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The following interesting theorem was recently obtained by H. Wielandt (Oral communication, see also J. Todd [3]):

Let M, N be two normal matrices of order n, and let r denote the rank of $M-N$. Let D be an arbitrary closed circular disk in the complex plane, If D contains exactly p eigenvalues of M, and exactly q eigenvalues of N, then $|p-q|\leq r$.

It is then natural to raise the following question: Without considering the rank of $M-N$, is it possible to compare the eigenvalues of M and N in a manner similar to that of Wielandt's theorem? The purpose of this Note is to present such a rank-free comparison theorem which includes Wielandt's theorem stated above.

THEOREM. *Let M, N be two normal matrices*[1] *of order n and let r be an integer such that $0\leq r<n$. Let $\varepsilon\geq 0$ be such that ε^2 is not less than the $(r+1)$th eigenvalue of $(M-N)^*(M-N)$, when the eigenvalues of $(M-N)^*(M-N)$ are arranged in descending order.*[2] *If a closed circular disk*

$$|z-z_0|\leq\rho$$

contains p eigenvalues of M, then the concentric disk

$$|z-z_0|\leq\rho+\varepsilon$$

contains at least $p-r$ eigenvalues of N.

While Wielandt's proof of his theorem uses geometric arguments involving convexity, the proof of our theorem will be based on an inequality (Lemma below). This difference in methods explains why our result is of more quantitative character than Wielandt's theorem.

LEMMA. *Let A, B be any two matrices*[3] *of order n. If $\{\alpha_i\}$, $\{\beta_i\}$, $\{\gamma_i\}$ are the eigenvalues of A^*A, B^*B and $(A+B)^*(A+B)$ respectively, each arranged in descending order*

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[1] The elements of all matrices considered here are real or complex numbers.

[2] As usual, the adjoint of a matrix A is denoted by A^*.

[3] Here A, B need not be normal.

$$\alpha_i \geq \alpha_{i+1}, \quad \beta_i \geq \beta_{i+1}, \quad \gamma_i \geq \gamma_{i+1}, \qquad (1 \leq i \leq n-1)$$

then the inequality

$$\sqrt{\gamma_{i+j+1}} \leq \sqrt{\alpha_{i+1}} + \sqrt{\beta_{j+1}}$$

holds for any two nonnegative integers i, j *such that* $i+j+1 \leq n$.

A more general form of this lemma (valid for completely continuous linear operators in a Hilbert space) has been given in [2], and is a generalization of a classical inequality of H. Weyl [4, p. 445] concerning eigenvalues of sum of two symmetric kernels of linear integral equations.

Proof of the theorem. Let $\{\mu_i\}$, $\{\nu_i\}$ denote the eigenvalues of M, N respectively and so arranged that

$$|\mu_i - z_0| \geq |\mu_{i+1} - z_0|, \quad |\nu_i - z_0| \geq |\nu_{i+1} - z_0|, \qquad (1 \leq i \leq n-1).$$

Let

$$A = M - z_0 I, \qquad B = N - M.$$

Then $A+B = N - z_0 I$. Let $\{\alpha_i\}$, $\{\beta_i\}$, $\{\gamma_i\}$ denote the eigenvalues of A^*A, B^*B and $(A+B)^*(A+B)$, each arranged in descending order. As M, N are both normal, we have

$$\alpha_i = |\mu_i - z_0|^2, \qquad \gamma_i = |\nu_i - z_0|^2, \qquad (1 \leq i \leq n).$$

By the above Lemma, we have

$$|\nu_{i+r} - z_0| \leq |\mu_i - z_0| + \sqrt{\beta_{r+1}}, \qquad (1 \leq i \leq n-r).$$

Using our hypothesis $\beta_{r+1} \leq \varepsilon^2$, we obtain

$$(1) \qquad |\nu_{i+r} - z_0| \leq |\mu_i - z_0| + \varepsilon, \qquad (1 \leq i \leq n-r).$$

Let p denote the number of eigenvalues μ_i of M contained in the disk $|z - z_0| \leq \rho$, and q the number of eigenvalues ν_i of N contained in the concentric disk $|z - z_0| \leq \rho + \varepsilon$. We shall prove that

$$(2) \qquad q \geq p - r.$$

If $n - q - r < 1$, then $q \geq n - r \geq p - r$. Thus we may assume $1 \leq n - q - r$. By (1),

$$|\nu_{n-q} - z_0| \leq |\mu_{n-q-r} - z_0| + \varepsilon.$$

But, according to the definition of q, we have

$$|\nu_{n-q} - z_0| > \rho + \varepsilon.$$

Therefore

$$|\mu_{n-q-r}-z_0|>\rho\ ,$$

which implies $n-q-r\leq n-p$ or (2). Our theorem is thus proved.

COROLLARY. *Let M, N be two normal matrices of order n and let r be an integer such that $0\leq r<n$. Let $x_1, x_2, \cdots, x_{n-r}$ be $n-r$ orthonormal vectors in the unitary n-space. If a closed circular disk $|z-z_0|\leq\rho$ contains p eigenvalues of M, then the concentric disk*

$$|z-z_0|\leq\rho+\Big(\sum_{i=1}^{n-r}\|(M-N)x_i\|^2\Big)^{\frac{1}{2}} \tag{3}$$

contains at least $p-r$ eigenvalues of N.

Proof. By a minimum property of eigenvalues of Hermitian matrices [**1**, Theorem 1], the expression

$$\sum_{i=1}^{n-r}\|(M-N)x_i\|^2=\sum_{i=1}^{n-r}\Big((M-N)^*(M-N)x_i,\ x_i\Big)$$

is not less than the sum of the last $n-r$ eigenvalues of $(M-N)^*(M-N)$, and consequently not less than the $(r+1)$th eigenvalue of $(M-N)^*(M-N)$. Thus the corollary follows directly from the theorem.

In case r is the rank of $M-N$, we can choose $n-r$ orthonormal vectors $x_1, x_2, \cdots, x_{n-r}$ such that

$$(M-N)x_i=0 \qquad (1\leq i\leq n-r).$$

Then the disk (3) becomes $|z-z_0|\leq\rho$ and the corollary reduces to Wielandt's theorem.

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