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Introduction. We consider a system S of algeraic difference equations with coefficients in a difference field $\mathscr{F}$ and involving also parameters λ_i. Well-known results concerning systems of algebraic equations and systems of algebraic differential equations would lead one to expect that, if S has solutions in some extension of the difference field formed by adjoining the parameters λ_i to $\mathscr{F}$, then the system resulting from S by assigning special values to the λ_i has solutions, provided only that the special values are chosen so as not to annul a certain difference polynomial. But the examples in [5, p. 510] show that this is not so.

The difficulty in these examples arises from the fact that a difference field $\mathscr{F}$ may have incompatible extensions, that is to say, extensions which cannot both be embedded isomorphically in any one of its extensions. In particular, it may happen that one can express in terms of a solution of the system S an element α, independent of the λ_i. Then α will be contained in the difference field formed by adjoining to $\mathscr{F}$ a solution of any system (possessing solutions) which arises by specializing the parameters of S. It will then not be possible to find solutions if one specializes the λ_i in such a way that the extension of $\mathscr{F}$ formed by adjoining the specialized values is incompatible with that formed by adjoining α.

The principal result of this paper is that one can restore the expected result concerning the specialization of parameters of S by imposing a suitable condition of compatibility. If the system S has solutions, then, in order to assure that the system obtained from S by specializing the parameters has solutions, it suffices to choose the specializations from an extension of $\mathscr{F}$ compatible with a certain extension $\mathscr{G}$ of $\mathscr{F}$ and not annulling a certain difference polynomial. In particular, if $\mathscr{F}$ is algebraically closed it has no incompatible extensions so that it suffices to choose specializations of the parameters not annulling a certain difference polynomial. Hence, in this case, one has the same freedom of specialization as with systems of algebraic equations. Even in the general case, there is considerable freedom as the compatibility condition will evidently be satisfied if the specialized values are chosen from $\mathscr{G}$ itself or any extension of $\mathscr{G}$. We turn now to a formal discussion of this theorem.

We consider a difference field $\mathscr{F}$ and extensions $\mathscr{G}$ and $\mathscr{H}$ of

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$\mathscr{F}$. Let a set S of elements α_i be selected from $\mathscr{G}$ and a set $\bar{S}$ of elements $\bar{\alpha}_i$ from $\mathscr{H}$, where the index i has the same range, finite or infinite, in each case. We shall say that the $\bar{\alpha}_i$ constitute a *specialization over* $\mathscr{F}$ of the α_i if there is a homomorphism of the difference ring[1] $\mathscr{F}\{S\}$ onto the difference ring $\mathscr{F}\{\bar{S}\}$, this homomorphism leaving the elements of $\mathscr{F}$ fixed and carrying each α_i into $\bar{\alpha}_i$.

We wish to discuss the following question. Let $\beta_1, \cdots, \beta_q; \gamma_1, \cdots, \gamma_p$ be a set of elements lying in an extension of the difference field $\mathscr{F}$ and such that no nonzero difference polynomial in $\mathscr{F}\{u_1, \cdots, u_q\}$ vanishes when we substitute β_i for u_i, $i=1, \cdots, q$. Let $\bar{\beta}_1, \cdots, \bar{\beta}_q$ constitute a specialization over $\mathscr{F}$ of $\beta_1, \cdots, \beta_q$. Under what circumstances do there exist elements $\bar{\gamma}_1, \cdots, \bar{\gamma}_p$ such that $\bar{\beta}_1, \cdots, \bar{\beta}_q; \bar{\gamma}_1, \cdots, \bar{\gamma}_p$ constitute a specialization over $\mathscr{F}$ of the set $\beta_1, \cdots, \beta_q; \gamma_1, \cdots, \gamma_p$? If such elements $\bar{\gamma}_j$ exist we shall say that the specialization of the β_i can be *extended* to a specialization of the β_i and γ_j. We have already indicated that, in order to insure the possibility of the extension, we must impose a condition of compatibility. Our principal result is contained in the following theorem.

THEOREM 1. *Given a difference field* $\mathscr{F}$ *and an extension*

$$\mathscr{H}=\mathscr{F}\langle\beta_1, \cdots, \beta_q; \gamma_1, \cdots, \gamma_p\rangle$$

of $\mathscr{F}$ *which is such that the degree of transformal transcendence of* $\mathscr{G}=\mathscr{F}\langle\beta_1, \cdots, \beta_q\rangle$ *over* $\mathscr{F}$ *is* q, *there exists a nonzero element* δ *in* $\mathscr{F}\{\beta_1, \cdots, \beta_q\}$ *such that any specialization* $\bar{\beta}_1, \cdots, \bar{\beta}_q$ *over* $\mathscr{F}$ *of* $\beta_1, \cdots, \beta_q$ *with the properties that*

(a) $\mathscr{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q\rangle$ *and* $\mathscr{H}$ *are compatible extensions of* $\mathscr{F}$,
(b) *the specialization of* δ *is not zero,*

can be extended to a specialization $\bar{\beta}_1, \cdots, \bar{\beta}_q; \bar{\gamma}_1, \cdots, \bar{\gamma}_p$ *over* $\mathscr{F}$ *of* $\beta_1, \cdots, \beta_q; \gamma_1, \cdots, \gamma_p$.

It is evident that Theorem 1 may be applied to show that *zeros of a reflexive prime difference ideal may be found for all assignments of values to its parametric indeterminates (if any) which lie in an extension of a certain field and do not annul a certain nonzero difference polynomial in the parametric indeterminates.*

The condition that $\beta_1, \cdots, \beta_q$ annul no nonzero difference polynomial

[1] For this and similar notations see [**5**, pp. 508 and 513]. Basic definitions will be found in [**9**], [**8**], [**1**] and [**5**].

with coefficients in $\mathscr{F}$ is essential in Theorem 1. Let β be an element transcendental over the field $\mathfrak{R}$ of rational numbers, and consider the difference field $\mathfrak{R}\langle\beta\rangle$, whose elements are their own transforms. We may extend $\mathfrak{R}\langle\beta\rangle$ to $\mathfrak{R}\langle\beta, \gamma\rangle$, where $\gamma^2=\beta$, $\gamma_1=-\gamma$ (subscripts now denote transforms). Then β may be specialized to the square of an element of $\mathfrak{R}$. No such specialization can be extended to γ. It is evident that this implies that no element δ exists with the properties prescribed in Theorem 1.

We give the proof of Theorem 1 in § 2 using preliminary lemmas proved in § 1. It is possible for a set S of elements α_i, $i=1, \cdots, n$, to specialize to a set $\bar{S}$ such that $\mathscr{F}\langle S\rangle$ and $\mathscr{F}\langle\bar{S}\rangle$ are incompatible extensions of $\mathscr{F}$. In § 3 we give an example of such a specialization and prove a theorem to the general effect that such specializations are scarce.

1. Proof of two lemmas.

1. 1. *Absolutely irreducible polynomials.* Let there be given a set S of elements λ_i, where the index i ranges over a suitable set of ordinals, and the λ_i lie in an extension of a field (not a difference field) $\mathscr{F}$ of characteristic 0. Let P be an absolutely irreducible polynomial in $\mathscr{F}(S)[x_1, \cdots, x_n]$. We shall show that almost every specialization of the λ_i specializes P into an absolutely irreducible polynomial. Specifically, we shall prove the following result.

LEMMA 1. *There is a nonzero element γ in $\mathscr{F}[S]$ snch that for any specialization of the λ_i over $\mathscr{F}$ for which γ does not specialize to zero, specializations of the coefficients of P are defined, and the polynomial $\bar{P}$ which is obtained by replacing the coefficients in P by their specializations is of the same degree as P in x_n and is absolutely irreducible.*

Proof. Using a device due to Kronecker [11, VI, p. 129] we introduce an auxiliary variable t and replace each x_i in P by $t^{m^{i-1}}$, where m is an integer exceeding the degree of P in any x_i. Then P goes over into a polynomial P^* in t. In the algebraic closure of $\mathscr{F}(S)$, P^* factors into (not necessarily distinct) linear factors

$$P^*=P_1 \cdots P_r .$$

Let S_i^*, $i=1, \cdots, 2^r-2=\nu$, denote the products of all subsets of from 1 to $r-1$ of the P_i. Let $T_i^*=P^*/S_i^*$.

