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ABSTRACT RIEMANN SUM

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1. Introduction. A theorem of B. Jessen [5] asserts that for f(x) of period one and Lebesgue integrable on [0, 1]

(1)
$$\lim_{n\to\infty} 2^{-n} \sum_{k=0}^{2^n-1} f(x+k2^{-n}) = \int_0^1 f(t)dt \text{ almost everywhere.}$$

We show that the theorem of Jessen is a special case of a theorem analogous to the Birkhoff ergodic theorem [1] but dealing with sums of the form

(2)
$$2^{-n} \sum_{k=0}^{2^{n-1}} f(T^{k/2^{n}} x).$$

In this form T is an operator on a σ -finite measure space such that $T^{1/2^n}$ exists as a one-to-one point transformation which is measure preserving for $n=0, 1, \cdots$, and f(x) is integrable with f(x)=f(Tx). We also obtain in §3 the analogues for abstract Riemann sums of the ergodic theorems of Hurewicz [4] and of Hopf [3].

We might remark that there is no use, due to the examples of Marcinkiewicz and Zygmund [6] and Ursell [8], in considering sums of the form

$$\frac{1}{n}\sum_{k=0}^{n-1}f(T^{k/n}x)$$

without further hypothesis on f(x). However we may replace 2^n throughout by $m_1m_2\cdots m_n$ with m_j integral and $m_j\geq 2$ without altering any argument.

In §4 necessary and sufficient conditions are obtained on a transformation T in order that the sums (2) have a limit as $n \to \infty$ for almost all x. These conditions are analogous to those of Ryll-Nardzewski [7] in the ergodic case. We use the necessary conditions to establish an analogue of a form of the Hurewicz ergodic theorem for two operators [2].

2. Notation. Let (S, Ω, μ) be a fixed σ -finite measure space. We consider throughout point transformations T which have measurable square roots of all orders, that is,

(3.1) There exist one-to-one point transformations T_n so that

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$$T_{n} = T; \quad T_{n}^{2} = T_{n-1}$$
 $n = 1, 2, \cdots$

(3.2) If
$$X \in \Omega$$
, then $T_n X \in \Omega$ and $T_n^{-1} X \in \Omega$, $n=0, 1, \cdots$.

No requirement is made of the uniqueness of the sequence T_n . For example in the theorem of Jessen, T is the identity transformation while $T_n x = x + 2^{-n} \pmod{1}$. We also suppose throughout that T is measure preserving

(3.3)
$$\mu(TX) = \mu(X) \quad for \quad X \in \Omega.$$

3. Limit theorems. Let ϕ be a finite valued set function defined on Ω and absolutely continuous with respect to μ . Form the sums

(4)
$$\Phi_n(X) = \sum_{k=0}^{2^n-1} \Phi(T_n^k X)$$
 $n=0, 1, \cdots,$

and

(5)
$$\mu_n(X) = \sum_{k=0}^{2^n-1} \mu(T_n^k X)$$
 $n=0, 1, \cdots$

Then Φ_n is absolutely continuous with respect to μ_n and there exists an averaging sequence of point functions $f_n(x)$ so that

(2)
$$\Phi_n(X) = \int_X f_n(x) \mu_n(dx), \qquad n = 0, 1, \cdots.$$

THEOREM 1. Let T be a transformation such that (3.1), (3.2) and (3.3) are satisfied. Let φ be a finite valued set function defined on Ω , absolutely continuous with respect to μ and such that $\varphi(TX) = \varphi(X)$. Then for almost all $x[\mu]$ the averaging sequence of point functions defined by (4), (5) and (6) has a limit as $n \to \infty$. The limit function F(x) has the following properties:

- (i) $F(T_n x) = F(x)$ almost everywhere $[\mu]$, $n=0, 1, \cdots$.
- (ii) F(x) is integrable over S.
- (iii) For any set X with $T_nX=X$, $n=0, 1, \cdots$ and $\mu(X) < \infty$

$$\int_{x} F(x)\mu(dx) = \int_{x} f(x)\,\mu(dx).$$

Proof. Note first that since $\Phi(TX) = \Phi(X)$,

(7)
$$\varphi_n(T_nX) = \sum_{k=0}^{2^n-1} \varphi(T_n^{k+1}X) = \varphi(X).$$

Likewise

862

(8)
$$\mu_n(T_nX) = \mu_n(X) .$$

Therefore for all X

$$\int_{X} f_{n}(T_{n}x)\mu_{n}(dx) = \int_{T_{n}x} f_{n}(x) \mu_{n}(dx) = \int_{X} f_{n}(x) \mu_{n}(dx)$$

and consequently

(9)
$$f_n(T_n x) = f_n(x)$$
 almost everywhere $[\mu_n]$.

