e, it follows that

$$\frac{\phi_i(\zeta)}{\zeta}=\frac{h_i^{1/\alpha}[h_e^{-1}(\zeta^\alpha)]}{\{h_e^{-1}(\zeta^\alpha)\}^{1/\alpha}}\left\{\frac{h_e^{-1}(\zeta^\alpha)}{\zeta^\alpha}\right\}^{1/\alpha}\to\left\{\frac{\lambda_i}{\lambda_e}\right\}^{1/\alpha} \qquad \text{as } \zeta\to 0$$

for unrestricted approach. Hence, $l\geqq\{\lambda_i\lambda_e^{-1}\}^{1/\alpha}>0$.

Finally, we note that

$$g(z)=\{\phi[h_e^{1/\alpha}(z)]\}^{\alpha}$$

and hence

$$\lim_{z\to 0}\frac{g(z)}{z}=l^{\alpha}\lambda_e$$

for non-tangential approach. This proves the lemma[3].

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UNIVERSITY OF MINNESOTA

[3] Another proof of Lichtenstein's theorem may be obtained from an asymptotic expansion due to R. S. Lehman, *Development of the mapping function at an analytic corner*, Stanford, Applied Math. Tech. Rep. No. 21, 1954. This would seem, however, more complicated than the proof given here. The author became aware of Lehman's work only after the present note was submitted for publication.

THE STRICT DETERMINATENESS OF CERTAIN INFINITE GAMES

PHILIP WOLFE

1. Introduction. Gale and Stewart [1] have discussed an infinite two-person game in extensive form which is the generalization of a game as defined by Kuhn [3] obtained by deleting the requirement of finiteness of the game tree and regarding as plays all unicursal paths of maximal length originating in the distinguished vertex x_0. In a *win-lose* game the set S of all plays is divided into two sets S_I and S_{II} such that player I wins the play s if $s \in S_I$ and player II wins it if $s \in S_{II}$. Gale and Stewart have shown that a two-person infinite win-lose game of perfect information with no chance moves (called a GS game here) is strictly determined if S_I belongs to the smallest Boolean algebra containing the open sets of a certain topology for S. Here we answer affirmatively the question posed by them: Is a GS game strictly determined if S_I is a G_δ (or, equivalently, an F_σ)? The notation and results of [1] are used throughout, as well as the partial ordering of X given by: $x > y$ if $f^n(x) = y$ for some $n \geqslant 1$.

2. Alternative description of S_I. Let Γ be the game $(x_0, X_I, X_{II}$ $X, f, S, S_I, S_{II})$, where

$$S_I = \bigcap_{n=1}^{\infty} E_n ,$$

$E_1 \supseteq E_2 \supseteq \cdots$, and E_n is open. Following [3], let the rank $rk(x)$, for $x \in X$, be the unique k such that $f^k(x) = x_0$. As in [1], $\mathfrak{U}(x)$ is the set of all plays passing through x (the topology for S is that in which $\mathfrak{U}(x)$ is a neighborhood of each play in it). Then for each n,

$$E_n = \bigcup \{\mathfrak{U}(y) : \mathfrak{U}(y) \subseteq E_n\} ;$$

and since for any $y \in X$ we have

$$\mathfrak{U}(y) = \bigcup \{\mathfrak{U}(z) : f(z) = y\} ,$$

with

$$rk(z) = 1 + rk(y) ,$$

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there exists for each n a subset Y_n of X such that $rk(y)>n$ for all $y\in Y_n$ and

$$E_n=\bigcup\{\mathfrak{U}(y): y\in Y_n\} .$$

Furthermore, since of any two neighborhoods having a non-void intersection, one is contained in the other, each Y_n may be chosen so that $\mathfrak{U}(y)$, $\mathfrak{U}(y')$ are disjoint for different y, y' in Y_n.

