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**SOME REMARKS ON  $p$ -RINGS AND THEIR BOOLEAN  
GEOMETRY**

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**Introduction.** In this paper the word *ring* will always mean a ring with identity, and the Boolean algebra associated with a Boolean ring  $B$  will mean the Boolean algebra corresponding to  $B$  in the one-to-one correspondence, described by Stone [10], between the set of all Boolean rings and the set of all Boolean algebras. In a Boolean algebra,  $\cap$ ,  $\cup$ ,  $'$ , will denote the operations of intersection, union, and complementation respectively.

A commutative ring  $R$  will be called a *Boolean valued ring* if there exists a Boolean algebra  $\mathfrak{B}$ , and a single valued mapping  $x \rightarrow \phi(x)$  of  $R$  into  $\mathfrak{B}$  satisfying :

- (i)  $\phi(x)=0$  if and only if  $x=0$ ,
- (ii)  $\phi(xy)=\phi(x) \cap \phi(y)$ ,
- (iii)  $\phi(x+y) \subseteq \phi(x) \cup \phi(y)$ .

When such a mapping exists it will be called a *valuation* for  $R$ . It is not difficult to show that a ring is a Boolean valued ring if and only if it is isomorphic to a subdirect sum of integral domains. Hence every commutative regular ring is Boolean valued.

In a Boolean valued ring the function  $d(x, y) = \phi(x - y)$  satisfies the usual requirements for a distance function, except that the "distance" is an element of a Boolean algebra. The investigation of the geometric properties of a Boolean ring with respect to the distance function defined above was begun by Ellis [3, 4] and has been extended by Blumenthal [1]. The present paper is mainly concerned with extending some of these results to a larger class of Boolean valued rings, namely the  $p$ -rings.

It seems that  $p$ -rings were first defined and studied by McCoy and Montgomery [7] in order to generalize the well known theorem of Stone on the structure of Boolean rings. In [7] it is shown that every  $p$ -ring is a subdirect sum of fields  $I_p$ . In any commutative ring  $R$  the idempotents form a Boolean ring with respect to the multiplication of

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$R$  and addition defined by  $x \oplus y = x + y - 2xy$  (see [6, Exercise 2, p. 211]). This Boolean ring will be called the Boolean ring of idempotents of  $R$ .

**1. A representation theorem for  $p$ -rings.** The main theorem of this section, Theorem 1, and its first corollary are due to Foster [5]. (This fact was unknown to the author until after this paper was presented to the Society.) The proof given here is different from Foster's and quite a bit shorter. Corollary 2 is, to the best of the author's knowledge, new. In connection with Corollary 2 reference is made to Stone's theorem [11, p. 383] on the automorphism group of a Boolean ring. It may be of some interest to note that it is a consequence of Theorem 1 that every  $p$ -ring is uniquely determined by the prime  $p$  and the Boolean ring of idempotents.

**THEOREM 1.** *Let  $B$  be a Boolean ring,  $p$  a fixed prime,  $R^*$  the set of all  $(p-1)$ -tuples of pairwise orthogonal elements of  $B$ . If addition and multiplication for elements of  $R^*$  are defined by*

$$(i) \quad (a_1, a_2, \dots, a_{p-1}) + (b_1, b_2, \dots, b_{p-1}) = (c_1, c_2, \dots, c_{p-1}),$$

where

$$c_i = \sum_{j=0}^{p-1} a_j b_{i-j}, \quad a_0 = 1 + \sum_{j=1}^{p-1} a_j, \quad b_0 = 1 + \sum_{j=1}^{p-1} b_j,$$

and the integers  $i$  and  $j$  are reduced mod  $p$ ; and

$$(ii) \quad (a_1, a_2, \dots, a_{p-1})(b_1, b_2, \dots, b_{p-1}) = (d_1, d_2, \dots, d_{p-1}),$$

where  $d_i = \sum_{j=1}^{p-1} a_j b_{j^{-1}i}$ , and  $j^{-1}$  is the least integer mod  $p$  satisfying  $jx \equiv 1 \pmod{p}$ , then  $R^*$  is a  $p$ -ring which has for its Boolean ring of idempotents a ring isomorphic to  $B$ . Further, every  $p$ -ring is isomorphic to a  $p$ -ring of this type.

**COROLLARY 1.** *Every element  $a$  in a  $p$ -ring may be uniquely expressed in the form  $a = a_1 + 2a_2 + \dots + (p-1)a_{p-1}$ , where  $2, \dots, p-1$  are the successive summands of 1 and the  $a_i$  are pairwise orthogonal idempotents.*

**COROLLARY 2.** *The automorphism group of a  $p$ -ring is isomorphic to the automorphism group of its Boolean ring of idempotents.*

*Proof.* The given Boolean ring  $B$  may be regarded as a subring of the ring of all functions defined on a set  $\Omega$  with values in the two element field  $I_2$ . For a given prime  $p$  consider the ring  $A_p$  of all functions defined on  $\Omega$  with values in the prime field  $I_p$ . Note that an idempotent

$f$  in  $A_p$  takes on only the values 0 or 1 at each point of  $\Omega$ . If there is an element  $g$  in  $B$  such that  $g(\omega)=0$  if and only if  $f(\omega)=0$ , then  $f$  will be said to *belong to*  $B$ . Denote by  $1, 2, \dots, p-1$  the identity of  $A_p$  and its successive summands and define a subset  $\bar{R}^*$  of  $A_p$  to be the set of all  $x$  for which the idempotents

$$x_i=1-(x-i)^{p-1}, \quad i=1, 2, \dots, p-1,$$

*belong to*  $B$ . Note that if  $x \in \bar{R}^*$  then  $x_0=1-\sum_{i=1}^{p-1}x_i$  is an idempotent and *belongs to*  $B$ . It is now easy to verify that

- (i)  $\bar{R}^*$  is a subring of  $A_p$ ,
- (ii) there is a one-to-one correspondence between  $\bar{R}^*$  and the set  $R^*$  which preserves the operations, and
- (iii) the Boolean ring of idempotents of  $\bar{R}^*$  is isomorphic to  $B$ .

