

# Pacific Journal of Mathematics

**THE SYMMETRY FUNCTION IN A CONVEX BODY**

SHERMAN K. STEIN

# THE SYMMETRY FUNCTION IN A CONVEX BODY

S. STEIN

Let  $K_n$  be an  $n$ -dimensional convex body in  $n$ -dimensional Euclidean space  $E_n$ . At each point  $P$  in  $K_n$  consider the largest subset  $S(P)$  of  $K_n$  radially symmetric with respect to the point  $P$ . This set is well-defined and convex for it is simply the intersection of  $K_n$  with its radial reflection through the point  $P$ . Let  $m(P)$  equal the measure of  $S(P)$  and let  $f(P)$  equal  $m(P)V_n^{-1}$  where  $V_n$  is the measure of  $K_n$ . Clearly  $0 \leq f(P) \leq 1$  for all  $P$  in  $K_n$  and  $f(P)=0$  only if  $P$  is on the boundary of  $K_n$ ; also  $f$  is continuous. Moreover  $f$  attains the value 1 only if  $K_n$  is radially symmetric. The object of this note is to present various properties of this function  $f$ .

**THEOREM 1.** (Besicovitch [1],  $n=2$ ). *There is a point  $P$  in  $K_2$  such that  $f(P)=2/3$ . (In [3, p. 46] this theorem is ascribed to S. S. Konvyer.)*

**THEOREM 2.** (Besicovitch [2],  $n=2$ ). *If  $K_2$  is of constant width then there is a point  $P$  in  $K_2$  such that  $f(P)=.840\dots$ .*

H. G. Eggleston [4] studied further the symmetric function in a body of constant width.

Using a result of P. C. Hammer [5] on the ratio which the centroid of a convex body divides the chords passing through it, F. W. Levi [6] obtained the following.

**THEOREM 3.** *If  $P$  is the centroid of  $K_n$  then*

$$f(P) \geq 2(1+n^n)^{-1}.$$

The following properties of  $f$  will be obtained.

**THEOREM 4.**  $\int_{K_n} f = 2^{-n} V_n$ .

**COROLLARY.** *There is a point  $P$  in  $K_n$  such that  $f(P) > 2^{-n}$ .*

**THEOREM 5.** *If  $a$  is a real number then the set of points  $P$  in  $K_n$  at which  $f(P) \geq a$  is convex. Furthermore  $f$  attains its maximum value at precisely one point.*

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COROLLARY (to proof of Theorem 5, suggested by referee). If  $0 \leq \lambda \leq 1$  and  $P$  and  $Q$  are in  $K_n$  then

$$f(\lambda P + (1-\lambda)Q) \geq \lambda f(P) + (1-\lambda)f(Q) .$$

THEOREM 6. If  $K_n$  is an  $n$ -dimensional simplex and  $P$  is its centroid, then  $f$  attains its maximum at  $P$  and  $f(P) = 2(n+1)^{-1}$ .

*Proof of Theorem 4.* Consider the set of points

$$K_{2n} = \{(P, Q) | P \in K_n, Q \in S(P)\} .$$

In a straightforward manner this set can be shown to be convex and hence measurable. By Fubini's theorem on the relation between iterated and multiple integrals, the volume  $V_{2n}$  of  $K_{2n}$  is seen to equal  $\int_{K_n} m$  and also  $\int_{K_n} h$  where  $h(Q)$  denotes the measure of the cross section of  $K_{2n}$  defined by

$$\{(P, Q) | (Q \text{ fixed}), S(P) \ni Q\} .$$

Now  $S(P) \ni Q$  only if  $P$  is less than half way from  $Q$  to the boundary of  $K_n$  along the line determined by  $P$  and  $Q$ . Thus  $h(Q) = 2^{-n} V_n$  independently of  $Q$  [7, p. 38]. Thus

$$\int_{K_n} f = V_n^{-1} \int_{K_n} h = V_n^{-1} 2^{-n} (V_n)^2 = 2^{-n} V_n .$$

*Proof of Corollary to Th. 4.* Since the average value of  $f$  on  $K_n$  is  $2^{-n}$  and since  $f(P) < 2^{-n}$  on (and near) the boundary of  $K_n$  there must be a point at which  $f$  exceeds  $2^{-n}$ .