Since $s\in S_I$ if and only if $s\in E_n$ for an infinite number of values of n, we have: $s\in S_I$ if and only if for infinitely many n there exists i (dependent on n) such that $s(i)\in Y_n$. Thus, since on the one hand $i=rk(s(i))>n$, and on the other for any n there is at most one i such that $s(i)\in Y_n$, letting

$$Y=\bigcup_{n=1}^{\infty} Y_n$$

we have: $s\in S_I$ if and only if $s(i)\in Y$ for infinitely many i.

3. Lemmas.

LEMMA 1. *If Γ is a GS game with*

$$\textstyle\sum_{II}^{W}(\Gamma)=\Lambda$$

and

$$T=S-\bigcup\{\mathfrak{U}(x): \textstyle\sum_{II}^{W}(\Gamma_x)\neq\Lambda\} ,$$

then

$$\Gamma_T=(x_0, X_I^T, X_{II}^T, X^T, f^T, T, S_I^T, S_{II}^T)$$

is a subgame of Γ,

$$\textstyle\sum_{I}^{W}(\Gamma_T)\neq\Lambda$$

implies

$$\textstyle\sum_{I}^{W}(\Gamma)\neq\Lambda ,$$

and

$$\textstyle\sum_{II}^{W}((\Gamma_T)_x)=\Lambda$$

for all $x\in X^T$.

Proof. Since T is a closed nonempty subset of S, Γ_T is a subgame of Γ by Theorem 5 of [1]. The second statement follows from assertion B [1, p. 260]. Finally suppose that

$$\textstyle\sum_{II}^{W}((\Gamma_T)_x)\neq\Lambda$$

for some $x\in X^T$. Letting, in assertion A [1, p. 260],

$$F=\mathfrak{U}(x)\cap T ,$$

and noting that F is closed and nonempty and that

$$(\Gamma_T)_x = (\Gamma_x)_F ,$$

we have

$$\sum_{II}^{W}(\Gamma_x) \neq \Lambda ,$$

which is impossible in view of the construction of T.

We assume hereafter that Γ is a GS game with S_I described in terms of $Y \subseteq X$ as in § 2, and that

$$\sum_{II}^{W}(\Gamma) = \Lambda ,$$

whence

$$\sum_{II}^{W}(\Gamma_T) = \Lambda$$

by Lemma 1. The strict determinateness of Γ will follow from Lemma 1 and the fact that

$$\sum_{I}^{W}(\Gamma_T) \neq \Lambda ,$$

proved in § 4.

LEMMA 2. *For $x \in X^T$, we have*

$$s \in S_I^{Tx}$$

if and only if

$$s \in S^{Tx} \quad \textit{and} \quad s(i) \in Y$$

for infinitely many i.

LEMMA 3. *For $x \in X^T$ there exists*

$$\sigma_x \in \sum_I((\Gamma_T)_x)$$

such that for any

$$\tau \in \sum_{II}((\Gamma_T)_x)$$

we have

$$\langle \sigma_x, \tau \rangle (i) \in Y$$

for some $i > rk(x)$.

Proof. Let Y_x be the set of all

$$y \in Y \cap X^T$$

such that $y > x$ and no members of Y fall between x and y. Let Γ' be the game

$$(x_0, X_I^{Tx}, X_{II}^{Tx}, X^{Tx}, f^{Tx}, S^{Tx}, S'_I, S'_{II}) ,$$

where

$$S'_I = S^{Tx} \cap \bigcup \{\mathfrak{U}(y) : y \in Y_x\}$$

and

$$S'_{II} = S^{Tx} - S'_I$$

(that is, the game in which I wins if the play passes through any member of Y following x). Noting that

$$S_I^{Tx} \subseteq S'_I ,$$

we have

$$S'_{II} \subseteq S_{II}^{Tx}$$

and hence

$$\sum_{II}^{W}(\Gamma') = \Lambda .$$

But S'_I is open in S^{Tx} and so Γ' is strictly determined by Corollary 10 of [1], whence there exists

$$\sigma_x \in \sum_I^W(\Gamma') ,$$

which satisfies the conclusion of the lemma.