This takes care of the first part of the theorem.

Now, let  $R$  be a  $p$ -ring and  $B$  its Boolean ring of idempotents. The ring  $R$  may be regarded as a subring of the ring of all functions defined on a set  $\Omega$  with values in  $I_p$ , and  $B$  as a subring of the ring of all functions defined on the same set  $\Omega$  with values in  $I_2$ . Note that for each  $x$  in  $R$ ,  $1-(x-i)^{p-1}$  is an idempotent for  $i=1, 2, \dots, p-1$ , and hence is an element of  $B$  (it should be pointed out that here the elements of  $B$  are a subset of  $R$ ). Further, note that  $x_i=1-(x-i)^{p-1}$  may be characterized as that function for which  $x_i(\omega)=1$  if  $x(\omega)=i$  and  $x_i(\omega)=0$  if  $x(\omega) \neq i$ . It follows readily from this observation that the  $p$ -ring  $\bar{R}^*$  constructed with  $B$  as in the first part of the theorem is precisely the given  $p$ -ring  $R$ .

The proof of Corollary 1 also follows readily from the observation made above. To prove Corollary 2 let  $R$  be a  $p$ -ring and  $B$  its Boolean ring of idempotents. Denote by  $\mathfrak{A}_R$  and  $\mathfrak{A}_B$  the automorphism groups of  $R$  and  $B$  respectively. Clearly, every  $T$  in  $\mathfrak{A}_R$  is a permutation of the elements of  $B$ . Further,

$$\begin{aligned} (a \oplus b)T &= (a + b - 2ab)T = aT + bT - 2TaTbT = aT + bT - 2aTbT \\ &= aT \oplus bT \end{aligned}$$

for every  $a, b \in B$ , so that  $T \in \mathfrak{A}_R$  determines an element  $T'$  in  $\mathfrak{A}_B$ . It is easily seen that the mapping  $T \rightarrow T'$  of  $\mathfrak{A}_R$  into  $\mathfrak{A}_B$  is a homomorphism. It remains to show that the mapping is an isomorphic mapping of  $\mathfrak{A}_R$  onto  $\mathfrak{A}_B$ . By Corollary 2, every  $a$  in  $R$  may be written

$$a = a_1 + 2a_2 + \dots + (p-1)a_{p-1},$$

where  $a_i = 1 - (a - i)^{p-1} \in B$ . For each  $T'$  in  $\mathfrak{A}_B$ , define a mapping  $T$  of  $R$  into  $R$  by

$$aT = a_1T' + 2(a_2T') + \cdots + (p-1)(a_{p-1}T').$$

Since  $T'$  has an inverse it follows that  $T$  also has an inverse, and hence that  $T$  is a one-to-one mapping of  $R$  onto  $R$ . Further, if  $b \in R$ , so that  $b = b_1 + 2b_2 + \cdots + (p-1)b_{p-1}$ , where  $b_i \in B$ , then by the theorem

$$a + b = c_1 + 2c_2 + \cdots + (p-1)c_{p-1},$$

where

$$c_i = a_j b_i \oplus a_i b_{i-1} \oplus \cdots \oplus a_{p-1} b_{i-(p-1)}.$$

Clearly,

$$c_i T' = a_j T' b_i T' \oplus a_i T' b_{i-1} T' \oplus \cdots \oplus a_{p-1} T' b_{i-(p-1)} T'.$$

Hence,

$$(a + b)T = c_1 T' + 2(c_2 T') + \cdots + (p-1)(c_{p-1} T') = aT + bT.$$

Similarly it is seen that  $(ab)T = (aT)(bT)$  for all  $a, b$  in  $R$ . Thus,  $T$  is an automorphism of  $R$ . It follows from the definition of  $T$  that  $aT = aT'$  in case  $a$  is an idempotent in  $R$ , and hence that the mapping  $T \rightarrow T'$  defined above is a mapping of  $\mathfrak{A}_R$  onto  $\mathfrak{A}_B$ . Finally, let  $T \in \mathfrak{A}_R$  such that  $T \rightarrow E'$ , the identity of  $\mathfrak{A}_B$ . Then  $T$  is an automorphism of  $R$  which maps every idempotent into itself. If  $a \in R$ , so that  $a = a_1 + 2a_2 + \cdots + (p-1)a_{p-1}$ , then

$$aT = a_1T + 2(a_2T) + \cdots + (p-1)(a_{p-1}T) = a_1 + 2a_2 + \cdots + (p-1)a_{p-1} = a.$$

Thus, the kernel of the homomorphic mapping defined above contains only the identity of  $\mathfrak{A}_R$ , and hence  $\mathfrak{A}_R$  and  $\mathfrak{A}_B$  are isomorphic.

If  $B$  is the Boolean ring of idempotents of a  $p$ -ring  $R$  and  $\mathfrak{B}$  the associated Boolean algebra, then the mapping  $a \rightarrow \phi(a) = a^{p-1}$  of  $R$  onto  $\mathfrak{B}$  obviously satisfies Conditions (i) and (ii) of the definition of a Boolean valued ring. That Condition (iii) is also satisfied is seen by verifying

$$(x + y)^{p-1} \subseteq x^{p-1} + y^{p-1} - x^{p-1}y^{p-1}$$

for all  $x, y$  in  $R$ , where the addition and multiplication are those of  $R$  and the inclusion that of  $\mathfrak{B}$ . This relation is equivalent to the identity

$$(x + y)^{p-1}(x^{p-1} + y^{p-1} - x^{p-1}y^{p-1}) = (x + y)^{p-1},$$

which is readily verified (as pointed out by the referee) by noting that

$$z = x^{p-1} + y^{p-1} - x^{p-1}y^{p-1}$$

is the identity element for the subring of *R* generated by *x* and *y*, so that  $(x+y)^t z = (x+y)^t$  for any positive integer *t*. It follows readily from the proof of Theorem 1 that

$$a^{p-1} = a_1 + a_2 + \dots + a_{p-1},$$

where  $a_i = 1 - (a-i)^{p-1}$ . This completes the proof of the following.