Relation (3.1) then implies

(10)
$$\begin{cases} \lim_{n \to \infty} f_n(T^j_m x) = \lim_{n \to \infty} f_n(x) \\ \lim_{n \to \infty} f_n(T^j_m x) = \lim_{n \to \infty} f_n(x) \end{cases} \text{ almost everywhere } \begin{bmatrix} \mu \end{bmatrix} \ j=1, \ \cdots, \ 2^m - 1 \\ m=1, \ 2, \ \cdots \end{cases}$$

Let

(11)
$$A = \{x | \sup_{0 \le n} f_n(x) \ge 0\}.$$

It is asserted that

(12)
$$\int_{A} f_{0}(x)\mu(dx) \geq 0.$$

We define the following sets:

$$P_{j} = \{x | f_{j}(x) \ge 0\} \qquad j = 0, \ 1, \ \cdots$$

$$A_{N} = \{x | \sup_{0 \le n \le N} f_{n}(x) \ge 0\} \qquad N = 0, \ 1, \ \cdots$$

$$C_{N, \ j} = P'_{N} \cap \cdots \cap P'_{j+1} \cap P_{j} \qquad j = 0, \ \cdots, \ N.$$

Now (9) together with (3.1) imply that $T_k P_j = P_j$ for $k \leq j$. Consequently

$$T_{j}C_{N, j} = C_{N, j}$$
 and $\Phi(C_{N, j}) = \Phi(T_{j}^{k}C_{N, j})$.

Therefore

$$2^{j} \varphi(C_{N, j}) = \sum_{k=0}^{2^{j}-1} \varphi(T_{j}^{k} C_{N, j}) = \varphi_{j}(C_{N, j})$$

and

$$2^{j} \varphi(C_{N,j}) = \int_{C_{N,j}} f_{j}(x) \mu_{j}(dx) \ge 0, \qquad j=0, \cdots, N.$$

Since the $C_{N,j}$ are disjoint for $j=0, \dots, N$, we have $\varphi(A_N) \ge 0$ and by a limiting process we obtain (12).

Likewise if

(13)

 $B = \{x | \inf_{0 \leq n} f_n(x) \ge 0\}$,

then

(14)
$$\int_{B} f_{0}(x)\mu(dx) \geq 0.$$

Inasmuch as the preceding argument made no use of the finiteness of Φ , we may apply the result to the set function $\Psi = \Phi - c\mu$ for any real c. Since

$$\Psi_n(X) = \int_X (f_n(x) - c) \mu_n(dx)$$

we deduce that for

(15) $A^{c} = \{x | \sup_{0 \leq n} f_{n}(x) \geq c\}$

we have

(16) $\varphi(A^c) \ge c \mu(A^c)$

and for

(17)
$$A_{a} = \{x | \inf_{0 \leq n} f_{n}(x) \leq d\}$$

we have

Let now for r > s

(19)
$$L_s^r = \{x | \lim_{n \to \infty} f_n(x) > r \text{ and } \lim_{n \to \infty} f_n(x) < s\}$$
.

From (10) we obtain

(20)
$$T_m^j L_s^r = L_s^r$$
 $j=0, 1, \cdots, 2^m-1; m=0, 1, \cdots$

Since L_s^r is invariant under each T_m we may consider it as a new space. The sets A^r and A_s relative to the new space are now the full space L_s^r . Hence if we apply (16) and (18) we obtain

$$arphi(L^r_s){\ge}r\mu(L^r_s)$$
 ; $\hspace{0.1cm} heta(L^r_s){\le}s\mu(L^r_s)$.

