

# Pacific Journal of Mathematics

**REMARK ON THE PRECEDING PAPER. ALGEBRAIC  
EQUATIONS SATISFIED BY ROOTS OF NATURAL NUMBERS**

ERNST GABOR STRAUS AND OLGA TAUSSKY

REMARK ON THE PRECEDING PAPER  
ALGEBRAIC EQUATIONS SATISFIED BY ROOTS OF  
NATURAL NUMBERS

E. G. STRAUS AND O. TAUSSKY

In the preceding paper [1] it was shown that the polynomials in question are factors of  $\Phi_n(x^k/n)$  where  $\Phi_n$  is the cyclotomic polynomial of order  $n$  and  $k, n$  are positive integers. The case  $k=2$  was settled in [1, Lemma 2]. It will now be shown that this is essentially the only nontrivial case. For a different treatment of a somewhat related question see K. T. Vahlen [2].

First let us remark that we can exclude the case  $n=m^d$  where  $d/k, d > 1$ ; since we may then set  $y=x^{k/d}/m$  so that  $\Phi_n(y^d)$  is either reducible with cyclotomic factors or equal to  $\Phi_{nd}(y)$ . We shall refer to  $n$  and  $\Phi_n(x^k/n)$  which satisfy the above exclusion as *simplified*.

**THEOREM.** *The simplified polynomial  $\Phi_n(x^k/n)$  is irreducible for all odd  $k$ . For  $k=2l$  the polynomial is reducible if and only if  $\Phi_n(x^2/n)$  is reducible. In that case we have*

$$(1) \quad \Phi_n(x^k/n) = g(x^l)g(-x^l),$$

where the polynomials on the right are irreducible.

The proof is based on the following lemma.

**LEMMA.** *If  $k > 2$  and  $n^{1/k}$  is simplified then  $n^{1/k}$  is not contained in a cyclotomic field.*

*Proof.* The Galois group of a cyclotomic field  $R(\zeta)$  is Abelian and hence all subfields of  $R(\zeta)$  are normal. The field  $R(n^{1/k})$  is, however, not a normal field for  $k > 2$ .

We can now prove the Theorem. Let  $\zeta_n$  be a primitive  $n$ th root of unity. A zero  $\omega$  of a simplified  $\Phi_n(x^k/n)$  is a zero of

$$(2) \quad x^k - n\zeta_n$$

and hence  $R(\omega)$  is an algebraic extension of  $R(\zeta_n)$ . If the degree of  $R(\omega)$  over  $R(\zeta_n)$  were  $k$  then its degree over  $R$  would be  $k\varphi(n)$ . Hence  $\Phi_n(x^k/n)$  is reducible if and only if (2) is reducible over  $R(\zeta_n)$ . Say

$$(3) \quad x^k - n\zeta_n = F(x)G(x) \quad F, G \in R(\zeta_n)[x].$$

Since all the roots of (2) are of the form  $n^{1/k}\zeta_{kn}^s$  we have

---

Received July 11, 1955.

$$F(0) = n^{1/k} \zeta \in R(\zeta_h) \qquad l = \deg F$$

where  $\zeta$  is a root of unity. In other words

$$(4) \qquad n^{1/k} \in R(\zeta_h, \zeta) = R(\zeta')$$

where  $\zeta'$  is a root of unity.

According to the lemma (4) is impossible if the reduced fraction  $1/k$  has denominator  $> 2$ . For  $k$  odd this means  $l=0$  or  $k$  and  $\Phi_h(x^k/n)$  irreducible. For  $k$  even and  $0 < l < k$  we can have only  $l=k/2$ . In this case

$$F(0) = \pm n^{1/2} \zeta_{hk}^s, \quad G(0) = \pm n^{1/2} \zeta_{hk}^t;$$

and since both  $F(0)G(0)$  and  $F(0)/G(0)$  are in  $R(\zeta_h)$  we obtain

$$s + t \equiv s - t \equiv 0 \pmod{k}.$$

Hence  $s \equiv t \equiv 0 \pmod{l}$  so that

$$(5) \qquad F(0) = \sqrt{n \zeta_h^u} \in R(\zeta_h).$$

But we noted in [1, Lemma 1] that (5) is necessary and sufficient for the reducibility of  $\Phi_h(x^2/n)$ . Thus we have

$$\Phi_h(x^2/n) = g(x)g(-x) \text{ and therefore}$$

$$\Phi_h(x^k/n) = g(x^l)g(-x^l)$$

as the complete factorization of  $\Phi_h(x^k/n)$  over  $R[x]$ .