4. Winning Γ_T. Let

$$Y' = (Y \cap X^T) \cup \{x_0\} .$$

For each $x \in Y'$ let σ_x be as given by Lemma 3, and let σ'_x be the restriction of σ_x to the set of all z in X^T such that $x \leq z$ and that there exists no y in Y' with $x < y \leq z$. We show that the domains of the σ'_x cover X^T and are disjoint: First, if $x_0 \in X_I^T$, then x_0 belongs to the domain of σ_{x_0}. For

$$z \in X_I^T - \{x_0\} ,$$

let

$$x = \max\{z' : z' \in Y' \ \& \ z' < z\} .$$

Then $x \in Y'$ and z belongs to the domain of σ'_x; thus the domains of the σ'_x cover X_I^T. Now suppose that $x_1, x_2 \in Y'$, $x_1 \neq x_2$, and that there exists x_3 common to the domains of σ'_{x_1} and σ'_{x_2}; then $x_1 \leq x_3$ and $x_2 \leq x_3$, so that either $x_1 < x_2 \leq x_3$ or $x_2 < x_1 \leq x_3$, which is impossible in view of the restriction imposed upon σ_x in obtaining σ'_x.

Since the domains of the σ'_x cover X_I^T and are disjoint, they have a common extension σ^*, which necessarily maps the elements of X_I^T on their immediate successors, and thus belongs to $\sum_I(\Gamma_T)$.

We show that σ^* wins Γ_T. Let

$$\tau\in\sum_{II}(\Gamma_T) .$$

For this τ and any x in Y', let $i(x)$ be the least i such that $\langle\sigma_x, \tau\rangle(i)\in Y'$, whose existence is given by Lemma 3. Define $\{x_n\}$ inductively by

$$x_{n+1}=\langle\sigma^*, \tau\rangle(i(x_n)) \qquad n=0, 1, \cdots$$

(x_0 is the distinguished vertex). Since

$$rk(x_{n+1})=i(x_n)>rk(x_n) ,$$

and x_n, x_{n+1} are on a common path, we have $x_{n+1}>x_n$ for all n, and so if $x_n\in Y'$ then

$$x_{n+1}=\langle\sigma^*, \tau\rangle(i(x_n))=\langle\sigma_{x_n}, \tau_{x_n}\rangle(i(x_n))\in Y' ,$$

where

$$\tau_{x_n}\in\sum_{II}((\Gamma_T)_{x_n})$$

is the restriction of τ to $X_{II}^{Tx_n}$. Thus by induction $x_n\in Y'$ for all n, and hence

$$\langle\sigma^*, \tau\rangle(i)\in Y$$

for infinitely many values of i, so that

$$\langle\sigma^*, \tau\rangle\in S_I^{T} .$$

Since τ is arbitrary,

$$\sigma^*\in\sum_I^W(\Gamma_T) ,$$

so that by Lemma 1, we have

$$\sum_I^W(\Gamma)\neq\Lambda .$$

As this is the consequence of the sole fact that

$$\sum_{II}^W(\Gamma)=\Lambda ,$$

Γ is strictly determined.

Reversing the roles of the players in the above gives the result that a GS game is strictly determined if S_I is an F_σ.

The strict determinateness of a two-person zero-sum game with G payoff having *chance moves* can be shown. The proof is more complicated, but uses the same ideas [4].

5. An application. Let

$$\Gamma=(x_0, X_I, X_{II}, X, f, S, \Phi)$$

be a zero-sum two-person infinite game of perfect information with no chance moves having payoff Φ such that there exists a real function h on X ($|h(x)|<K<\infty$) with

$$\Phi(s)=\limsup_{i\to\infty} h(s(i)) \quad \text{for all} \quad s\in S .$$

Γ is the result of an attempt to reduce the following situation to a game: The tree K of a GS game and a function h as above are given; the two players make choices in K in the belief that every play will terminate in some unknown, but distant, vertex x, at which time player I will receive the amount $h(x)$ from player II. A payoff function Φ is sought such that $\Phi(s)$ ($-\Phi(s)$) expresses the utility to player I (II) of a play s in K.