The finiteness of Φ together with the assumption r > s implies $\mu(L_s^r) = 0$. Thus $\lim f_n(x)$ exists almost everywhere $[\mu]$.

Property (i) of the limit function F(x) follows immediately from (10). Utilizing (i) the proofs of (ii) and (iii) are now identical with

the corresponding proofs by Hurewicz [4, p. 201] in the ergodic case.

The theorem for abstract Riemann sums analogous to the Hopf ergodic theorem is now deducible as a corollary.

COROLLARY 1. Let T be a transformation such that (3.1) and (3.2) are satisfied and in addition

(21)
$$\mu(T_n X) = \mu(X)$$
 $n = 0, 1, \cdots$

Then for any integrable f(x) with f(Tx)=f(x) and any g(x)>0 with g(Tx)=g(x)

(22)
$$\lim_{n \to \infty} \sum_{k=0}^{2^{n-1}} f(T_n^k x) \\ \sum_{k=0}^{2^{n-1}} g(T_n^k x)$$

exists for almost every $x \ [\mu]$. The limit function h(x) is integrable, satisfies $h(T_n x) = h(x)$ for almost all $x \ [\mu]$, and for sets Y with $\mu(Y) < \infty$ and $T_m Y = Y$, $m = 0, 1, \cdots$

(23)
$$\int_{Y} h(x)g(x)\mu(dx) = \int_{Y} f(x)\mu(dx).$$

Proof. Introduce the measure

$$\nu(X) = \int_X g(x)\mu(dx),$$

and the set function

$$F(X) = \int_X f(x)\mu(dx).$$

The function F is absolutely continuous with respect to ν and is finite valued. Condition (21) implies that

$$F_n(X) = \int_X^{2^{n-1}} \int_{x=0}^{2^{n-1}} f(T_n^k x) \mu(dx)$$

and

$$\nu_n(X) = \int_X \sum_{k=0}^{2^n-1} g(T_n^k x) \mu(dx).$$

Thus from the representation

$$F_n(X) = \int_X f_n(x) \nu_n(dx)$$

we deduce that

$$f_n(x) = rac{\sum\limits_{k=0}^{2^n-1} f(T_n^k x)}{\sum\limits_{k=0}^{2^n-1} g(T_n^k x)}$$
 almost everywhere $[\mu].$

The corollary is then an immediate consequence of Theorem 1.

The theorem of Jessen now follows from the version of Corollary 1 with g(x)=1 with the T_n as noted in § 2.

4. Invariant measure and two operators. It is possible for the conclusion of Corollary 1 to hold when g(x)=1 but T does not satisfy (21). If we introduce

(24)
$$R_n(A, Y) = 2^{-n} \sum_{k=0}^{2^n - 1} \mu(Y \cap T_n^{-k} A)$$

we obtain the following theorem.

THEOREM 2. If T is a transformation such that (3.1) and (3.2) are satisfied, then the following statements are equivalent:

(25.1) For every integrable f(x) with f(Tx)=f(x),

$$\lim_{n\to\infty}2^{-n}\sum_{k=0}^{2^n-1}f(T_n^kx)$$

exists for almost every $x \ [\mu]$.

- (25.2) For each Y with $\mu(Y) \leq \infty$, $\lim_{n \to \infty} R_n(A, Y) \leq K \mu(A)$.
- (25.3) For each Y with $\mu(Y) \leq \infty$, $\lim_{n \to \infty} R_n(A, Y) \leq K \mu(A)$.
- (25.4) For an increasing sequence of sets Y_j with $\bigcup_{j=1}^{\infty} Y_j = S$,

$$\lim_{n\to\infty} R_n(A, Y_j) \leq K \mu(A) .$$

(25.5) There exists a countably additive measure ν with the properties:

(i) $0 \le \nu(X) \le K \mu(X)$ (ii) If $A = T_n A$, $n = 1, 2, \dots, \nu(A) = \mu(A)$ (iii) $\nu(A) = \nu(T_n A)$, $n = 1, 2, \dots$.