**THEOREM 2.** *The mapping*

$$x \rightarrow \phi(x) = x^{p-1} = \sum_{i=1}^{p-1} [1 - (a-i)^{p-1}]$$

*of a p-ring R onto its Boolean algebra B of idempotents is a valuation for R.*

It may be of interest to mention that the principal ideals of a *p*-ring *R* form a Boolean algebra with respect to ideal union and intersection. This is a special case of a result of von Neumann [9] which states that the principal ideals of any commutative regular ring form a Boolean algebra. Further, it may be shown that the mapping  $(x) \rightarrow x^{p-1}$  of the set of principal ideals of *R* onto its Boolean algebra of idempotents is an isomorphism. A proof of this may be obtained from the following two facts, (i) if  $x^{p-1}$  and  $y^{p-1}$  are any two idempotents in *R* then

$$z = x^{p-1} + y^{p-1} - x^{p-1}y^{p-1}$$

is their Boolean algebra union; and (ii) if  $(x)$  and  $(y)$  are any two principal ideals of *R* then  $(xy)$  and  $(z)$  are their intersection and union respectively.

**2. The matrix ring  $B_{p-1}$ .** It was mentioned in the introduction that a Boolean valued ring admits a distance function. This notion is made more precise by the following.

**DEFINITION.** An abstract set  $\mathfrak{M}$  is called a *Boolean distance space* (or simply a Boolean space) if with each pair of elements *a, b* there is associated a unique element  $d(a, b)$  of a Boolean algebra  $\mathfrak{B}$  satisfying:

- (i)  $d(a, b) = d(b, a)$ ,
- (ii)  $d(a, b) = 0$  if and only if  $a = b$ ,
- (iii)  $d(a, b) \subseteq d(a, c) \cup d(c, b)$  for all  $a, b, c$  in  $\mathfrak{M}$ .

It is readily verified that any Boolean valued ring becomes a Boolean space by defining  $d(a, b) = \phi(b-a)$ . It follows from Theorem 2 that every *p*-ring *R* is a Boolean space. Further, if in the representation of *R* by

the elements of  $R^*$ , the elements of  $B$  in a particular  $(p-1)$ -tuple are thought of as "coordinates", then the sum of the coordinates is the distance between the given element and zero.

It is desirable at this point to consider a certain ring of matrices associated with a  $p$ -ring  $R$ . Let  $B$  be the Boolean ring of idempotents of  $R$  and denote by  $B_{p-1}$  the set of all  $(p-1) \times (p-1)$  matrices with elements in  $B$ . Some of the matrices in  $B_{p-1}$  may be used to define transformations of  $R$  into itself as follows. Let  $a \in R$  and  $a^*$  the element of  $R^*$  corresponding to  $a$  in the isomorphism of Theorem 1, let  $M \in B_{p-1}$ , and form the matrix product  $a^*M$ , using the addition  $\oplus$  of the Boolean ring  $B$ . Clearly  $a^*M$  is a  $(p-1)$ -tuple of elements of  $B$ , but it may or may not be in  $R^*$ . If  $a^*M \in R^*$ , let  $b$  be the element of  $R$  corresponding to  $a^*M$  and write  $b = aM$ . If  $x^*M \in R^*$  for all  $x$  in  $R$ , that is,  $xM$  is defined for all  $x$  in  $R$ , then  $M$  defines a transformation of  $R$  into itself. It is not difficult to see that a necessary and sufficient condition that a matrix  $M = (a_{ij})$  in  $B_{p-1}$  define a transformation of  $R$  is that  $a_{is}a_{it} = 0$  for  $i, s, t = 1, 2, \dots, p-1$ ,  $s \neq t$ , in other words, that each row of  $M$  be an element of  $R^*$ .

Before the next definition is given it should be recalled that for every matrix in the ring of  $n \times n$  matrices over an arbitrary commutative ring, a determinant may be computed in the usual way. Further, it may be shown that such a matrix is nonsingular if and only if its determinant has an inverse in the given ring (see [6] or [8]). Thus, since in a Boolean ring the identity is the only element which has an inverse,  $M$  in  $B_{p-1}$  is nonsingular if and only if  $\det(M) = 1$ .

DEFINITION. A nonsingular matrix  $M = (a_{ij})$  in  $B_{p-1}$  for which

$$a_{is}a_{it} = 0, \quad i, s, t = 1, 2, \dots, p-1, s \neq t,$$

is called *orthogonal* if  $\phi(xM) = \phi(x)$  for all  $x$  in  $R$ .

It is readily verified that the set of orthogonal matrices in  $B_{p-1}$  is a subgroup of the group of nonsingular matrices. The next theorem will show that the set of orthogonal matrices coincides with the set of all nonsingular matrices for which  $a_{is}a_{it} = 0$ ,  $s \neq t$ , that is, all nonsingular matrices which define transformations of  $R$ . (The original version of Theorem 3 stated only that (i) and (iii) are equivalent. The author is indebted to the referee for pointing out that (ii) may be included, thus making possible a considerable simplification.)

THEOREM 3. Let  $M = (a_{ij}) \in B_{p-1}$  for which  $a_{is}a_{it} = 0$ ,  $i, s, t = 1, 2, \dots, p-1$ ,  $s \neq t$ , then the following are equivalent: (i)  $M$  is orthogonal, (ii)  $M$  is nonsingular, (iii)  $MM' = I$ .