In each S_i^* and T_i^* the powers of t may be replaced in a unique way by power products of the x_i which correspond to them by the substitution of the preceding paragraph and are of degree less than m in each x_i. Let polynomials S_i and T_i result from these replacements.

The absolute irreducibility of P is equivalent to its irreducibility in the algebraic closure of $\mathscr{F}(S)$ and this, in turn, is equivalent to the statement that none of the polynomials $Q_i = P - S_i T_i$, $i=1, \cdots, \nu$, is zero. Let ϕ_i, $i=1, \cdots, \nu$, be the coefficient of a term which appears effectively in Q_i. Let $\phi = \phi_1 \cdots \phi_\nu$. Let θ be the coefficient of a term of P which is of highest degree in x_n.

There exists an element γ in $\mathscr{F}[S]$ such that for any specialization of the λ_i for which γ does not specialize to zero:

(a) Specializations exist for all the coefficients of P,
(b) The specialization may be extended so as to define specializations for each coefficient occurring in the S_i and T_i,
(c) $\phi\theta$ does not specialize to zero under the extended specialization.

γ has the properties claimed in the statement of Lemma 1. For the existence of specializations of the coefficients of P is guaranteed in (a). The equality of the degrees of P and $\overline{P}$ in x_n follows from (c). It follows from (b) that polynomials $\overline{Q_i}$, $\overline{S_i}$, $\overline{T_i}$, $\overline{S_i}^*$ and $\overline{T_i}^*$ may be defined as the polynomials resulting by replacements of the coefficients of the Q_i, S_i, T_i, S_i^* and T_i^* respectively by their specializations. By Condition (c) no $\overline{Q_i}$ is zero. This implies the absolute irreducibility of $\overline{P}$. For $\overline{P^*} = \overline{P_1} \cdots \overline{P_r}$, where the $\overline{P_i}$ (which coincide with certain $\overline{S_i}^*$) result from the specialization of the coefficients of the P_i. Since the $\overline{P_i}$ are of degree zero[2] or one in t, factors of $\overline{P}$ in any extension of its coefficient field can be found by the method of Kronecker from the $\overline{P_i}$ of first degree. The $\overline{Q_i}$ relate to the $\overline{P_i}$ in the same way as the Q_i to the P_i. Hence if $\overline{P}$ had a proper factorization in any field, then some $\overline{Q_i}$ would be zero. This completes the proof of Lemma 1.

1. 2. *Absolutely irreducible manifolds.* Let Σ be a prime p.i.[3] (polynomial ideal) in $\mathscr{F}(S)[u_1, \cdots, u_q; x_1, \cdots, x_p]$, the u_i constituting a set of parametric indeterminates for Σ. Let $A_1, \cdots, A_p$ be a characteristic set of Σ with A_i introducing x_i, $i=1, \cdots, p$. We suppose that the manifold $\mathfrak{M}$ of Σ is absolutely irreducible. Then the following generalization of Lemma 1 may be proved.

LEMMA 2. *There is a nonzero element γ in $\mathscr{F}[S]$ such that for any*

[2] Actually no $\overline{P_i}$ is of zero degree, for this would imply that some $\overline{Q_i} = 0$.

[3] We use this term as in [**9**, Chapter IV], to designate ideals in polynomial rings as distinguished from difference ideals.

specialization $\bar{\lambda}_i$ *of the* λ_i *for which* γ *does not specialize to zero, specializations of the coefficients of* $A_1, \cdots, A_p$ *are defined, and the polynomials* $\bar{A}_1, \cdots, \bar{A}_p$ *which are obtained by replacing the coefficients of* $A_1, \cdots, A_p$*, respectively, by their specializations form a characteristic set of a prime* p.i. $\bar{\Sigma}$ *in* $\mathscr{F}(\bar{S})[u_1, \cdots, u_q; x_1, \cdots, x_p]$*, where* $\bar{S}$ *denotes the set of* $\bar{\lambda}_i$. *The manifold of* $\bar{\Sigma}$ *is absolutely irreducible. Each* $\bar{A}_i$ *is of the same degree as* A_i *in* x_i. *The* $\bar{\lambda}_i$ *and a generic zero of* $\bar{\Sigma}$ *constitute a specialization over* $\mathscr{F}$ *of the* λ_i *and a generic zero of* Σ.

Proof. Let

$$w = \sum_{i=1}^{p} a_i x_i ,$$

the a_i integers, be a resolvent unknown for Σ; let G be the corresponding resolvent, Π the prime p. i.

$$\left(\Sigma ,\ w - \sum_{i=1}^{p} a_i x_i\right) .$$

Then Π contains polynomials $M_i x_i - N_i$, $i=1, \cdots, p$, where the M_i and the N_i are polynomials in w and the u_i of lower degree in w than G, and the M_i are nonzero.

$$(1) \qquad G;\ M_1 x_1 - N_1, \cdots, M_p x_p - N_p$$

is a characteristic set of Π corresponding to the ordering $u_1, \cdots, u_q$; w; $x_1, \cdots, x_p$ of the indeterminates, which we use throughout the following discussion.

G is absolutely irreducible. For, by [5, p. 514], the reducibility of G in any field would imply the reducibility of Σ in some extension of $\mathscr{F}$. Hence, by the preceding lemma, there is a nonzero element γ_1 of $\mathscr{F}[S]$ such that, for all specializations of the λ_i for which γ_1 does not vanish the coefficients of G specialize, and the polynomial $\bar{G}$ which is obtained by replacing the coefficients of G by their specializations is absolutely irreducible and is of the same degree as G in w.

Each coefficient of the M_i, N_i and A_i may be written as a quotient of elements of $\mathscr{F}[S]$. Let δ be the product of the denominators of these quotients. Let η_i/δ, $i=1, \cdots, p$, the $\eta_i \neq 0$, be coefficients of terms of the M_i. Let η be the product of the η_i.

Let I be the product of the initials of the A_i, and J the remainder of I with respect to (1). J is a nonzero polynomial in w and the u_i. Some coefficient of J has the form κ/δ^t, where $\kappa \neq 0$ is in $\mathscr{F}[S]$ and t is a positive integer.

We let $\gamma = \gamma_1 \delta \eta \kappa$. For any specialization of the λ_i to a set $\bar{S}$ of elements $\bar{\lambda}_i$ for which γ does not specialize to zero we may define polynomials $\overline{M}_i$, $\overline{N}_i$, $\overline{A}_i$ and $\overline{G}$ which result from the M_i, N_i, A_i and G respectively by the specialization of their coefficients. $\overline{G}$ is absolutely irreducible. The $\overline{M}_i$ are not zero and are reduced with respect to $\overline{G}$ since $\overline{G}$ is of the same degree as G in w. Hence

$$(2) \qquad \overline{G}; \quad \overline{M}_1x_1-\overline{N}_1, \cdots, \overline{M}_px_p-\overline{N}_p$$

is a characteristic set of a prime p.i. $\overline{\Pi}$ in $\mathscr{F}(\overline{S})[u_1, \cdots, u_q; w; x_1, \cdots, x_p]$. Each $\overline{A}_i$ is of the same degree as A_i in x_i, and its initial results from the specialization of the coefficients in the initial of A_i. The $\overline{A}_i$ are in $\overline{\Pi}$. For the A_i are in Π and hence have zero remainders with respect to (1). The equations which express this go over upon specialization into equations which show that the $\overline{A}_i$ have zero remainders with respect to (2). In saying this we make use of the fact that each coefficient in these equations may be written as an element of $\mathscr{F}[S]$ divided by a power of δ.