The payoff Φ defined above arises from ascription to players I and II respectively of "optimistic" and "pessimistic" behaviors in this way: Player I assumes that the play s will terminate in some "distant" vertex $s(i)$ at which h assumes nearly its supremum on all "distant" vertices of s; he thus makes his choices so as to maximize the expression

$$\limsup_{i\to\infty} h(s(i))=\Phi(s) ;$$

and player II supposes that s will terminate in some "distant" vertex at which his gain $-h(s(i))$ assumes nearly its infimum for all such vertices, and thus seeks to maximize

$$\liminf_{i\to\infty} -h(s(i))=-\Phi(s) ,$$

that is, to minimize Φ. The derived game is thus zero-sum. Ascription, however, of such "optimistic" or "pessimistic" payoffs to both players yields, in general, a non-zero sum game.

We show now that the game Γ of this section is strictly determined, using the method of Theorem 15 of [1] which asserts the strict determinateness of Γ for the more special case of continuous Φ. (Gillette [2] has shown the strict determinateness of an infinite game of perfect information with chance moves which consists in repeated play from a finite set of finite games and has payoff

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} g_n(s) ,$$

where $g_n(s)$ is the gain from the nth game played.)

First, as a converse to the equivalence of § 2, let $Y\subseteq X$, and denote by Y_n the set of all members of Y having rank greater than n. Then

$$\begin{aligned}\{s: s(i)\in Y \text{ for infinitely many } i\} &= \bigcap_n \{s: s(i)\in Y_n \text{ for some } i\}\\ &= \bigcap_n \bigcup \{\mathfrak{U}(y) ; y\in Y_n\} ,\end{aligned}$$

which is a G_δ.

Now in Γ', for t real, let

$$S_I^t = \{s : h(s(i)) > t \text{ for infinitely many } i\} ,$$

and $S_{II}^t = S - S_I^t$. Then S_I^t is a G_δ, and thus the GS game

$$\Gamma_t = (x_0, X_I, X_{II}, X, f, S, S_I^t, S_{II}^t)$$

is strictly determined. Let

$$v = \sup \{t : \textstyle\sum_I^W(\Gamma_t) \neq \Lambda\} .$$

Since $S_I^K = \Lambda$, $S_I^{-K} = S$, and S_I^t is a decreasing function of t, we have

$$-K \leqslant v \leqslant K , \quad \textstyle\sum_I^W(\Gamma_t) \neq \Lambda \quad \text{if} \quad t < v ,$$

and

$$\textstyle\sum_{II}^W(\Gamma_t) \neq \Lambda \quad \text{if} \quad t > v .$$

Given $\varepsilon > 0$, choose

$$\sigma_0 \in \textstyle\sum_I^W(\Gamma_{v-\varepsilon}) \quad \text{and} \quad \tau_0 \in \textstyle\sum_{II}^W(\Gamma_{v+\varepsilon}) .$$

Then for any

$$\sigma \in \textstyle\sum_I(\Gamma) , \quad \tau \in \textstyle\sum_{II}(\Gamma) ,$$

we have

$$h(\langle \sigma_0, \tau \rangle (i)) > v - \varepsilon \quad \text{for infinitely many } i$$

and do not have

$$h(\langle \sigma, \tau_0 \rangle (i)) > v + \varepsilon \quad \text{for infinitely many } i ;$$

so that

$$\Phi(\langle \sigma_0, \tau \rangle) \geqslant v - \varepsilon \quad \text{and} \quad \Phi(\langle \sigma, \tau_0 \rangle) < v + 2\varepsilon .$$

Hence

$$v - \varepsilon \leqslant \sup_\sigma \inf_\tau \Phi(\langle \sigma, \tau \rangle) \leqslant \inf_\tau \sup_\sigma \Phi(\langle \sigma, \tau \rangle) \leqslant v + 2\varepsilon ;$$

thus Γ is strictly determined, and has value v.

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