*Proof.* That (i) implies (ii) is trivial. Suppose next that  $M=(a_{ij})$  is any nonsingular matrix for which  $a_{is}a_{it}=0, s \neq t$ . Then  $M'$  is nonsingular, as is  $M'M=(b_{jk})$ . Note however that

$$b_{jk} = \sum_{i=1}^{p-1} a_{ij}a_{ik} = 0$$

if  $j \neq k$ , so that  $M'M$  is diagonal. Let the diagonal elements be  $d_1, d_2, \dots, d_{p-1}$ , then since 1 is the only element of  $B$  which has an inverse,  $\det(M'M)=d_1d_2 \cdots d_{p-1}=1$ , hence each  $d_i=1$ , or  $M'M=I$ . It follows that  $M'=M^{-1}$ , and hence  $MM'=I$ . Thus, (ii) implies (iii). Finally, let  $M=(a_{ij})$  be a matrix with  $a_{is}a_{it}=0, s \neq t$ , and suppose that  $MM'=I$ . Then  $M$  is nonsingular and defines a transformation of  $R$ . Let  $a \in R$ , and let  $(a_1, a_2, \dots, a_{p-1})$  be the element of  $R^*$  corresponding to  $a$  in the isomorphism of Theorem 1, so that  $aM$  in  $R$  corresponds to the  $(p-1)$ -tuple  $(b_1, b_2, \dots, b_{p-1})$ , where  $b_i = \sum_{j=1}^{p-1} a_j a_{ji}$ . By Theorem 2 and since  $\sum_{i=1}^{p-1} a_{ji} = 1$ ,

$$\phi(aM) = \sum_{i=1}^{p-1} b_i = \sum_{i=1}^{p-1} \left( \sum_{j=1}^{p-1} a_j a_{ji} \right) = \sum_{j=1}^{p-1} a_j \left( \sum_{i=1}^{p-1} a_{ji} \right) = \sum_{j=1}^{p-1} a_j = \phi(a).$$

Thus  $M$  is orthogonal, (iii) implies (i) and this completes the proof of the theorem.

**3. The group of motions of  $R$ .** The group of orthogonal matrices in  $B_{p-1}$  will be used to describe the motions (isometries) of the Boolean space of a  $p$ -ring  $R$ . This is done in Theorem 4, which also contains (thanks to the referee) a geometric characterization of transformations  $x \rightarrow xM$  of  $R$  defined by arbitrary matrices in  $B_{p-1}$ . First, two lemmas and a definition are needed. The lemmas are obvious and their proofs are omitted.

LEMMA 1. *In a Boolean algebra if  $ax=0$  implies  $ay=0$  then  $y \subseteq x$ .*

LEMMA 2. *Let  $R$  be a  $p$ -ring,  $B$  its Boolean ring of idempotents, and  $B_{p-1}$  the matrix ring described in the last section. If  $z \in B, a \in R$ , and  $M \in B_{p-1}$  such that  $xM$  is defined for all  $x$  in  $R$  then  $z(aM)=(za)M$ .*

DEFINITION. A one-to-one mapping  $x \rightarrow f(x)$  of a Boolean space  $\mathfrak{M}$  onto itself is called a *motion (isometry)* of  $\mathfrak{M}$  if  $d(f(x), f(y))=d(x, y)$  for all  $x, y$  in  $\mathfrak{M}$ .

THEOREM 4. *Let  $R, B, B_{p-1}$  be defined as in Lemma 2. The mapping  $x \rightarrow f(x)$  of  $R$  into  $R$  has the properties*



- (i)  $f(0)=0$  ,
- (ii)  $d(f(x), f(y)) \subseteq d(x, y)$  ,

if and only if there exists an  $M=(a_{ij})$  in  $B_{p-1}$  with  $a_{is}a_{it}=0$ ,  $s \neq t$ , such that  $f(x)=xM$  for all  $x$  in  $R$ . Further, the mapping is a motion if and only if  $M$  is orthogonal.

**COROLLARY.** *The mapping  $x \rightarrow f(x)$  of  $R$  into  $R$  satisfies  $d(f(x), f(y)) \subseteq d(x, y)$  if and only if  $f(x)=xM+a$  for some  $M$  in  $B_{p-1}$  with  $a_{is}a_{it}=0$ ,  $s \neq t$ , and  $a$  in  $R$ . Further, the mapping is a motion if and only if  $M$  is orthogonal.*

*Proof.* Let  $M=(a_{ij}) \in B_{p-1}$  with  $a_{is}a_{it}=0$ ,  $s \neq t$ , and consider the transformation  $f(x)=xM$ . That  $f(0)=0$  is trivial. Let  $a, b \in R$  and choose  $z$  in  $B$  so that  $z \cdot \phi(b-a)=0$ . Then  $\phi(zb-za)=0$ , hence  $zb=za$  and  $(zb)M=(za)M$ . Thus, by Lemma 2,

$$z(bM-aM)=0 \quad , \quad z \cdot \phi(bM-aM)=0 \quad ,$$

and hence by Lemma 1,  $d(f(b), f(a)) \subseteq d(b, a)$ . Further, if  $M$  is orthogonal (recall that, by Theorem 3, orthogonality for such an  $M$  is equivalent to nonsingularity) and if  $y$  is chosen in  $B$  so that  $y \cdot \phi(bM-aM)=0$  then by Lemma 2,  $(yb)M=(ya)M$ . Since  $M$  is nonsingular this implies  $yb=ya$  and hence that  $y \cdot \phi(b-a)=0$ . Thus,  $d(b, a) \subseteq d(f(b), f(a))$  which, together with the other inequality, gives  $d(f(b), f(a))=d(b, a)$ . Since  $M$  has an inverse it follows that  $x \rightarrow f(x)$  is a motion of the Boolean space of  $R$ .

Next, suppose that  $x \rightarrow f(x)$  is a transformation of  $R$  with the properties (i) and (ii) stated in the theorem. Then  $\phi(f(x)) \subseteq \phi(x)$  for all  $x$  in  $R$ . Let  $a_i=f(i)$ ,  $i=1, 2, \dots, p-1$ , and let  $(a_{i1}, a_{i2}, \dots, a_{i,p-1})$  be the element in  $R^*$  corresponding to  $a_i$  in the isomorphism of Theorem 1. Define  $M$  in  $B_{p-1}$  to be the matrix whose  $i$ th row is  $(a_{i1}, a_{i2}, \dots, a_{i,p-1})$  and note that  $M$  defines a transformation of  $R$ . Now, let  $x \in R$ , then clearly

$$\phi(f(x)-xM) \subseteq \phi(f(x)) \cup \phi(xM) \subseteq \phi(x) \quad .$$

Further,

$$\begin{aligned} &\phi(f(x)-xM) \\ &= \phi(f(x)-f(i)+iM-xM) \subseteq \phi(f(x)-f(i)) \cup \phi(iM-xM) \subseteq \phi(x-i) \quad , \end{aligned}$$