Let $\overline{\Sigma}$ denote the prime p.i. consisting of those polynomials of $\overline{\Pi}$ which are free of w. The $\overline{A}_i$ are in $\overline{\Sigma}$. Let $B_1, \cdots, B_p$ be a characteristic set of $\overline{\Sigma}$ with B_i introducing x_i. The product of the degrees of the B_i in the indeterminates they introduce equals the degree of $\overline{G}$ in w. This is the degree of G in w and hence equals the product of the degrees of the A_i in the respective x_i. This product, in turn, equals the product of the degrees of the $\overline{A}_i$ in the respective x_i. Hence the product of the degrees of the B_i in their respective x_i equals the corresponding product formed for the $\overline{A}_i$. It follows that the chain $B_1, \cdots, B_p$ cannot be lower than the chain $\overline{A}_1, \cdots, \overline{A}_p$. The latter is therefore a characteristic set of $\overline{\Sigma}$.

The absolute irreducibility of the manifold of $\overline{\Sigma}$ is a consequence of the absolute irreducibility of $\overline{G}$ since $\overline{G}$ is a resolvent for $\overline{\Sigma}$. It remains only to prove the last statement of the lemma. Let P be any polynomial of Σ whose coefficients are in $\mathscr{F}[S]$. On specialization of its coefficients P becomes a polynomial $\overline{P}$. The equation which shows that the remainder of P with respect to $A_1, \cdots, A_p$ is 0 goes over into an equation showing that the remainder of $\overline{P}$ with respect to $\overline{A}_1, \cdots, \overline{A}_p$ is 0. Hence $\overline{P}$ is in $\overline{\Sigma}$. This is equivalent to the statement that any algebraic relation between the λ_i and a generic zero of Σ goes over on specialization into a relation between the $\bar{\lambda}_i$ and a generic zere of $\overline{\Sigma}$. This completes the proof of Lemma 2.

1. 3. *Adjunction of a generic zero.* Our application of the preceding lemma will arise in the following situation. Let Π be a prime p.i. in $\mathscr{F}[u_1, \cdots, u_q; y_1, \cdots, y_p]$, the u_i constituting a parametric set. Let $\mathscr{H}$ be the field obtained by adjoining a generic zero $u_i = \alpha_i$, $i=1, \cdots, q$; $y_j = \beta_j$, $j=1, \cdots, p$, of Π to $\mathscr{F}$. The manifold $\mathfrak{M}$ of Π is the union of manifolds $\mathfrak{M}_1, \cdots, \mathfrak{M}_r$ irreducible over $\mathscr{H}$. Let $\mathfrak{M}_1$ be an $\mathfrak{M}_i$ containing the generic zero named above. Then $\mathfrak{M}_1$ is absolutely irreducible.

To prove this statement we consider the field $\mathscr{G}$ consisting of those elements of $\mathscr{H}$ which are algebraic over $\mathscr{F}$. Let $\mathfrak{M}'$ be the least manifold over $\mathscr{G}$ which contains $\mathfrak{M}_1$, and let Π' be the ideal of $\mathfrak{M}'$. Evidently Π' is prime. $u_i = \alpha_i$, $i=1, \cdots, q$; $y_j = \beta_j$, $j=1, \cdots, p$, is a zero of Π'. Now Π' contains Π and hence is of at most the dimension q of Π. Since the degree of transcendence of

$$\mathscr{G}(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_p) = \mathscr{H}$$

with respect to $\mathscr{G}$ is the same as the degree of transcendence of $\mathscr{H}$ with respect to $\mathscr{F}$, and the latter is q, it follows that $u_i = \alpha_i$, $i=1, \cdots, q$; $y_j = \beta_j$, $j=1, \cdots, p$ is actually a generic zero of Π', and that Π' is of dimension q.

It suffices to prove the absolute irreducibility of $\mathfrak{M}'$. For, since $\mathfrak{M}'$ is of the same dimension as $\mathfrak{M}_1$ and contains $\mathfrak{M}_1$, its absolute irreducibility would imply that it coincides with $\mathfrak{M}_1$, and hence that the latter is absolutely irreducible.

Suppose $\mathfrak{M}'$ is not absolutely irreducible. Then there is an element γ algebraic over $\mathscr{G}$ such that $\mathfrak{M}'$ is reducible over $\mathscr{G}(\gamma)$. Let γ be of degree d over $\mathscr{G}$. Then γ is also of degree d over $\mathscr{H}$ because every element of $\mathscr{H}$ algebraic over $\mathscr{G}$ is in $\mathscr{G}$.[4] Evidently γ is also of degree d over $\mathscr{G}_1 = \mathscr{G}(\alpha_1, \cdots, \alpha_q)$. Let e be the degree of $\mathscr{H}$ over $\mathscr{G}_1$, and let f be the degree of $\mathscr{G}_1(\gamma; \beta_1, \cdots, \beta_p)$ with respect to $\mathscr{G}_1(\gamma)$. The reducibility of $\mathfrak{M}'$ over $\mathscr{G}(\gamma)$ implies that $f < e$. On the other hand the degree of $\mathscr{G}_1(\gamma; \beta_1, \cdots, \beta_p)$ with respect to $\mathscr{G}_1$ is given both by de and df, so that $e=f$. This is a contradiction which establishes our claim that $\mathfrak{M}_1$ is absolutely irreducible.

2. Proof of Theorem 1.

2. 1. *A special case.* We return to the notation in which Theorem 1 was stated. We treat first the case that $p=1$, and that γ_1, which we shall now denote by η, using subscripts to denote its transforms, is algebraic over $\mathscr{G}$. Without loss of generality we may suppose that $\mathscr{F}$ is inversive.[5]

[4] One applies Lemma 2 of [5] to the prime p.i. in $\mathscr{G}[y]$ whose generic zero is γ.

[5] This is an easy consequence of the fact, proved in [3], that every difference field has an inversive extension.

Suppose first that η and, hence, its transforms are algebraic over $\mathscr{F}$. Then the compatibility of the extension $\mathscr{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q\rangle$ and $\mathscr{H}$ implies the existence of a field[6] $\mathscr{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q; \eta\rangle$. We say that $\bar{\beta}_1, \cdots, \bar{\beta}_q; \eta$ constitutes a specialization of $\beta_1, \cdots, \beta_q; \eta$. For, if P is a polynomial in $\mathscr{F}\{u_1, \cdots, u_q; y\}$ which vanishes when we replace u_i by β_i, $i=1, \cdots, q$, and y by η, then each coefficient of P as a polynomial in the u_{ij} is a difference polynomial of $\mathscr{F}\{y\}$ which has the zero η. If this were not the case a set of β_{ij} would be algebraically dependent over $\mathscr{F}\langle\eta\rangle$ and hence over $\mathscr{F}$, which is not so. It follows that P vanishes when we replace the u_i by the corresponding $\bar{\beta}_i$ and y by η. Hence in this case the assertion of Theorem 1 holds with $\delta=1$.

2. 2. *Conclusion of the algebraic case.* We proceed to complete the proof of the algebraic case by induction. We shall suppose that the conclusion of Theorem 1 has be been verified for algebraic functions of the β_{ij}, $i=1, \cdots, q; j=0, 1, \cdots, n-1$. Let η be algebraic over the field formed by adjoining to $\mathscr{F}$ the β_{ij}, $i=1, \cdots, q; j=0, 1, \cdots, n$.

We denote by S_k, $k=0, 1, \cdots$, the set of β_{ij}, $i=1, \cdots, q; j=k, \cdots, k+n-1$[7]; and by T_k the set of β_{ij}, $i=1, \cdots, q; j=k, \cdots, k+n$. Then η is algebraic over $T(=T_0)$. Let those elements of $\mathscr{H}=\mathscr{G}\langle\eta\rangle$ which are algebraic over any $\mathscr{F}(S_k)$ be adjoined to $\mathscr{G}$. There results a difference field whose inversive extension we denote by $\mathscr{G}'$. Let η be of degree d over $\mathscr{G}'$. Evidently there is an element σ of $\mathscr{H}$, algebraic over $\mathscr{F}(S)$, such that some transform η_t of η is of degree d over $\mathscr{G}\langle\sigma\rangle$. Let $\mathscr{G}^*$ be the difference field formed by adjoining to $\mathscr{G}$ elements whose tth transforms are respectively σ and the β_i, $i=1, \cdots, q$. Then η is of degree d over $\mathscr{G}^*$. Let Π be the reflexive prime difference ideal in $\mathscr{G}^*\{y\}$ whose generic zero is η. We claim that the characteristic set of Π consists of a single polynomial.