The proof is almost identical with that of Ryll-Nardzewski [7] in

866

the ergodic case, and is omitted. The existence of an invariant measure implies, as in the ergodic case [2], the following theorem with two operators (or two sequences of roots of the same operator).

THEOREM 3. Let T and U each satisfy (3.1), (3.2), (3.3) and (25.1), and let

$$\sum_{k=0}^{2^n-1}\mu(T^k_nX)$$

be absolutely continuous with respect to

$$\mu_n(X) = \sum_{k=0}^{2^{n-1}} \mu(U_n^k X), \qquad n = 0, \ 1, \ \cdots.$$

For any finite valued set function Φ absolutely continuous with respect to μ and with $\Phi(TX) = \Phi(X)$ form

$$\varphi_n(X) = \sum_{k=0}^{2^n-1} \varphi(T_n X).$$

Then in the representation

$$\varphi_n(X) = \int_X f_n(x) \mu_n(dx),$$

the averaging sequence of point functions $f_n(x)$ tends to a limit as $n \to \infty$ for almost every x [μ].

As a consequence of Theorem 3 we obtain the following corollary in the same fashion as Corollary 1 was derived from Theorem 1.

COROLLARY 2. Let T and U each satisfy (3.1) and (3,2), and in addition

(26)
$$\mu(V_n X) = \mu(X) \qquad n = 0, \cdots$$

for V=T and V=U. Then for any integrable f(x) with f(Tx)=f(x)and any g(x)>0 with g(Ux)=g(x)

$$\lim_{n \to \infty} \frac{\sum\limits_{k=0}^{2^{n}-1} f(T_{n}^{k}X)}{\sum\limits_{k=0}^{2^{n}-1} g(U_{n}^{k}X)}$$

exists for almost all $x [\mu]$.

PAUL CIVIN

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Pacific Journal of Mathematics Vol. 5, No. 6 , 1955