for  $i=1, 2, \dots, p-1$ . Hence

$$\phi(f(x)-xM) \subseteq \prod_{k=0}^{p-1} \phi(x-k) = \phi \left[ \prod_{k=0}^{p-1} (x-k) \right] = \phi(x^p-x) = 0 \quad ,$$

and hence  $f(x)=xM$ . If, in addition,  $x \rightarrow f(x)$  is a motion, then, since  $\phi(i)=1, i=1, 2, \dots, p-1$ , it follows that

$$\sum_{j=1}^{p-1} a_{ij} = \phi(a_i) = 1 .$$

Let  $z_{ijk} = a_{ik}a_{jk}, i, j, k=1, 2, \dots, p-1, i \neq j$ , and note that  $z_{ijk}a_i = z_{ijk}a_j = kz_{ijk}$ , whence  $z_{ijk}(a_i - a_j) = 0$ . Since

$$\phi(a_i - a_j) = \phi(f(i) - f(j)) = \phi(i - j) = 1 ,$$

it follows that  $a_i - a_j$  has an inverse in  $R$ . Thus,  $a_{ik}a_{jk} = z_{ijk} = 0, i \neq j$ , and hence  $MM' = I$ . By Theorem 3,  $M$  is orthogonal and this completes the proof of the theorem.

The corollary is obtained by an obvious application of the theorem.

In case  $p=2$  it is clear that  $B_{p-1}$  contains only one orthogonal element. Thus, the corollary to Theorem 4 generalizes a result of Ellis [4] which states that any motion  $x \rightarrow f(x)$  of the Boolean space of a Boolean ring may be written  $f(x) = x + a$ . This result can also be easily proved without reference to Theorem 4, thus, if  $R$  is a Boolean ring and  $x \rightarrow f(x)$  a motion of the Boolean space of  $R$  then, since  $d(x, y) = x - y, f(x) - f(y) = x - y$ , and hence  $f(x) = x + f(0)$ .

**4. Superposability.** Two subsets  $\mathfrak{A}$  and  $\mathfrak{B}$  of a Boolean space  $\mathfrak{M}$  are said to be *congruent* if there is a one-to-one mapping of  $\mathfrak{A}$  onto  $\mathfrak{B}$  which preserves distances. If the congruent mapping of  $\mathfrak{A}$  onto  $\mathfrak{B}$  may be extended to a motion of  $\mathfrak{M}$ , then  $\mathfrak{A}$  and  $\mathfrak{B}$  are said to be *superposable*. In case every two congruent subsets of  $\mathfrak{M}$  are superposable  $\mathfrak{M}$  is said to have the property of *free mobility*. Ellis [3] has shown that the Boolean space of a Boolean ring has the property of free mobility. It will be shown in this section that this is in general not true for a  $p$ -ring with  $p > 2$ . In fact the following theorem and its corollary will be proved.

**THEOREM 5.** *Let  $R$  be a  $p$ -ring,  $p > 2, B$  its Boolean ring of idempotents and  $\mathfrak{B}$  the Boolean algebra associated with  $B$ . A necessary and sufficient condition that the Boolean space of  $R$  have the property of free mobility is that  $\mathfrak{B}$  be a complete Boolean algebra.*

**COROLLARY.** *Every two congruent, finite subsets of the Boolean space of a  $p$ -ring are superposable.*

The following two lemmas are needed in the proof of the theorem. It should be pointed out that the validity and proof of Lemma 4 are

unchanged if the matrix ring  $B_{p-1}$  is replaced by the ring of  $n \times n$  matrices over any Boolean ring.

LEMMA 3. *Let  $a, b$  be elements of a Boolean valued ring  $S$ . If  $ab=0$  then*

$$\phi(a+b)=\phi(a)\cup\phi(b).$$

*Proof.* By commutativity  $ba=ab=0$ , so that

$$\phi(a+b)[\phi(a)\cup\phi(b)]=\phi(a+b)\phi(a)\cup\phi(a+b)\phi(b)=\phi(a^2)\cup\phi(b^2)=\phi(a)\cup\phi(b).$$

Hence,  $\phi(a)\cup\phi(b)\subseteq\phi(a+b)$ . This last relation, together with  $\phi(a+b)\subseteq\phi(a)\cup\phi(b)$ , implies  $\phi(a+b)=\phi(a)\cup\phi(b)$ .

LEMMA 4. *Let  $R, B, B_{p-1}$  be defined as in Lemma 2. If  $M=(a_{ij})\in B_{p-1}$  for which  $a_{ij}a_{kj}=0$  and  $a_{ji}a_{jk}=0$ , for  $i, j, k=1, 2, \dots, p-1, i\neq k$ , then there exists a matrix  $C=(c_{ij})$  in  $B_{p-1}$  such that*

- (i)  $M+C$  is orthogonal,
- (ii)  $c_{ir}c_{is}=0$ , for  $i, r, s=1, 2, \dots, p-1, r\neq s$ ,
- (iii)  $a_{ir}c_{is}=0$ , for  $i, r, s=1, 2, \dots, p-1$ .

*Proof.* (The following proof is due to the referee. It is much more simple and considerably shorter than the author's.) Suppose first that  $B$  is the field  $I_2$  so that  $M$  is a matrix with at most a single 1 in each row and each column. Then the desired matrix  $C$  must satisfy (i)  $M+C$  is nonsingular, (ii)  $C$  has at most a single 1 in each row, and (iii)  $C$  has a zero row if the corresponding row of  $M$  is not zero. It is not difficult to see that there exists a matrix  $C$  satisfying (ii) and (iii) and such that  $M+C$  has exactly one 1 in each row and column. Next suppose that  $B$  is an arbitrary Boolean ring. Then the elements  $a_{ij}$  of  $M$  together with 1 generate a finite Boolean ring  $B'\subseteq B$ . It is sufficient to find a matrix  $C$  with elements in  $B'$ . However, since  $B'$  is a complete direct sum of fields  $I_2$ , the desired matrix  $C$  may be obtained by applying the process above to each summand in the direct sum.