Evidently, the first polynomial of this characteristic set is of order zero and degree d in y. To prove that it is the only polynomial of the characteristic set we must show that, for any $r>0$, the degree of η_r over $\mathscr{G}^*(\eta_0, \cdots, \eta_{r-1})$ is d. For any $r>0$, η_r satisfies an irreducible algebraic equation of degree $d'\leq d$ with coefficients in $\mathscr{G}^*(\eta_0, \cdots, \eta_{r-1})$. Since η_r is algebraic over $\mathscr{F}(T_r)$ these coefficients may be chosen to be algebraic over $\mathscr{F}(T_r)$. The coefficients are rational combinations with coefficients in $\mathscr{F}$ of certain transforms and inverse transforms of the β_i and σ, and of $\eta_0, \cdots, \eta_{r-1}$. The β_{ij} involved, either directly, or

[6] The field $\mathscr{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q; \eta\rangle$, and other fields arising in similar situations, is not necessarily determined to within isomorphisms. It is a field which contains and is generated by subfields isomorphic to $\mathscr{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q\rangle$ and $\mathscr{F}\langle\eta\rangle$. Our notation is intended to indicate that one such field is selected and held fixed throughout the discussion.

[7] S_k is to denote the empty set if $n=0$.

because the σ_k involved or $\eta_0, \cdots, \eta_{r-1}$ are algebraic functions of them, are finite in number. We may specify a positive integer p such that for all β_{ij} involved, we have $-t \leq j \leq p$.

We now specialize the β_{ij}, $-t \leq j < r$, to integers. This is to be a specialization in the sense of algebra only; the operation of transforming need not be preserved by the specialization. If the integer values are appropriately chosen the specialization may be extended to the β_{ij}, $r \leq j \leq p$, and to the σ_k involved in the coefficients and $\eta_0, \cdots, \eta_r$ in such a way that these β_{ij} remain algebraically independent over $\mathscr{F}$. It follows that the coefficients of the irreducible equation for η_r, the σ_k involved, $k \geq r$, and η_r itself are unaltered by the specialization. That is to say, the specializations of these elements, and of the β_{ij}, $j \geq r$, satisfy precisely the same set of algebraic relations over $\mathscr{F}$ as did the corresponding unspecialized elements.

The σ_k, $-t \leq k < r$, and $\eta_0, \cdots, \eta_{r-1}$ specialize to elements algebraic over $\mathscr{F}(S_r)$. There is then an element λ, algebraic over $\mathscr{F}(S_r)$ such that these specializations lie in $\mathscr{F}(S_r, \lambda)$. Evidently, then, η_r is of degree at most d' over $\mathscr{G}^*(\lambda)$. Hence if d'' donotes the degree of η_r over $\mathscr{G}'(\lambda)$ we must certainly have $d'' \leq d'$.

Now λ is algebraic over the field $\mathscr{K}$ consisting of elements of $\mathscr{G}'$ which are algebraic over $\mathscr{F}(S_r)$. Let its degree over $\mathscr{K}$ be h. Every element of $\mathscr{G}'(\eta_r)$ algebraic over $\mathscr{K}$ is in $\mathscr{K}$, as follows from the descriptions of these fields. Hence λ is also of degree h over $\mathscr{G}'(\eta_r)$ in consequence of Lemma 2 of [5]. Then λ is also of degree h over $\mathscr{G}'$. Hence the degree of $\mathscr{G}'(\eta_r, \lambda)$ with respect to $\mathscr{G}'$ must equal dh and also hd''. Hence $d''=d'=d$. Thus we have shown that the characteristic set of Π consists of a single polynomial. We denote this polynomial by F. We may choose F so that its coefficients are in $\mathscr{F}\{\beta_{1,-t}, \cdots, \beta_{q,-t}; \sigma_{-t}\}$.

Let μ denote the initial of F. Some transform of μ is algebraic over $\mathscr{F}\langle\beta_1, \cdots, \beta_q\rangle$. Hence there is an element $\delta_1 \neq 0$ of $\mathscr{F}\{\beta_1, \cdots, \beta_q\}$ such that any specialization of the β_{ij} in the sense of algebra which does not annul δ_1 cannot be extended to a specialization to zero of this transform of μ. By the induction hypothesis, if $n \geq 1$, or by the special case proved in 2.1 if $n=0$, there is a $\delta_2 \neq 0$ in $\mathscr{F}\{\beta_1, \cdots, \beta_q\}$ such that any specialization $\bar{\beta}_1, \cdots, \bar{\beta}_q$ of $\beta_1, \cdots, \beta_q$ such that δ_2 does not specialize to zero and that $\mathscr{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q\rangle$ and $\mathscr{G}\langle\sigma\rangle$ are compatible extensions of $\mathscr{F}$, can be extended to a specialization $\bar{\beta}_1, \cdots, \bar{\beta}_q; \bar{\sigma}$ of $\beta_1, \cdots, \beta_q; \sigma$ over $\mathscr{F}$.

Let $\delta = \delta_1\delta_2$. We shall show that δ has the properties specified in Theorem 1. Let $\bar{\beta}_1, \cdots, \bar{\beta}_q$ be any specialization of $\beta_1, \cdots, \beta_q$ such that δ does not specialize to zero and that $\mathscr{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q\rangle$ and $\mathscr{H}$

are compatible extensions of $\mathscr{F}$. Since σ is in $\mathscr{H}$, $\mathscr{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q\rangle$ and $\mathscr{G}\langle\sigma\rangle$ are compatible extensions of $\mathscr{F}$. Hence the specialization of the β_i to the $\bar{\beta}_i$ can be extended to a specialization of σ to $\bar{\sigma}$. Let F become $\bar{F}$ when we replace σ by $\bar{\sigma}$ in its coefficients.[8] Because δ_1 does not specialize to zero $\bar{F}$ is of the same degree as F and its initial $\bar{\mu}$ is the specialization of μ. We let $\bar{\eta}$ be any solution of the difference equation $\bar{F}=0$. We shall show that $\bar{\beta}_1, \cdots, \bar{\beta}_q; \bar{\eta}$ constitutes a specialization over $\mathscr{F}$ of $\beta_1, \cdots, \beta_q; \eta$.

Let P be any polynomial in $\mathscr{F}\{u_1, \cdots, u_q; y\}$ which vanishes when we put $u_i=\beta_i$, $i=1, \cdots, q$; $y=\eta$. When the u_i are replaced by the β_i, P goes over into a polynomial P' of $\mathscr{G}\{y\}$, and η is a zero of P'. Hence P' is in Π. Then $\phi P'$, where ϕ is a product of powers of transforms of μ, is a linear combination of F and its transforms with coefficients which are polynomials of $\mathscr{G}^*\{y\}$. By a consideration of the process of forming the remainder we see that these coefficients are actually in $\mathscr{F}\langle\sigma_{-t}\rangle\{\beta_{1,-t}, \cdots, \beta_{q,-t}; y\}$ and hence, since the transforms of σ are algebraic over $\mathscr{G}$, they are in $\mathscr{F}\{\sigma_{-t}; \beta_{1,-t}, \cdots, \beta_{q,-t}; y\}$. Hence specializations may be defined for them.

From the relation of the preceding paragraph we obtain on specializing the β_i to the $\bar{\beta}_i$ and σ to $\bar{\sigma}$ an expression for $\bar{\phi}\bar{P}'$, where $\bar{\phi}\neq 0$ is the specialization of ϕ, and $\bar{P}'$ is the polynomial obtained from P by replacing the u_i by the $\bar{\beta}_i$, as a linear combination of $\bar{F}$ and its transforms. Hence $\bar{\eta}$ is a zero of $\bar{P}'$. This implies that P vanishes when the u_i, $i=1, \cdots, q$, are replaced by the corresponding $\bar{\beta}_i$ and y is replaced by $\bar{\eta}$. Thus Theorem 1 is proved in the algebraic case.