Densities Auslander, The use of forms in variational calculation Paul Civin, Abstract Riemann sum Paul Civin, Some ergodic theorems involving two operator Paul Civin, Some ergodic theorems involving two operator Eckford Cohen, The number of solutions of certain cubic congruence Paul Civin, Some ergodic theorems involving two operator Richard M. Cohn, Specializations over difference field Paul Civin, Some ergodic theorem for eigenvalues of normal matrice Bichard P. Gosselin, On the convergence behaviour of trigonometric interpolating polynomial Peter K. Richard P. Gosselin, On the convergence behaviour of trigonometric interpolating polynomial Peter K. Meyer Jerison, An algebra associated with a compact grou Wilhelm Magnus, Infinite determinants associated with Hill's equatio G. Power and D. L. Scott-Hutton, The slow steady motion of liquid past a semi-elliptical bos Peter characterization of fixed ideals in certain function ring C. T. Rajagopal, Additional note on some Tauberian theorems of O. Szás Peter Schenkman, On the tower theorem for finite group Shigeo Sasaki and Kentaro Yano, Pseudo-analytic vectors on pseudo-Kählerian manifold Peter Schenkman, On the tower theorem for finite group P. Stein and John E. L. Peck, On the numerical solution of Poisson's equation over a rectangl Poisson's theorem for finite generation of the preserve divisio Morgan Ward, The mappings of the positive integers into themselves which preserve divisio Pres	Nesmith Cornett Ankeny and Theodore Joseph Rivlin, <i>On a theorem of S.</i> <i>Bernstei</i>	8
 Paul Civin, Abstract Riemann sum. Paul Civin, Some ergodic theorems involving two operator. Eckford Cohen, The number of solutions of certain cubic congruence Richard M. Cohn, Specializations over difference field. Jean Dieudonné, Pseudo-discriminant and Dickson invarian Ky Fan, A comparison theorem for eigenvalues of normal matrice Richard P. Gosselin, On the convergence behaviour of trigonometric interpolating polynomial Peter K. Henrici, On generating functions of the Jacobi polynomial Meyer Jerison, An algebra associated with a compact grou. Wilhelm Magnus, Infinite determinants associated with Hill's equatio. G. Power and D. L. Scott-Hutton, The slow steady motion of liquid past a semi-elliptical bos Lyle E. Pursell, An algebraic characterization of fixed ideals in certain function ring C. T. Rajagopal, Additional note on some Tauberian theorems of O. Szás Louis Baker Rall, Error bounds for iterative solutions of Fredholm integral equation Shigeo Sasaki and Kentaro Yano, Pseudo-analytic vectors on pseudo-Kählerian manifold Eugene Schenkman, On the tower theorem for finite group P. Stein and John E. L. Peck, On the numerical solution of Poisson's equation over a rectangl Morgan Ward, The mappings of the positive integers into themselves which preserve divisio In our set of the provide the preserve divisio 	Louis Auslander, <i>The use of forms in variational calculation</i>	8
 Paul Civin, Some ergodic theorems involving two operator Eckford Cohen, The number of solutions of certain cubic congruence Richard M. Cohn, Specializations over difference field Jean Dieudonné, Pseudo-discriminant and Dickson invarian Ky Fan, A comparison theorem for eigenvalues of normal matrice Richard P. Gosselin, On the convergence behaviour of trigonometric interpolating polynomial Peter K. Henrici, On generating functions of the Jacobi polynomial Meyer Jerison, An algebra associated with a compact grou Wilhelm Magnus, Infinite determinants associated with Hill's equatio G. Power and D. L. Scott-Hutton, The slow steady motion of liquid past a semi-elliptical bos Lyle E. Pursell, An algebraic characterization of fixed ideals in certain function ring C. T. Rajagopal, Additional note on some Tauberian theorems of O. Szás Louis Baker Rall, Error bounds for iterative solutions of Fredholm integral equation Shigeo Sasaki and Kentaro Yano, Pseudo-analytic vectors on pseudo-Kählerian manifold Eugene Schenkman, On the tower theorem for finite group P. Stein and John E. L. Peck, On the numerical solution of Poisson's equation over a rectangl Morgan Ward, The mappings of the positive integers into themselves which preserve divisio In us Weisner, Group-theoretic origin of certain generating function 	Paul Civin, Abstract Riemann sum	8
 Eckford Cohen, <i>The number of solutions of certain cubic congruence</i> Richard M. Cohn, <i>Specializations over difference field</i>	Paul Civin, Some ergodic theorems involving two operator	8
 Richard M. Cohn, Specializations over difference field	Eckford Cohen, The number of solutions of certain cubic congruence	8
 Jean Dieudonné, Pseudo-discriminant and Dickson invarian	Richard M. Cohn, Specializations over difference field	8