*Proof of Theorem 5.* Let  $R$  be a  $p$ -ring for which the Boolean algebra  $\mathfrak{B}$  associated with the Boolean ring of idempotents is complete. Let  $S_1$  and  $T_1$  be any two subsets of  $R$  which are congruent under the mapping  $x\rightarrow h_1(x)$  of  $S_1$  onto  $T_1$ . For some  $a$  in  $S_1$  consider the motions  $x\rightarrow s(x)=x-a$ , and  $x\rightarrow t(x)=x-h_1(a)$ . The subsets  $S_1$  and  $T_1$  are mapped by these motions into subsets  $S=s(S_1)$  and  $T=t(T_1)$  which are congruent under the mapping

$$x \rightarrow h(x) = h_1(x+a) - h_1(a) .$$

Clearly *S* and *T* both contain 0, and  $h(0)=0$ . It follows that  $\phi(h(x))=\phi(x)$  for *x* in *S*. To facilitate the following discussion let  $\bar{x}=h(x)$  for each *x* in *S*, and let  $(x_1, x_2, \dots, x_{p-1})$  and  $(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{p-1})$  be the elements in  $R^*$  corresponding respectively to *x* and  $\bar{x}$  in the isomorphism of Theorem 1. For each *i, j=1, 2, \dots, p-1* define  $a_{ij} = \bigcup_{x \in S} x_i \bar{x}_j$ , and let  $M=(a_{ij})$ . Note that even though  $a_{ij}$  is defined by an operation of  $\mathfrak{B}$  it is nevertheless an element of *B*. For fixed *i* and  $j \neq k$  and any *y, z* in *S* consider the product  $b=(y_i \bar{y}_j)(z_i \bar{z}_k)$ . Clearly,  $by_i = b\bar{y}_j = bz_i = b\bar{z}_k = b$ . Since the elements in any  $(p-1)$ -tuple in  $R^*$  are pairwise orthogonal, it follows that  $by_s = by_i y_s = 0$  for  $s \neq i$ . Similarly,  $b\bar{y}_s = 0$  for  $s \neq j$ ,  $bz_s = 0$  for  $s \neq i$ , and  $b\bar{z}_s = 0$  for  $s \neq k$ . Hence,

$$by = b(y_1 + 2y_2 + \dots + (p-1)y_{p-1}) = iby_i = ib .$$

Similarly,  $bz = ib$ ,  $b\bar{y} = jb$ , and  $b\bar{z} = kb$ . Since  $x \rightarrow \bar{x}$  is a congruent mapping of *S* onto *T*,  $\phi(y-z) = \phi(\bar{y} - \bar{z})$ , and since  $j \neq k$ ,  $\phi(j-k) = 1$ . Hence,

$$\begin{aligned} b &= b \cdot \phi(j-k) = \phi(jb - kb) = \phi(b\bar{y} - b\bar{z}) = b\phi(\bar{y} - \bar{z}) = b\phi(y-z) \\ &= \phi(by - bz) = \phi(ib - ib) = 0 . \end{aligned}$$

Thus,

$$a_{ij} a_{ik} = \left( \bigcup_{y \in S} y_i \bar{y}_j \right) \left( \bigcup_{z \in S} z_i \bar{z}_k \right) = 0$$

in  $\mathfrak{B}$  and hence also in *B*. Similarly it may be shown that  $a_{ij} a_{kj} = 0$  for  $i, j, k=1, 2, \dots, p-1, i \neq k$ . Thus, *M* satisfies the hypotheses of Lemma 4 and hence there exists a matrix *C* in  $B_{p-1}$  such that  $M+C$  is orthogonal. The matrix  $M+C$  defines a motion of *R*, and the matrix *M* defines, at least, a transformation of *R* into *R*, as described in § 2. The transformation defined by *M* maps *S* onto a subset  $S^*$ , which will now be examined. For *s* in *S*, let  $s^* = sM$ , and note that  $a_{ij} \supseteq s_i \bar{s}_j$  follows from the definition of  $a_{ij}$ . Thus,  $s_i a_{ij} \supseteq s_i \bar{s}_j$ , and since for pairwise orthogonal elements  $x_i$  in  $\mathfrak{B}$ ,  $\bigcup x_i = \sum x_i$  in *B*, it follows that

$$s_j = \sum_{i=1}^{p-1} s_i a_{ij} \supseteq \sum_{i=1}^{p-1} s_i \bar{s}_j = \phi(s) \bar{s}_j = \phi(\bar{s}) \bar{s}_j = \bar{s}_j ,$$

or

$$(1) \quad s_j^* \supseteq \bar{s}_j , \quad j=1, 2, \dots, p-1 .$$

Further,

$$\phi(s^*) = \sum_{j=1}^{p-1} \sum_{i=1}^{p-1} s_i a_{ij} = \sum_{i=1}^{p-1} s_i \left( \sum_{j=1}^{p-1} a_{ij} \right) \subseteq \sum_{i=1}^{p-1} s_i = \phi(s) = \phi(\bar{s}) ,$$

and from (1) it follows that  $\phi(s^*) \supseteq \phi(\bar{s})$ . Thus,

$$(2) \quad \phi(s^*) = \phi(\bar{s}) .$$

If  $r \neq j$ , it follows from (1) that  $s_r^* \bar{s}_j \subseteq s_r^* s_j^* = 0$ , and hence that  $s_r^* \bar{s}_j = 0$ . From (2),

$$\sum_{i=1}^{n-1} s_i^* = \sum_{i=1}^{n-1} \bar{s}_i ,$$

whence

$$s_j^* = s_j^* \sum_{i=1}^{n-1} s_i^* = s_j^* \sum_{i=1}^{n-1} \bar{s}_i = s_j^* \bar{s}_j .$$

It follows that  $s_j^* \subseteq \bar{s}_j$ , and this together with (1) gives  $s_j^* = \bar{s}_j$ , hence  $sM = s^* = \bar{s} = h(s)$ . Thus, the transformation defined by  $M$  maps  $S$  onto  $T$  and coincides with the congruence  $s \rightarrow h(s)$ .