2. 3. *Completion of the proof of Theorem 1.* We now revert to the situation in which there are no restrictions on $\gamma_1, \cdots, \gamma_p$. We shall show that, without loss of generality, we may assume that each γ_j is transformally algebraic over $\mathscr{G}$. For, if this is not so, let, say, $\gamma_1, \cdots, \gamma_k$ constitute a basis of transformal transcendency[9] for the γ_j. If the theorem can be proved under the restriction just mentioned there is a $\delta'\neq 0$ in $\mathscr{F}\{\beta_1, \cdots, \beta_q; \gamma_1, \cdots, \gamma_k\}$ such that any specialization of $\beta_1, \cdots, \beta_q; \gamma_1, \cdots, \gamma_k$ for which δ' does not specialize to zero, and which is such that $\mathscr{H}$ and the field formed by adjoining the specialized elements to $\mathscr{F}$ are compatible extensions of $\mathscr{F}$, can be extended to a specialization of $\gamma_{k+1}, \cdots, \gamma_p$. We write δ' as a polynomial in $\gamma_1, \cdots, \gamma_k$ with coefficients in $\mathscr{F}\{\beta_1, \cdots, \beta_q\}$. Let $\delta\neq 0$ be a coefficient

[8] There is no difficulty in defining any needed inverse transforms of $\bar{\sigma}$.

[9] A basis of transformal transcendency of a set of elements (over a given difference field) is a maximal subset of the elements not annulling any nonzero difference polynomial with coefficients in the field.

of this polynomial. Then δ has the properties specified in Theorem 1. For, let $\bar{\beta}_1, \cdots, \bar{\beta}_q$ be a specialization over $\mathscr{F}$ of $\beta_1, \cdots, \beta_q$ which is such that δ does not specialize to 0 and that $\mathscr{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q\rangle$ and $\mathscr{H}$ are compatible extensions of $\mathscr{F}$. We extend $\mathscr{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q\rangle$ by means of successive transformally transcendental adjunctions of elements $\bar{\gamma}_1, \cdots, \bar{\gamma}_k$. Then it is evident that $\bar{\beta}_1, \cdots, \bar{\beta}_q; \bar{\gamma}_1, \cdots, \bar{\gamma}_k$ constitutes a specialization of $\beta_1, \cdots, \beta_q; \gamma_1, \cdots, \gamma_k$ such that δ' does not specialize to zero, and that $\mathscr{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q; \bar{\gamma}_1, \cdots, \bar{\gamma}_k\rangle$ and $\mathscr{H}$ are compatible extensions of $\mathscr{F}$. Hence it is possible to extend this specialization to a specialization of $\gamma_{k+1}, \cdots, \gamma_p$. We shall deal henceforth only with the restricted case.

Since the case that no β_i exist is trivial and may be dismissed, $\mathscr{G}$ contains an element, β_1, which is distinct from all its transforms. Because of this and the restriction that the γ_j are transformally algebraic over $\mathscr{G}$ the Theorem of [4] implies that $\mathscr{H}$ contains an element

$$\theta=\sum_{j=1}^{p} \mu_j \gamma_{js},$$

$s\geq 0$ an integer, the μ_j in $\mathscr{F}\{\beta_1, \cdots, \beta_q\}$, such that θ is of equal order and effective order over $\mathscr{G}$, and for some integer $k\geq s$, and each j, $j=1, \cdots, p$, γ_{jk} is in $\mathscr{G}\langle\theta\rangle$. There exist difference polynomials P_j, $j=1, \cdots, p$, and Q in $\mathscr{G}\{w\}$ such that θ is not a zero of Q and that each quotient P_j/Q becomes γ_{jk} when w is replaced by θ. We may and shall choose the P_j and Q to be in $\mathscr{F}\{\beta_1, \cdots, \beta_q; w\}$.

Let Π denote the reflexive prime difference ideal in $\mathscr{G}\{w\}$ with generic zero θ, and let $A_0, A_1, \cdots$ be a characteristic sequence for Π. A_0 is of equal order and effective order. We choose an integer m such that the order m' of A_m is not less than the order of the last polynomial of a characteristic set of Π and also not less than the order of Q. Let $h=m'-m$. Then A_j is of order h. We may assume without loss of generality that the coefficients of

$$(1) \qquad A_0, A_1, \cdots, A_m$$

are in $\mathscr{F}\{\beta_1, \cdots, \beta_q\}$. For, if this is not the case, it can be brought about by multiplying these polynomials by a suitable element of $\mathscr{F}\{\beta_1, \cdots, \beta_q\}$.

Let $\mathscr{G}'$ denote the subfield of $\mathscr{H}$ consisting of those of its elements which are algebraic over $\mathscr{G}$. By the Theorem of [6] there is an element[10] τ in $\mathscr{G}'$ such that $\mathscr{G}'=\mathscr{G}\langle\tau\rangle$. Since τ and its trans-

[10] We see from [6] that there is a finite set of elements τ_i which generate $\mathscr{G}'$ when adjoined, together with their transforms, to $\mathscr{G}$. Since the τ_i are algebraic over $\mathscr{G}$ it follows that there is a linear combination of them which will serve as τ.

forms are algebraic over $\mathscr{G}$, $\mathscr{G}\langle\tau\rangle=\mathscr{G}\{\tau\}$.

The manifold of A_0, regarded as a manifold over $\mathscr{G}'$, is the union of components of which at least one contains θ. Let Π' denote a reflexive prime difference ideal in $\mathscr{G}'\{w\}$ whose manifold is this component. Π' contains A_0 and is of order and effective order h.

We shall construct a beginning

$$C_0, \cdots, C_m \tag{2}$$

of a characteristic sequence of Π' in such a way that the coefficients of each polynomial of (2) are in the ring

$$\mathscr{R}=\mathscr{F}\{\beta_1, \cdots, \beta_q; \tau\}$$

and that each is obtained from the preceding by the procedure described in [1, pp. 142–145], all polynomials entering the computations having coefficients in $\mathscr{R}$. There is no trouble about C_0. We need merely start with the first polynomial of a characteristic sequence for Π and multiply it by a suitable element of $\mathscr{R}$. We specify that C_0 is to be irreducible. Suppose $C_0, \cdots, C_i$ have already been determined. Let B_{i+1} denote the remainder of the transform C_{i1} of C_i with respect to $C_0, \cdots, C_i$ considered as a chain of algebraic polynomials. Then

$$B_{i+1}=D_iC_{i1}-\sum_{j=0}^{i} L_{ij}C_j,$$

where D_i is a product of powers of initials of $C_0, \cdots, C_i$, and the L_{ij} are polynomials of $\mathscr{G}'\{w\}$. An examination of the remainder process shows that the L_{ij} and B_{i+1} are actually in $\mathscr{R}\{w\}$.

Now C_{i+1} is either equal to B_{i+1}, so that

$$C_{i+1}=D_iC_{i1}-\sum_{j=0}^{i} L_{ij}C_j, \tag{3}$$

or there is a relation

$$E_i[T_iB_{i+1}-C_{i+1}H_i]=\sum_{j=0}^{i} M_{ij}C_j, \tag{4}$$

where E_i is a product of powers of initials of $C_0, \cdots, C_i$, the M_{ij} are in $\mathscr{G}'\{w\}$, H_i is in $\mathscr{G}'\{w\}$ and is of order $h+i+1$, T_i is in $\mathscr{G}'\{w\}$ and is of order $h+1$, and C_{i+1} and T_i are reduced with respect to $C_0, \cdots, C_i$, while H_i is a product of polynomials reduced with respect to this chain. We see that by multiplying the polynomials defined by (4) by suitable elements of $\mathscr{R}\{w\}$ it is possible to obtain a relation of the form of (4) in which all polynomials present are in $\mathscr{R}\{w\}$. We assume this to be done. Then $C_0, \cdots, C_m$ as defined by relations (3) or (4) have the stated properties.