#### REFERENCES

1. A. J. Hoffman, M. Newman, E. G. Straus, O. Taussky, *The number of absolute points of a correlation*, Pacific J. Math., **6** (1956).
2. K. T. Vahlen, *Über reductible Binome*, Acta. Math., **19** (1895).

UNIVERSITY OF CALIFORNIA, LOS ANGELES  
NATIONAL BUREAU OF STANDARDS

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

H. L. ROYDEN  
Stanford University  
Stanford, California

R. P. DILWORTH  
California Institute of Technology  
Pasadena 4, California

E. HEWITT  
University of Washington  
Seattle 5, Washington

E. G. STRAUS  
University of California  
Los Angeles 24, California

## ASSOCIATE EDITORS

E. F. BECKENBACH  
C. E. BURGESS  
H. BUSEMANN  
H. FEDERER

M. HALL  
P. R. HALMOS  
V. GANAPATHY IYER  
R. D. JAMES

M. S. KNEBELMAN  
I. NIVEN  
T. G. OSTROM  
M. M. SCHIFFER

J. J. STOKER  
G. SZEKERES  
F. WOLF  
K. YOSIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
OREGON STATE COLLEGE  
UNIVERSITY OF OREGON  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF UTAH  
WASHINGTON STATE COLLEGE  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
CALIFORNIA RESEARCH CORPORATION  
HUGHES AIRCRAFT COMPANY

Printed in Japan by Kokusai Bunken Insatsusha  
(International Academic Printing Co., Ltd.), Tokyo, Japan

# Pacific Journal of Mathematics

Vol. 6, No. 1

November, 1956

David Blackwell, <i>An analog of the minimax theorem for vector payoffs</i> . . . . .	1
L. W. Cohen, <i>A non-archimedean measure in the space of real sequences</i> . . . . .	9
George Bernard Dantzig, <i>Constructive proof of the Min-Max theorem</i> . . . . .	25
Jim Douglas, <i>On the numerical integration of quasilinear parabolic differential equations</i> . . . . .	35
James Michael Gardner Fell, <i>A note on abstract measure</i> . . . . .	43
Isidore Isaac Hirschman, Jr., <i>A note on orthogonal systems</i> . . . . .	47
Frank Harary, <i>On the number of dissimilar line-subgraphs of a given graph</i> . . . . .	57
Newton Seymour Hawley, <i>Complex bundles with Abelian group</i> . . . . .	65
Alan Jerome Hoffman, Morris Newman, Ernst Gabor Straus and Olga Taussky, <i>On the number of absolute points of a correlation</i> . . . . .	83
Ernst Gabor Straus and Olga Taussky, <i>Remark on the preceding paper. Algebraic equations satisfied by roots of natural numbers</i> . . . . .	97
Ralph D. James, <i>Summable trigonometric series</i> . . . . .	99
Gerald R. Mac Lane, <i>Limits of rational functions</i> . . . . .	111
F. Oberhettinger, <i>Note on the Lerch zeta function</i> . . . . .	117
Gerald C. Preston, <i>On locally compact totally disconnected Abelian groups and their character groups</i> . . . . .	121
Vikramaditya Singh and W. J. Thron, <i>On the number of singular points, located on the unit circle, of certain functions represented by C-fractions</i> . . . . .	135
Sherman K. Stein, <i>The symmetry function in a convex body</i> . . . . .	145
Edwin Weiss, <i>Boundedness in topological rings</i> . . . . .	149
Albert Leon Whiteman, <i>A sum connected with the series for the partition function</i> . . . . .	159
Alfred B. Willcox, <i>Some structure theorems for a class of Banach algebras</i> . . . . .	177
Joseph Lawrence Zemmer, <i>Some remarks on p-rings and their Boolean geometry</i> . . . . .	193