It remains to show that  $sM = s(M+C)$  for  $s$  in  $S$ . By Lemma 4,  $c_{ij} a_{ir} = 0$ ,  $i, r, j = 1, 2, \dots, p-1$ . For  $s$  in  $S$  let  $b = s_i c_{ij}$ , then  $b \cdot a_{ir} = 0$ . Since

$$a_{ir} = \bigcup_{x \in S} x_i \bar{x}_r \supseteq s_i \bar{s}_r ,$$

it follows that

$$0 = b a_{ir} \supseteq b s_i \bar{s}_r = b \bar{s}_r ,$$

or that  $b \bar{s}_r = 0$ ,  $r = 1, 2, \dots, p-1$ . Thus,  $b\phi(s) = b\phi(\bar{s}) = 0$ , whence  $bs_i = 0$ . Consequently  $s_i c_{ij} = b = bs_i = 0$  for  $i, j = 1, 2, \dots, p-1$ . Thus,  $s(M+C) = sM$  for  $s$  in  $S$ , and the motion of  $R$  defined by  $M+C$  coincides with  $h(s)$  on  $S$ . Finally, let  $\alpha, \beta, \gamma$  be the motions of  $R$  defined by the mappings  $x \rightarrow s(x) = x - a$ ,  $x \rightarrow x(M+C)$ ,  $x \rightarrow t(x) = x - h_i(a)$ , respectively, and note that the motion  $\alpha\beta\gamma^{-1}$  coincides on  $S_1$  with the congruence  $x \rightarrow h_i(x)$  of  $S_1$  onto  $T_1$ .

To prove the necessity it will be shown that a  $p$ -ring,  $p > 2$ , whose Boolean algebra of idempotents is not complete does not have the property of free mobility. Let  $\mathfrak{B}$  be a Boolean algebra which is not complete, and let  $X$  be a subset of  $\mathfrak{B}$  for which no least upper bound exists. Since  $x \subset 1$  for all  $x$  in  $X$ , the set  $X^*$  of all upper bounds to  $X$  is not vacuous. Let  $Y$  be the set of complements of elements of  $X^*$ . It will be shown that if  $x, y$  are any upper bounds to  $X, Y$  respectively then  $xy \neq 0$ . Suppose on the contrary that  $xy = 0$ , then since  $x$  is not a least upper bound to  $X$ , there exists a  $z \subset x$  which is an upper bound to  $X$ . Then  $z' \in Y$ , hence  $z' \subseteq y$ , and  $xz' \subseteq xy = 0$ , or  $xz' = 0$ , whence  $xz = x$ . It follows that  $x \subseteq z \subset x$ , a contradiction. Thus,  $xy \neq 0$  as stated. Note, however, that for all  $a$  in  $X, b$  in  $Y, ab = 0$ .

Now, let  $R$  be a  $p$ -ring,  $p > 2$ , with  $\mathfrak{B}$  as its Boolean algebra of idempotents, and let  $X, Y$  be the subsets of  $\mathfrak{B}$  described above. Suppose, without loss of generality, that the cardinality of  $Y$  is greater than or equal to the cardinality of  $X$ . Then there is a one-to-one correspondence between  $X$  and a subset  $Y_1$  of  $Y$ , say  $x \longleftrightarrow f(x)$ . Denote by  $Y_2$  the subset of  $Y$  consisting of those elements which are not in  $f(X)$ , and define subsets  $A$  and  $B$  of  $R$  as follows:  $A$  contains 0, each  $y$  in  $Y_2$ , and for each  $x$  in  $X$ , the element  $x + f(x)$ ;  $B$  contains 0,  $2y$  for each  $y$  in  $Y_2$ , and for each  $x$  in  $X$ , the element  $x + 2f(x)$ . Consider the mapping  $z \rightarrow F(z)$  of  $A$  onto  $B$  defined by

$$F(z) = \begin{cases} 0 & \text{if } z = 0, \\ 2y & \text{if } z = y, \\ x + 2f(x) & \text{if } z = x + f(x), \end{cases}$$

To see that

$$\phi(F(z_1) - F(z_2)) = \phi(z_1 - z_2),$$

for all  $z_1, z_2$  in  $A$ , note first that  $\phi(F(z)) = \phi(z) = z$  for all  $z$  in  $A$ , and hence that if either  $z_1 = 0$  or  $z_2 = 0$ , the equality is immediate. Also, the equality is obvious if  $z_1, z_2 \in Y_2 \subset A$ . If  $z_1 = x_1 + f(x_1)$  and  $z_2 = x_2 + f(x_2)$  then

$$\phi(F(z_1) - F(z_2)) = \phi[(x_1 - x_2) + 2(f(x_1) - f(x_2))],$$

and since  $(x_1 - x_2)(f(x_1) - f(x_2)) = 0$ , it follows from Lemma 3 that

$$\phi(F(z_1) - F(z_2)) = \phi(x_1 - x_2) + \phi(f(x_1) - f(x_2)).$$

Similarly,

$$\phi(z_1 - z_2) = \phi(x_1 - x_2) + \phi(f(x_1) - f(x_2)).$$

Finally, if  $z_1 = x + f(x)$  and  $z_2 = y \in Y_2$ , then, again by the use of Lemma 3,

$$\begin{aligned} \phi(F(z_1) - F(z_2)) &= \phi[x + 2(f(x) - y)] = \phi(x) + \phi(f(x) - y) \\ &= \phi(x + f(x) - y) = \phi(z_1 - z_2). \end{aligned}$$

Thus,  $z \rightarrow F(z)$  is a congruent mapping of  $A$  onto  $B$ . Suppose that  $A$  and  $B$  are superposable. Then there exists an orthogonal matrix  $M = (m_{ij})$  in  $B_{p-1}$  such that the motion  $x \rightarrow xM$  coincides with  $F(x)$  on  $A$ , or  $F(x) = xM$  for all  $x$  in  $A$ . Thus,

$$(3) \quad \begin{cases} \text{(i)} & x + 2f(x) = [x + f(x)]M & \text{for } x \text{ in } X, \\ \text{(ii)} & 2y = yM & \text{for } y \text{ in } Y_2. \end{cases}$$