*Proof of Theorem 5.* Let  $P$  and  $Q$  be distinct points of  $K_n$  such that  $f(P) = f(Q)$ . We shall show<sup>1</sup> that  $f((P+Q)/2) > f(P)$ . This fact, combined with the fact that  $\{P | f(P) \geq a\}$  is closed, would prove the theorem. Consider the convex body  $(S(P) + S(Q))/2$ . This body is symmetric, and, if so translated that  $(P+Q)/2$  is its center, lies within  $K_n$ . By the Brunn-Minkowski theorem [7, p. 88] the measure of this set is strictly larger than  $m(P)$  if  $S(P)$  is not congruent to  $S(Q)$  by a translation. If  $S(P)$  is congruent to  $S(Q)$  by a translation, consider the convex hull of the set union of  $S(P)$  and  $S(Q)$ . This set is clearly symmetric with respect to the point  $(P+Q)/2$ , lies in  $K_n$ , and has a measure greater than  $m(P)$ . Thus  $f((P+Q)/2) > f(P) = f(Q)$ .

*Proof of Corollary to Th. 5.* A continuous function which satisfies

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<sup>1</sup> If  $P$  and  $Q$  are on the boundary of  $K_n$  it may happen that  $f((P+Q)/2) = f(P)$ .

$$f(\lambda P + (1-\lambda)Q) \geq \lambda f(P) + (1-\lambda)f(Q)$$

for  $\lambda=1/2$  and all  $P, Q$  in a line segment satisfies the inequality for all  $\lambda$ ,  $0 \leq \lambda \leq 1$ , and  $P, Q$ , in the line segment.

*Proof of Theorem 6.* Since affine transformations preserve symmetry, centroids, and ratio of volumes it will be sufficient to consider the case where  $K_n$  is regular.

Let  $Q$  be the point in  $K_n$  maximizing  $f$ . If  $T$  is an orthogonal transformation interchanging two of the vertices of  $K_n$ , and leaving the remaining vertices fixed then  $f(Q)=f(T(Q))$ . Thus, by Theorem 5,  $T(Q)=Q$ . Since this is true for each pair of vertices of  $K_n$ ,  $Q$  must be equidistant from all the vertices of  $K_n$ . Thus  $Q=P$ .

Now to compute  $f(P)$ .

Let  $K'_n$  be the reflection of  $K_n$  through  $P$  of altitude  $h$  and volume  $V$ . The boundary of  $K_n \cap K'_n$  is readily seen to be composed of  $2(n+1)$  congruent  $n-1$  dimensional sets  $B_i$ ,  $1 \leq i \leq 2(n+1)$  each of volume  $V^*$ . Let  $S$  denote the volume of  $K_n \cap K'_n$ .

Considering  $K_n \cap K'_n$  as being composed of  $2(n+1)$  congruent joins with the common vertex  $P$ , bases  $B_i$ , and altitude  $h(n+1)^{-1}$  one obtains

$$(1) \quad S = 2(n+1)h(n+1)^{-1}V^*n^{-1}.$$

On the other hand, considering  $K_n \cap K'_n$  as being obtained from  $K_n$  by the removal of  $n+1$  congruent sets, each of which is a join of a vertex of  $K_n$  with a  $B_i$  and has an altitude  $(n-1)(n+1)^{-1}h$ , one obtains

$$(2) \quad S = V - (n+1)(n-1)(n+1)^{-1}hV^*n^{-1}.$$

Elimination of the product  $hV^*$  from (1) and (2) yields

$$S = 2(n+1)^{-1}V$$

and thus

$$f(P) = 2(n+1)^{-1}.$$

## REFERENCES

1. A. S. Besicovitch, *Measure of asymmetry of convex curves*, J. London Math. Soc., **23** (1948), 237-240.
2. ———, *Measure of asymmetry of convex curves II*, J. London Math. Soc., **26** (1951), 280-293.
3. I. Iaglom and V. G. Boltianskii, *Vypuklye Figury (Convex Figures)*, Moscow, 1951.
4. H. G. Eggleston, *Measure of asymmetry of convex curves of constant width and restricted radii of curvature*, Quart. J. Math., Ser (2), **3** (1952), 63-72.
5. P. C. Hammer, *The centroid of a convex body*, Proc. Amer. Math. Soc., **2** (1951), 522-525.

6. F. W. Levi, *Über zwei Sätze von Herrn Besicovitch*, Arch. Math., **3** (1952), 125-129.
7. T. Bonnesen and W. Fenchel, *Konvexe Körper*, Chelsea, New York, 1948.

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