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1. Introduction and Summary. This study proceeds from a theorem of Favard that states that polynomial sets obeying a certain type of three-term recursion formula are, with respect to some weight function, orthogonal over some set on the real axis. For a rather wide subclass of these polynomial sets, it is shown that the orthogonality set consists of a discrete but infinite set of points of the real axis. The orthogonality set and the weight function are given by certain entire functions that can be constructed from the polynomials by a limiting process. A set of modified Lommel polynomials is given as an explicit example of the general theory. They are orthogonal over the reciprocals of the roots of certain Bessel functions.

2. General theory and existence of orthogonality. In 1935, Favard [2] sketched the proof of the following theorem:

If there is a sequence of real polynomials $\{\varphi_n(x)\}$ possessing a recursion relation of the form

$$(1) \qquad \varphi_{n+1}(x) = (x - a_n)\varphi_n(x) - \lambda_n\varphi_{n-1}(x) , \qquad (n \geq 0) ,$$

with

$$(2) \qquad \varphi_0(x) = 1 , \quad \varphi_1(x) = x - a_0 , \quad \varphi_{-1}(x) = 0$$

and

$$(3) \qquad \lambda_n > 0 ,$$

then the polynomials are orthogonal.

This powerful theorem seems to have escaped attention; it suggests that whenever a relation of the form (1) exists, a weight function and orthogonality domain could probably be found by detailed study. We want to accomplish this for a wide class of polynomials for which the orthogonality range comes out to be a denumerable set of points.

The class of polynomials mentioned is obtained by imposing upon (2) and (3) the restrictions that

$$(4) \qquad\qquad a_n = 0 \qquad\qquad (n \geq 0)$$

and that the limit B exists where

(5)
$$\sum_{n=1}^{\infty} \lambda_n = B \ .$$

The restriction (4) implies that the polynomials are alternately even and odd and that we may take our weight function $d\mu(x)$ such that

$$\int_{-\infty}^{+\infty} x^{2n+1} d\mu(x) = 0$$

for positive integral n. We may take infinite limits of integration since, by defining some factor of the integrand as zero at suitable points, all real integrals may be written with infinite limits. We also take $\mu(x)$ to be a nondecreasing function such that

(6)
$$\int_{-\infty}^{+\infty} d\mu(x) = 1 \ .$$

From (1) and (2), the leading coefficients of our polynomials are unity and thus

$$\int_{-\infty}^{+\infty} x\varphi_{n-1}(x)\varphi_n(x) d\mu(x) = \int_{-\infty}^{+\infty} \varphi_n^2(x) d\mu(x) \ .$$

Hence after multiplying (1) by $\varphi_{n-1}(x) d\mu(x)$ and integrating we have

$$0 = \int_{-\infty}^{\infty} \varphi_n^2(x) d\mu(x) - \lambda_n \int_{-\infty}^{\infty} \varphi_{n-1}^2(x) d\mu(x) \ , \qquad (n > 0).$$

The solution of this recursion relation subject to the condition (6) is

(7)
$$\int_{-\infty}^{+\infty} \varphi_n^2(x) d\mu(x) = \prod_{i=1}^{n} \lambda_i$$

where for $n=0$, the product on the right is to be taken as unity.

We now introduce a second set of polynomials $F_n(z)$ by the substitutions

(8)
$$z = 1/x \ ,$$

(9)
$$F_n(z) = z^n \varphi_n(1/z) \ .$$

Because of (1), these polynomials obey the recursion relation

(10)
$$F_{n+1}(z) - F_n(z) = -\lambda_n z^2 F_{n-1}(z) \ , \qquad (n \geq 0).$$

We deduce from this relation the following

THEOREM 1. *There exists an entire function $F(z)$ such that $|F_n(z)| \leq F(z)$ for all n and real z.*

Proof. We define $G_n(z)$ as the larger of the absolute values of $F_n(z)$

and $F_{n-1}(z)$. This, with (10), yields

$$|F_{n+1}(z)| \leqq (1 + \lambda_n |z|^2) G_n(z) \ ,$$

and hence

$$G_{n+1}(z) \leqq (1 + \lambda_n |z|^2) G_n(z) \ .$$

From this and the easily verified identities

$$F_0(z) = F_1(z) = G_1(z) = 1 \ ,$$

we have, for all n,

$$|F_n(z)| \leqq \prod_{n=1}^{\infty} (1 + \lambda_n |z|^2) \ .$$

Here the convergence of the infinite product is assured by (5). If we define an entire function $F(z)$ by

$$F(z) = \prod_{n=1}^{\infty} (1 + \lambda_n z^2) \ ,$$

then we have, for real z and all n, that

(11) $$|F_n(z)| \leqq F(z) \ .$$

THEOREM 2. *The polynomials $F_n(z)$ converge toward an entire function $E(z)$ whose zeros form a denumerable set on the real axis. For all z in any bounded set, the convergence is uniform.*