It follows from (3) (i) that

$$x + 2f(x) = [x + f(x)]m_{11} + [x + f(x)]m_{12},$$

or that

$$x = [x + f(x)]m_{11}, \quad f(x) = [x + f(x)]m_{12},$$

whence  $x = xm_{11}$ ,  $f(x) = f(x)m_{12}$ , so that

$$(4) \quad (i) \quad x \subseteq m_{11}, \quad (ii) \quad f(x) \subseteq m_{12}, \quad \text{for all } x \text{ in } X.$$

Similarly, from (3) (ii) it follows that

$$(5) \quad y \subseteq m_{12}, \quad \text{for all } y \text{ in } Y_2.$$

Relations (4) and (5) state that  $m_{11}$  is an upper bound to  $X$ , and  $m_{12}$  an upper bound to  $Y$ . But  $m_{11}m_{12} = 0$ , and this contradicts the choice of  $X$  and  $Y$ . Thus, the congruent subsets  $A$  and  $B$  of  $R$  are not superposable. This completes the proof of the theorem.

*Proof of the corollary.* If the congruent subsets  $S_1$  and  $T_1$  in the sufficiency part of the proof are finite then

$$a_{ij} = \bigcup_{x \in S} x_i \bar{x}_j$$

exists whether  $\mathfrak{B}$  is complete or not. The sufficiency proof then shows that  $S_1$  and  $T_1$  are superposable.

**5. Betweenness and linearity.** Let  $R$  be a  $p$ -ring,  $B$  its Boolean ring of idempotents, and  $\mathfrak{B}$  the Boolean algebra associated with  $B$ . Since  $\phi(a-b) = a \oplus b$  for all  $a, b$  in  $B$ , it follows that the subset  $B$  of  $R$  is congruent to the autometrized Boolean algebra  $\mathfrak{B}$  (autometrized Boolean algebra is the name given by Ellis [3] to what is here called the Boolean space of a Boolean ring (2-ring)). The same is true for the image of  $B$  under any motion of  $R$ . The subset  $f(B)$ , where  $f$  is any motion of  $R$ , will be called a *one-dimensional subspace* of  $R$ . Note that in view of Theorem 5 the set of all one-dimensional subspaces of  $R$  is not necessarily the same as the set of all subsets of  $R$  congruent to  $\mathfrak{B}$ , unless  $\mathfrak{B}$  is a complete Boolean algebra. In any event, all of the results of Blumenthal [1] are applicable to a one-dimensional subspace of  $R$ . For example, one is led to define betweenness for elements of  $R$  as follows:

DEFINITION. Let  $a, b, c \in R$ , then  $b$  is said to be *between*  $a$  and  $c$  if and only if

$$(i) \quad a \neq b \neq c,$$

- (ii)  $a, b, c$  are contained in a one-dimensional subspace of  $R$ ,
- (iii)  $\phi(b-a) \cup \phi(c-b) = \phi(c-a)$ .

The symbol  $\beta(a, b, c)$  will mean that  $b$  is between  $a$  and  $c$ .

Following Blumenthal [1] a set of  $m$  pairwise distinct elements of  $R$  is said to be a  $\beta$ -linear  $m$ -tuple provided there exists a labeling,  $a_1, a_2, \dots, a_m$  such that  $\beta(a_{i_1}, a_{i_2}, a_{i_3})$  holds for all  $1 \leq i_1 < i_2 < i_3 \leq m$ .

The following theorem now follows almost immediately from the corresponding theorem for an autometrized Boolean algebra [1, Theorem 4.2, p. 9].

**THEOREM 6.** *If each triple of pairwise distinct elements of an  $m$ -tuple,  $m > 4$ , is  $\beta$ -linear then the  $m$ -tuple is  $\beta$ -linear.*

*Proof.* Since each triple is congruent to a subset of the autometrized Boolean algebra  $\mathfrak{B}$ , whose elements are the idempotents of  $R$ , it follows from a theorem of Ellis [3, Theorem 5.1, p. 92] that the  $m$ -tuple is congruent to an  $m$ -tuple of  $\mathfrak{B}$ , for which all triples are  $\beta$ -linear. Hence, by the theorem of Blumenthal referred to above, the given  $m$ -tuple is  $\beta$ -linear.

**6. Two unsolved problems.** A set of  $k$  elements,  $a_1, a_2, \dots, a_k$ , of a Boolean space is called a *metric basis* for the space if  $x$  is the only point with distances  $d(a_i, x)$  from the  $a_i$ . It is not difficult to show that in the Boolean space of a  $p$ -ring  $R$  the elements  $1, 2, \dots, p-1$  form a metric basis. However, necessary and sufficient conditions that a subset  $A \subseteq R$  form a metric basis are not known.

Another unsolved problem is the extension to the Boolean space of a  $p$ -ring,  $p > 2$ , of the result of Ellis used in the proof of Theorem 6. Ellis calls an abstract set  $\Sigma$  a *B-metrized space* if with each  $x, y$  in  $\Sigma$  there is associated an element  $d(x, y)$  of a Boolean algebra  $\mathfrak{B}$ , satisfying: (i)  $d(x, y) = 0$ , if and only if  $x = y$ , and (ii)  $d(x, y) = d(y, x)$  for all  $x, y$  in  $\Sigma$ . Thus, a Boolean space is a *B-metrized space* in which  $d(x, z) \subseteq d(x, y) \cup d(y, z)$  holds for all  $x, y, z$ . Ellis has shown in [3] that a given abstract *B-metrized space*  $\Sigma$  is congruent to a subset of the Boolean space of a Boolean ring  $R$  if every three points of  $\Sigma$  are congruent to some set of three points in  $R$ , and further, that three is the smallest integer for which this is true. Whether or not there exists such an integer in case  $R$  is a  $p$ -ring,  $p > 2$ , is not known. If such an integer  $n$  exists for a  $p$ -ring  $R$ , then  $n$  is called the *best congruence order* of the Boolean space of  $R$  with respect to the class of *B-metrized spaces*. The reader is referred to Blumenthal [2] for a discussion of congruence orders of Euclidean spaces, and the metric characterization problem.



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