We now treat the w_i as a set of indeterminates in the sense of algebra, and the difference fields as fields. The polynomials of Π which are of order not exceeding m' form a prime p.i. Π_m in $\mathscr{G}\{w_0, \cdots, w_{m'}\}$, while the polynomials of Π' of order not exceeding m' form a prime p.i, Π'_m in $\mathscr{G}'\{w_0, \cdots, w_{m'}\}$. Both Π_m and Π'_m have dimension h. $A_0, \cdots, A_m$ is a characteristic set for Π_m, and $C_0, \cdots, C_m$ is a characteristic set for Π'_m.

We say that the manifold of Π'_m is absolutely irreducible. For, by the definition of τ, every element of $\mathscr{G}'(\theta_0, \cdots, \theta_{m'})$ not in $\mathscr{G}'$ is transcendental over $\mathscr{G}'$. Hence $C_0, \cdots, C_m$ is the characteristic set of a prime p.i. Π''_m in $\mathscr{G}'(\theta_0, \cdots, \theta_{m'})[w_0, \cdots, w_{m'}]$ whose manifold is that of Π'_m.[11] But $\theta_0, \cdots, \theta_{m'}$ is a generic zero of Π'_m and a zero of Π''_m, since it annuls $C_0, \cdots, C_m$ but not the initials of these polynomials. By the remark after the proof of Lemma 2 above it follows that the manifold of Π''_m is absolutely irreducible.

Lemma 2 now shows that $\mathscr{R}$ contains an element δ_0 such that for any specialization in the sense of algebra of the β_{ij} and the transforms of τ for which δ_0 does not vanish, (2) specializes to a characteristic set $\overline{C}_0, \cdots, \overline{C}_m$ of a prime p.i. over the field formed by adjoining to $\mathscr{F}$ the specializations of the β_{ij} and the transforms of τ. Now C_0 is absolutely irreducible. For it follows from the absolute irreducibility of the manifold of Π'_m that C_0 has no factors other than itself which involve w_h, whatever extension of $\mathscr{G}'$ is used as the coefficient field; while the irreducibility of C_0 in $\mathscr{G}'$ shows that in no field does it have factors other than field elements which are free of w_h. Hence, by Lemma 1, there is a δ_1 in $\mathscr{R}$ such that for any specialization of the β_{ij} and the transforms of τ for which δ_1 does not vanish, C_0 specializes to an absolutely irreducible polynomial.

Since $\delta_0\delta_1$ is algebraic over $\mathscr{F}\{\beta_1, \cdots, \beta_q\}$, this ring contains a δ_2 such that any specialization in the sense of algebra of the β_{ij} for which δ_2 does not specialize to 0 cannot be extended to a specialization of $\delta_0\delta_1$ in which this product specializes to 0.

By the special case of Theorem 1 proved in 2.2 there is a δ_3 in $\mathscr{F}\{\beta_1, \cdots, \beta_q\}$ such that any specialization of the β_i over $\mathscr{F}$ to elements $\overline{\beta}_1, \cdots, \overline{\beta}_q$ such that $\mathscr{F}\langle\overline{\beta}_1, \cdots, \overline{\beta}_q\rangle$ and $\mathscr{G}'=\mathscr{F}\langle\beta_1, \cdots \beta_q; \tau\rangle$ are compatible extensions of $\mathscr{F}$, and that δ_3 does not specialize to 0, can be extended to a specialization of the β_i and τ.

The polynomials of Π' which are in $\mathscr{G}\{w\}$ form a reflexive prime

[11] To prove the identity of the manifolds we consider a generic zero of a component of the manifold of Π''_m which is irreducible over $\mathscr{G}'(\theta_0, \cdots, \theta_{m'})$. This generic zero must annul the C_i. Because the dimension of the component equals the dimension of Π'_m the generic zero cannot annul the initial of any C_i. Hence it annuls the polynomials of Π''_m. Hence the component is contained in the manifold of Π''_m. Our statement follows readily from this.

difference ideal of dimension h with zero θ. Evidently this must be Π. Hence Q and the product J of the initials of the polynomials of (1), which are not in Π, are not in Π'. Let S be the product of the T_i and the initials of the H_i of (4). Then S is not in Π'. The remainder R of JQS with respect to the chain (2) is therefore not 0. Let $\delta_4 \neq 0$ be a coefficient of R. Then δ_4 is in the ring $\mathscr{R}$, and there is a δ_5 in $\mathscr{F}\{\beta_1, \cdots, \beta_q\}$ such that any specialization of $\beta_1, \cdots, \beta_q$ for which δ_5 does not vanish cannot be extended to a specialization of δ_4 to zero.

We let $\delta = \delta_2\delta_3\delta_5$. We shall show that δ has the properties specified in the statement of Theorem 1.

Let $\bar{\beta}_1, \cdots, \bar{\beta}_q$ be a specialization of $\beta_1, \cdots, \beta_q$ over $\mathscr{F}$ which is such that $\mathscr{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q\rangle$ and $\mathscr{H}$ are compatible extensions of $\mathscr{F}$ and that δ does not vanish under the specialization. Then $\mathscr{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q\rangle$ and $\mathscr{G}'$ are compatible extensions of $\mathscr{F}$, and δ_3 does not vanish under the specialization. Hence there is a $\bar{\tau}$ such that $\bar{\beta}_1, \cdots, \bar{\beta}_q; \bar{\tau}$ constitutes a specialization over $\mathscr{F}$ of $\beta_1, \cdots, \beta_q; \tau$. Let the polynomials of (2) become $\bar{C}_0, \cdots, \bar{C}_m$ when their coefficients are subjected to this specialization. The non-vanishing of δ_2 shows that $\bar{C}_0, \cdots, \bar{C}_m$ is a characteristic set of a prime p.i. over the field $\mathscr{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q; \bar{\tau}\rangle$, with $w_0, \cdots, w_{h-1}$ constituting a set of parametric indeterminates, and that $\bar{C}_0$ is irreducible. The initials of the C_i specialize to the initials of the $\bar{C}_i$.

Because δ_5 does not vanish the specialization carries R into a non-zero polynomial $\bar{R}$ reduced with respect to $\bar{C}_0, \cdots, \bar{C}_m$. Hence J, Q and S are carried by the specialization of their coefficients into polynomials $\bar{J}$, $\bar{Q}$ and $\bar{S}$ respectively which are annulled by no regular zero of the chain $\bar{C}_0, \cdots, \bar{C}_m$. Hence the T_i and the initials of the H_i do not vanish when their coefficients are specialized, so that the relations (3) and (4) are carried by the specialization into relations of the same type. It follows that $\bar{C}_0, \cdots, \bar{C}_m$ is the beginning of a characteristic sequence of one or more reflexive prime difference ideals whose manifolds are components of the general solution of $\bar{C}_0$. Let $\bar{\Pi}$ be one of these ideals, and $\bar{\theta}$ a generic zero of $\bar{\Pi}$. Evidently $\bar{J}$ and $\bar{Q}$ do not have $\bar{\theta}$ as a zero.

Let the P_i, $i=1, \cdots, p$, be carried into polynomials $\bar{P}_i$ by the specialization of their coefficients. We define $\bar{\gamma}_{is}$, $i=1, \cdots, p$, to be the result obtained by replacing w by $\bar{\theta}$ in $\bar{P}_i/\bar{Q}$.

We say that $\bar{\beta}_1, \cdots, \bar{\beta}_q; \bar{\gamma}_{1s}, \cdots, \bar{\gamma}_{ps}$ constitutes a specialization over $\mathscr{F}$ of $\beta_1, \cdots, \beta_q; \gamma_{1s}, \cdots, \gamma_{ps}$. For let F be a polynomial of

$\mathscr{F}\{u_1, \cdots, u_q; y_1, \cdots, y_p\}$ which is free of y_{ij}, $j<s$, and which vanishes when we replace each u_i, $i=1, \cdots, q$, by β_i, and each y_{jk}, $j=1, \cdots, p$; $k=s, s+1, \cdots$, by γ_{jk}. Let the u_i in F be replaced by the β_i, the y_{js} by P_j/Q, and their transforms by transforms of these expressions. After multiplication by a suitable product of powers of transforms of Q there results a polynomial G in $\mathscr{F}\{\beta_1, \cdots, \beta_q; w\}$. Evidently G is in Π.