Proof. We first derive the Cauchy criterion for the set using equations (10) and (5) and Theorem 1.

$$|F_{n+M}(z) - F_n(z)| \leqq \sum_{i=0}^{M-1} |F_{n+M-i}(z) - F_{n+M-1-i}(z)|$$

$$\leqq |z|^2 F(z) \sum_{i=0}^{M-1} \lambda_{n+M-1-i}$$

$$\leqq |z|^2 F(z) [S(n) - S(n+M)] \ ,$$

where

$$S(n) = \sum_{i=n}^{\infty} \lambda_i \ .$$

Since $S(n)$ is a convergent series, the analytic functions $F_n(z)$ converge uniformly toward a function $E(z)$ if z is bounded. Hence $E(z)$ is an analytic entire function of z. It cannot vanish identically because from (2), (9), and (10), we have that

(12) $$E(0) = 1 \ .$$

From a theorem of Hurwitz (see [4], p. 119 *and* [3], Theorem 1.91.3) it follows that if z_0 is a zero of $E(z)$ of multiplicity q and $N(z_0)$ some neighborhood of z_0, then for n sufficiently large, $N(z_0)$ contains precisely q zeros of $F_n(z)$. Now for any positive integer p, there exists a circle C_p centered on the origin that has in its exterior at least p zeros of $\varphi_{p+1}(x)$. From the Sturmian properties of the $\varphi_n(x)$, it follows that for $n > p+1$, $\varphi_n(x)$ also has at least p zeros in the exterior of C_p. That is, there exists a circle C_p' (the reciprocal circle of C_p) that contains in its interior at least p zeros of each of the polynomials $F_n(z)$ for $n > p+1$. Hence for this arbitrary integer p, $E(z)$ contains at least p zeros in or on C_p'. (A zero of multiplicity q we count as q zeros.) Thus $E(z)$ has an infinite number of zeros.

The reality of the zeros of the $F_n(z)$ can be seen from the Sturmian properties of the $\varphi_n(x)$ and, using the Hurwitz theorem again, it quickly follows that the zeros of $E(z)$ are real.

THEOREM 3. *The derivative of $\mu(x)$ is everywhere zero except at the reciprocals of the zeros of $E(z)$.*

A theorem of this type appears in the literature. (See [3], Theorem 6.1.1.) In the following a proof applicable to the present situation is given.

Proof. From Theorem 2 and (9), it follows that

$$(13) \qquad \lim_{n \to \infty} \frac{\varphi_n(x)}{x^n} = E\left(\frac{1}{x}\right)$$

uniformly in every bounded domain.

Suppose we fix our attention on the interval between two successive positive zeros, p_ν and $p_{\nu+1}$ of $E(1/x)$. Let us consider an interval slightly smaller than this, $[a, b]$, so that $E^2(1/x)$ will have a positive lower bound over the interval. We may then use the two preceding theorems to develop the following estimate:

$$\int_a^b x^{2n} E^2(1/x) d\mu(x)$$

$$= \int_a^b \varphi_n^2(x) d\mu(x) + \int_a^b [x^{2n} E^2(1/x) - \varphi_n^2(x)] d\mu(x)$$

$$\leq \int_a^b \varphi_n^2(x) d\mu(x) + \int_a^b x^{2n} |E(1/x) - F_n(1/x)| \cdot |E(1/x) + F(1/x)| d\mu(x)$$

$$\leq \int_a^b \varphi_n^2(x) d\mu(x) + 2S(n) \int_a^b x^{2(n-1)} F'^2(1/x) d\mu(x) .$$

This may be written in the form

$$\int_a^b x^{2n} \left\{ E^2(1/x) - \frac{2S(n)}{x^2} F^2(1/x) \right\} d\mu(x) \leq \int_a^b \varphi_n^2(x) d\mu(x) .$$

Now we can choose n sufficiently large to make the term in the bracket on the left positive; for the interval was chosen so as to give $E^2(1/x)$ a lower bound and the negative term contains the convergent factor $S(n)$. The inequality is then preserved if we replace the factor x^{2n} by a^{2n}. On the right-hand side, we may replace the limits by $-\infty$ and $+\infty$ and use (5) and (7) to estimate the normalization integral. This yields

$$\int_a^b E^2(1/x) d\mu(x) \leq 2S(n) \int_a^b \frac{1}{x^2} F^2(1/x) d\mu(x) + \frac{1}{a^{2n}} \prod_{i=1}^n \lambda_i .$$

Both the right-hand terms tend to zero as n increases while the left-hand side is independent of n. Hence, the integral must vanish identically. This reasoning can be repeated for every interval between the roots of $E(1/x)$. Hence, the contribution to $\dfrac{d\mu}{dx}$ arises entirely from the zeros as stated by the theorem.