 Ky Fan, A comparison theorem for eigenvalues of normal matrice Richard P. Gosselin, On the convergence behaviour of trigonometric interpolating polynomial Peter K. Henrici, On generating functions of the Jacobi polynomial Meyer Jerison, An algebra associated with a compact grou Wilhelm Magnus, Infinite determinants associated with Hill's equatio G. Power and D. L. Scott-Hutton, The slow steady motion of liquid past a semi-elliptical bos Lyle E. Pursell, An algebraic characterization of fixed ideals in certain function ring C. T. Rajagopal, Additional note on some Tauberian theorems of O. Szás Louis Baker Rall, Error bounds for iterative solutions of Fredholm integral equation Shigeo Sasaki and Kentaro Yano, Pseudo-analytic vectors on pseudo-Kählerian manifold P. Stein and John E. L. Peck, On the numerical solution of Poisson's equation over a rectangl Morgan Ward, The mappings of the positive integers into themselves which preserve divisio	Jean Dieudonné, Pseudo-discriminant and Dickson invarian	9
 Richard P. Gosselin, On the convergence behaviour of trigonometric interpolating polynomial	Ky Fan, A comparison theorem for eigenvalues of normal matrice	9
 interpolating polynomial	Richard P. Gosselin, On the convergence behaviour of trigonometric	
 Peter K. Henrici, On generating functions of the Jacobi polynomial Meyer Jerison, An algebra associated with a compact grou Wilhelm Magnus, Infinite determinants associated with Hill's equatio G. Power and D. L. Scott-Hutton, The slow steady motion of liquid past a semi-elliptical bos Lyle E. Pursell, An algebraic characterization of fixed ideals in certain function ring C. T. Rajagopal, Additional note on some Tauberian theorems of O. Szás Louis Baker Rall, Error bounds for iterative solutions of Fredholm integral equation Shigeo Sasaki and Kentaro Yano, Pseudo-analytic vectors on pseudo-Kählerian manifold P. Stein and John E. L. Peck, On the numerical solution of Poisson's equation over a rectangl Morgan Ward, The mappings of the positive integers into themselves which preserve divisio	interpolating polynomial	9
 Meyer Jerison, An algebra associated with a compact grou	Peter K. Henrici, On generating functions of the Jacobi polynomial	9
 Wilhelm Magnus, Infinite determinants associated with Hill's equatio G. Power and D. L. Scott-Hutton, The slow steady motion of liquid past a semi-elliptical bos Lyle E. Pursell, An algebraic characterization of fixed ideals in certain function ring C. T. Rajagopal, Additional note on some Tauberian theorems of O. Szás Louis Baker Rall, Error bounds for iterative solutions of Fredholm integral equation Shigeo Sasaki and Kentaro Yano, Pseudo-analytic vectors on pseudo-Kählerian manifold P. Stein and John E. L. Peck, On the numerical solution of Poisson's equation over a rectangl Morgan Ward, The mappings of the positive integers into themselves which preserve divisio	Meyer Jerison, An algebra associated with a compact grou	ç
 G. Power and D. L. Scott-Hutton, <i>The slow steady motion of liquid past a semi-elliptical bos</i>. Lyle E. Pursell, <i>An algebraic characterization of fixed ideals in certain function ring</i>. C. T. Rajagopal, <i>Additional note on some Tauberian theorems of O. Szás</i>. Louis Baker Rall, <i>Error bounds for iterative solutions of Fredholm integral equation</i>. Shigeo Sasaki and Kentaro Yano, <i>Pseudo-analytic vectors on pseudo-Kählerian manifold</i>. Eugene Schenkman, <i>On the tower theorem for finite group</i>. P. Stein and John E. L. Peck, <i>On the numerical solution of Poisson's equation over a rectangl</i>. Morgan Ward, <i>The mappings of the positive integers into themselves which preserve divisio</i>. Louis Weisner, <i>Group-theoretic origin of certain generating function</i>. 	Wilhelm Magnus, Infinite determinants associated with Hill's equatio	ç
 Lyle E. Pursell, An algebraic characterization of fixed ideals in certain function ring	G. Power and D. L. Scott-Hutton, <i>The slow steady motion of liquid past a semi-elliptical bos</i>	ç
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 Shigeo Sasaki and Kentaro Yano, <i>Pseudo-analytic vectors on pseudo-Kählerian manifold</i> Eugene Schenkman, <i>On the tower theorem for finite group</i> P. Stein and John E. L. Peck, <i>On the numerical solution of Poisson's equation over a rectangl</i> Morgan Ward, <i>The mappings of the positive integers into themselves which preserve divisio</i> Seth Warner, <i>Weak locally multiplicatively-convex algebra</i> Louis Weisner, <i>Group-theoretic origin of certain generating function</i> 	equation	9
Eugene Schenkman, On the tower theorem for finite group P. Stein and John E. L. Peck, On the numerical solution of Poisson's equation over a rectangl Morgan Ward, The mappings of the positive integers into themselves which preserve divisio Seth Warner, Weak locally multiplicatively-convex algebra Iouis Weisner, Group-theoretic origin of certain generating function	Shigeo Sasaki and Kentaro Yano, <i>Pseudo-analytic vectors</i> on pseudo-Kählerian manifold	ç
 P. Stein and John E. L. Peck, On the numerical solution of Poisson's equation over a rectangl. Morgan Ward, The mappings of the positive integers into themselves which preserve divisio. Seth Warner, Weak locally multiplicatively-convex algebra. Louis Weisner. Group-theoretic origin of certain generating function. 	Eugene Schenkman. On the tower theorem for finite group	ç
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Louis Weisner, Group-theoretic origin of certain generating function	Seth Warner Weak locally multiplicatively-convex algebra	10
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