Let $\bar{G}$ denote the polynomial obtained from G by specializing its coefficients, and let $\bar{A}_0, \cdots, \bar{A}_m$ denote the polynomials so obtained from the polynomials of (1). The A_i have 0 remainder with respect to (2) considered as a chain of polynomials in the indeterminates $w_0, \cdots, w_{h+m}$. By specialization we see that the $\bar{A}_j$ have 0 remainder with respect to the chain $\bar{C}_0, \cdots, \bar{C}_m$, and hence have the zero $\bar{\theta}$. Similarly G has zero remainder with respect to the chain $A_0, \cdots, A_m$. By specialization we see that $\bar{J}\bar{G}$ has the zero $\bar{\theta}$. Hence $\bar{G}$ has the zero $\bar{\theta}$. If we replace the u_i in F by the $\bar{\beta}_i$ and the y_{js} and their transforms by the $\bar{P}_j/\bar{Q}$ and their transforms we shall also obtain $\bar{G}$. Hence F has zero $u_i=\bar{\beta}_i$, $y_{jk}=\bar{\gamma}_{jk}$, $k\geq s$. This proves our statement concerning the $\bar{\gamma}_{js}$. If we define $\bar{\gamma}_j$, $j=1, \cdots, p$, as an element whose sth transform is $\bar{\gamma}_{js}$, we see that $\bar{\beta}_1, \cdots, \bar{\beta}_q; \bar{\gamma}_1, \cdots, \bar{\gamma}_p$ is a specialization of $\beta_1, \cdots, \beta_q; \gamma_1, \cdots, \gamma_p$. The proof of Theorem 1 is now complete.

2. 4. *Corollaries to Theorem 1.*

Corollary 1. *The specialization of $\gamma_1, \cdots, \gamma_p$ whose existence is proved in Theorem* 1 *may be made in such a way that if a basis of transformal transcendency for $\gamma_1, \cdots, \gamma_p$ is selected in advance, then its elements specialize into a basis of transformal transcendency for $\bar{\gamma}_1, \cdots, \bar{\gamma}_p$. Furthermore the effective order of $\gamma_1, \cdots, \gamma_p$ with respect to the pre-assigned basis equals the effective order of $\bar{\gamma}_1, \cdots, \bar{\gamma}_p$ with respect to the basis obtained by specialization.*

Proof. Let $\gamma_1, \cdots, \gamma_k$ be the pre-assigned basis of transformal transcendency. The first statement follows immediately from the construction used in the proof of Theorem 1, since $\bar{\gamma}_1, \cdots, \bar{\gamma}_k$ are so chosen as to annul no nonzero difference polynomial with coefficients in $\mathscr{F}\langle\bar{\beta}_1, \cdots, \bar{\beta}_q\rangle$. The second statement follows from the fact that Π and Π' are of equal effective order.[12]

[12] No such statement holds for orders. For let $\mathscr{F}$ be an inversive difference field containing an aperiodic element. Let $u=\beta$, $y=\gamma$ be a generic zero of the ideal $\{y_1-u\}$ of $\mathscr{F}\{u, y\}$. Then $\mathscr{F}\langle\beta, \gamma\rangle$ is of first order over $\mathscr{F}\langle\beta\rangle$, but if β is specialized to an element of $\mathscr{F}$, γ specializes to an element of $\mathscr{F}$. The specialization of β can be chosen so as not to annul any pre-assigned nonzero element of $\mathscr{F}\{\beta\}$.

Corollary 2. *Let $\mu \neq 0$ be an element of $\mathscr{F}\{\beta_1, \cdots, \beta_q; \gamma_1, \cdots, \gamma_p\}$. For an appropriate choice of δ of Theorem 1 the specialization of $\gamma_1, \cdots, \gamma_p$ whose existence is proved in Theorem 1 may be made in such a way that μ does not specialize to 0 and that the requirements of Corollary 1 are satisfied.*

Proof. Let A be a polynomial of $\mathscr{F}\{u_1, \cdots, u_q; y_1, \cdots, y_p\}$ which goes into μ when the u_i, $i=1, \cdots, q$, are replaced by the β_i and the y_j, $j=1, \cdots, p$, by the γ_j. Suppose first that each[13] γ_j is transformally algebraic over $\mathscr{G}$. If we replace the u_i in A by the β_i and the y_j by the P_j/Q, and multiply the result by a suitable product of powers of transforms of Q, we obtain a polynomial T of $\mathscr{G}\{w\}$ which is not in Π. We redefine R as the remainder of $JQST$ with respect to the chain (2), and redefine δ_4 and δ_5 correspondingly. Evidently $\delta = \delta_2\delta_3\delta_5$ has the desired properties.

To complete the proof of the corollary we proceed, as in the proof of Theorem 1, to obtain a δ'. In consequence of what has just been proved δ' may be so chosen that any specialization of $\beta_1, \cdots, \beta_q; \gamma_1, \cdots, \gamma_k$ for which δ' does not vanish, and which satisfies the usual compatibility requirement, can be extended to a specialization of the remaining γ_j in such a way that μ does not specialize to 0. We form δ from δ' as in the proof of Theorem 1.

3. Proof of a partial converse.

3. 1. *A counterexample.* It is not necessarily the case that the extensions of a ground field $\mathscr{F}$ generated by a set of elements and by one of its specializations over $\mathscr{F}$ are compatible. To show this we take for the ground field the field $\mathfrak{R}$ of rational numbers and consider polynomials in $\mathfrak{R}\{y\}$. Let A be the polynomial $1+y^2$ and let F be $A^2+A_1^2$. Then y_2-y is a factor of F_1-F. Hence F, y_2-y is a characteristic set of a reflexive prime difference ideal[14] Π in $\mathfrak{R}\{y\}$. Let η be a generic zero of Π.

To $\mathfrak{R}$ we adjoin an element i such that $i^2=-1$, and define the transform of i to be itself. Then $\mathfrak{R}\langle i\rangle$ and $\mathfrak{R}\langle\eta\rangle$ are incompatible. For $1+\eta^2\neq 0$, since η, as a generic zero of Π, satisfies no zero order difference equation. Hence $\mathfrak{R}\langle\eta\rangle$ contains an element

$$\lambda = (1+\eta_1^2)/(1+\eta^2).$$

Since

$$(1+\eta^2)^2+(1+\eta_1^2)^2=0$$

[13] We are here using the symbolism of the proof of Theorem 1.

[14] It is easy to establish the irreducibility of F. Then one applies Theorem 3 of [**1**].

we see that $\lambda^2=-1$. From $\eta_2=\eta$ we readily derive the relation $\lambda\lambda_1=1$. These imply $\lambda_1=-\lambda$. Hence[15], $\mathfrak{R}\langle\lambda\rangle$ is incompatible with $\mathfrak{R}\langle i\rangle$. Evidently this implies that $\mathfrak{R}\langle\eta\rangle$ is incompatible with $\mathfrak{R}\langle i\rangle$. But i is a specialization of η over $\mathfrak{R}$. For the substitution $y=i$ annuls F and y_2-y, but does not annul their initials. Hence it annuls every polynomial of Π.

3. 2. *Compatibility of "most" specializations.* Let $\mathscr{F}\langle\eta_1, \cdots, \eta_n\rangle$ be an extension of the difference field $\mathscr{F}$. The following theorem provides a restriction on the specializations of $\eta_1, \cdots, \eta_n$ over $\mathscr{F}$ which generate extensions of $\mathscr{F}$ incompatible with $\mathscr{F}\langle\eta_1, \cdots, \eta_n\rangle$.

THEOREM 2. *There is an element $\gamma\neq0$ in $\mathscr{F}\{\eta_1, \cdots, \eta_n\}$ such that if $\bar\eta_1, \cdots, \bar\eta_n$ is a specialization over $\mathscr{F}$ of $\eta_1, \cdots, \eta_n$, and the corresponding specialization of γ is not 0, then $\mathscr{F}\langle\eta_1, \cdots, \eta_n\rangle$ and $\mathscr{F}\langle\bar\eta_1, \cdots, \bar\eta_n\rangle$ are compatible extensions of $\mathscr{F}$.*

Proof. Let Π be the reflexive prime difference ideal in $\mathscr{F}\{y_1, \cdots, y_n\}$ with generic zero $\eta_1, \cdots, \eta_n$. Theorem 2 is equivalent to the statement that there is a polynomial Q in $\mathscr{F}\{y_1, \cdots, y_n\}$, but not in Π, such that if $\lambda_1, \cdots, \lambda_n$ is a zero of Π but not of Q, then $\mathscr{F}\langle\lambda_1, \cdots, \lambda_n\rangle$ and $\mathscr{F}\langle\eta_1, \cdots, \eta_n\rangle$ are compatible extensions of $\mathscr{F}$. We shall prove this statement

We denote by $\mathscr{G}$ the subfield of $\mathscr{F}\langle\eta_1, \cdots, \eta_n\rangle$ consisting of those of its elements which are algebraic over $\mathscr{F}$. By [6] there is a finite set of elements of $\mathscr{G}$ which generate $\mathscr{G}$ when adjoined, with their transforms, to $\mathscr{F}$. Since these elements are algebraic over $\mathscr{F}$ it follows that there is an element δ such that $\mathscr{G}=\mathscr{F}\langle\delta\rangle$. There exist polynomials P, Q in $\mathscr{F}\{y_1, \cdots, y_n\}$, Q not in Π, such that δ is obtained by replacing the y_i in P/Q by the corresponding η_i. We shall show that Q has the properties claimed in the preceding paragraph.

Let Σ be the reflexive prime difference ideal in $\mathscr{F}\{w\}$ with generic zero δ. Let $B_0, B_1, \cdots, B_r$ be a characteristic set for Σ. When the w_i, $i=0, 1, \cdots$, are replaced by P_i/Q_i in the polynomials $B_0, \cdots, B_r$ and the resulting expressions are multiplied by an appropriate product of powers of transforms of Q, there results a set, $C_0, \cdots, C_r$, of polynomials of Π.

For a zero $\lambda_1, \cdots, \lambda_n$ of Π which is not a zero of Q we define the element δ' to be the result of replacing the y_i in P/Q by the corresponding λ_i. Since the λ_i annul $C_0, \cdots, C_r$ it is easy to see that δ' is a zero of $B_0, \cdots, B_r$. Because B_0 is of zero order, any zero of B_0 is

[15] The incompatibility of these extensions of $\mathfrak{R}$ is discussed in Ritt [**10**].

the generic zero of a prime ideal whose manifold is an ordinary manifold of B_0. Hence δ' is the generic zero of an ideal Σ' of this description. But Σ' must be Σ. For no other such ideal contains every B_i, $i=0, \cdots, r$.

It follows that $\mathscr{G}$ and $\mathscr{F}\langle\delta'\rangle$ are isomorphic under a mapping which leaves fixed the elements of $\mathscr{F}$. Let $\mathscr{G}'$ denote the field consisting of those elements of $\mathscr{F}\langle\lambda_1, \cdots, \lambda_n\rangle$ which are algebraic over $\mathscr{F}$. Evidently $\mathscr{G}'$ is an extension of $\mathscr{F}\langle\delta'\rangle$. It follows from the definition of compatibility and from the preceding statement that $\mathscr{G}$ and $\mathscr{G}'$ are compatible extensions of $\mathscr{F}$. By results obtained in proving Theorem 1 of [5] this implies that $\mathscr{F}\langle\lambda_1, \cdots, \lambda_n\rangle$ and $\mathscr{F}\langle\eta_1, \cdots, \eta_n\rangle$ are compatible extensions of $\mathscr{F}$. This proves Theorem 2.

3. 3. *Alternate proof of Theorem 2.* We give another proof of Theorem 2 in the case that Π has dimension $n-1$. This proof has the advantage of furnishing a polynomial Q explicitly.

Let $y_1, \cdots, y_{n-1}$ be a parametric set of indeterminates for Π. With the ordering $y_1, \cdots, y_n$ of the indeterminates, let F be the first polynomial of a characteristic set of Π. *Then the y_n-separant of F may be used as Q.*

Proof. Let S denote this separant and h the effective order of F in y_n. Let $\lambda_1, \cdots, \lambda_n$ be a zero of Π which is not a zero of S.

In the field $\mathscr{F}\langle\lambda_1, \cdots, \lambda_n\rangle$ the manifold of F has a component $\mathfrak{M}$ containing $\lambda_1, \cdots, \lambda_n$. Let Σ be the reflexive prime difference ideal in $\mathscr{F}\langle\lambda_1, \cdots, \lambda_n\rangle\{y_1, \cdots, y_n\}$ whose manifold is $\mathfrak{M}$. Let $\alpha_1, \cdots \alpha_n$ be a generic zero of Σ. Since $\lambda_1, \cdots, \lambda_n$ is not a zero of S, $y_1, \cdots, y_{n-1}$ constitute a parametric set for Σ, and Σ is of effective order h in y_n. We denote by Σ' the reflexive prime difference ideal $\Sigma \cap \mathscr{F}\{y_1, \cdots, y_n\}$.

Since Σ' contains F its manifold is either a component of F or is properly contained in a component of F. The latter case is impossible because Σ' contains no nonzero polynomial of effective order less than h in y_n or free of y_n. Since $\lambda_1, \cdots, \lambda_n$ is a zero of both Π and Σ', but not a zero of S, it follows from [7] that Π and Σ' are identical. Hence $\alpha_1, \cdots, \alpha_n$ is a zero of Π, and evidently a generic zero. Then $\mathscr{F}\langle\alpha_1, \cdots, \alpha_n\rangle$ and $\mathscr{F}\langle\eta_1, \cdots, \eta_n\rangle$ are isomorphic under a mapping which leaves fixed the elements of $\mathscr{F}$ and carries α_i into η_i, $i=1, \cdots, n$. Since $\mathscr{F}\langle\lambda_1, \cdots, \lambda_n; \alpha_1, \cdots, \alpha_n\rangle$ is defined this implies that $\mathscr{F}\langle\lambda_1, \cdots, \lambda_n\rangle$ and $\mathscr{F}\langle\eta_1, \cdots, \eta_n\rangle$ are compatible extensions of $\mathscr{F}$. This completes the proof.

REFERENCES

1. R.M. Cohn, *Manifolds of difference polynomials*, Trans. Amer. Math. Soc., **64** (1948), 133–172.
2. ———, *A note on the singular manifolds of a difference polynomial*, Bull. Amer. Math. Soc., **54** (1948), 917–922.
3. ———, *Inversive difference fields*, Bull. Amer. Math. Soc., **55** (1949), 597–603.
4. ———, *On extensions of difference fields and the resolvents of prime difference ideals*, Proc. Amer. Math. Soc., **3** (1952), 178–182.
5. ———, *Extensions of difference fields*, Amer. J. Math., **74** (1952), 507–530.
6. ———, *Finitely generated extensions of difference fields*, Proc. Amer. Math. Soc., **6** (1955), 3-5.
7. ———, *On the intersection of the components of a difference polynomial*, Proc. Amer. Math. Soc., **6** (1955), 42-45.
8. J.L. Doob and J.F. Ritt, *Systems of algebraic difference equations*, Amer. J. Math., **55** (1933), 505–514.
9. J.F. Ritt, *Differential algebra*, Amer. Math. Soc. Colloquium Publications, vol. 33.
10. ———, *Complete difference ideals*, Amer. J. Math., **63** (1941), 683-684.
11. L. B. van der Waerden, *Moderne algebra*, 1st edition, Berlin 1931.

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