THEOREM 4. *The polynomials $\varphi_n(x)$ are orthogonal over a denumerable set of points of the real axis.*

This is a trivial consequence of Theorem 3 and the properties of $E(z)$ enumerated in Theorem 2.

3. Construction of the weight function. We first need to generalize some of the previous definitions.

We will consider the polynomials that satisfy

$$(14) \qquad \varphi_{n+1}^{(s)}(x) = x\varphi_n^{(s)}(x) - \lambda_{n+s}\varphi_{n-1}^{(s)}(x) , \qquad (n \geq 0)$$

where s is a nonnegative integer, where λ_n is restricted as in (3) and (5), and where

$$(15) \qquad \varphi_1^{(s)}(x) = x, \ \varphi_0^{(s)}(x) = 1, \ \varphi_{-1}^{(s)}(x) = 0 .$$

From (9) and (10) we have

$$(16) \qquad F_{n+1}^{(s)}(z) = F_n^{(s)}(z) - \lambda_{n+s} z^2 F_{n-1}^{(s)}(z) , \qquad (n \geq 0),$$

where

$$(17) \qquad F_n^{(s)}(z) = z^n \varphi_n^{(s)}(1/z) .$$

Let us obtain a contiguous function recurrence relation for the $F_n^{(s)}(z)$. From (16), we may write

$$F_{n+1}^{(s)}(z) = F_n^{(s)}(z) - \lambda_{n+s} z^2 F_{n-1}^{(s)}(z) , \qquad (n \geq 0),$$

$$F_n^{(s+1)}(z) = F_{n-1}^{(s+1)}(z) - \lambda_{n+s} z^2 F_{n-2}^{(s+1)}(z) , \qquad (n \geq 1),$$

$$F_{n-1}^{(s+2)}(z) = F_{n-2}^{(s+2)}(z) - \lambda_{n+s} z^2 F_{n-3}^{(s+2)}(z) , \qquad (n \geq 2).$$

Now if we define $\psi_n^{(s)}(z)$ by

$$\psi_{n+1}^{(s)}(z) = A(z, s) F_{n+1}^{(s)}(z) + B(z, s) F_n^{(s+1)}(z) + C(z, s) F_{n-1}^{(s+2)}(z) , \qquad (n \geq 0),$$

where $A(z, s)$, $B(z, s)$, and $C(z, s)$ are arbitrary functions independent of n, we have that

$$\psi_{n+1}^{(s)}(z) = \psi_n^{(s)}(z) - \lambda_{n+s} z^2 \psi_{n-1}^{(s)}(z) , \qquad (n \geq 2).$$

Let us set

$$A(z, s) = 1 ,$$

$$B(z, s) = -1 ,$$

$$C(z, s) = \lambda_{s+1} z^2 .$$

It follows, using (15) and (17), that $\psi_1^{(s)}(z) = 0$ and $\psi_2^{(s)}(z) = 0$. Thus we must have that $\psi_n^{(s)}(z) = 0$ for $n \geq 1$. That is, we have the contiguous function recurrence relation

$$(18) \qquad F_n^{(s)}(z) - F_{n-1}^{(s+1)}(z) + \lambda_{s+1} z^2 F_{n-2}^{(s+2)}(z) = 0 , \qquad (n \geq 1).$$

Now we have that

$$(19) \qquad \lim_{n \to \infty} F_n^{(s)}(z) = E^{(s)}(z) ,$$

where $E^{(s)}(z)$ is an entire function of the sort described in Theorem 2. The recurrence relation for these entire functions

$$(20) \qquad E^{(s)}(z) - E^{(s+1)}(z) + \lambda_{s+1} z^2 E^{(s+2)}(z) = 0$$

follows from (18) and (19).

Now (18) and (20) look very similar. We may exploit this similarity to obtain a relation between the polynomials and the entire functions. If we multiply (18) by $E^{(s+1)}(z)$, multiply (20) by $F_{n-1}^{(s+1)}(z)$, and subtract the resulting expressions, we obtain an expression that may be written

$$[E^{(s+1)}(z) F_n^{(s)}(z) - E^{(s)}(z) F_{n-1}^{(s+1)}(z)]$$

$$= [E^{(s+2)}(z) F_{n-1}^{(s+1)}(z) - E^{(s+1)}(z) F_{n-2}^{(s+2)}(z)] \lambda_{s+1} z^2 .$$

Here, the similarity of the bracketed terms suggests iteration. Iterating it $(n-1)$ times, we obtain

$$[E^{(s+1)}(z)F_n^{(s)}(z) - E^{(s)}(z)F_{n-1}^{(s+1)}(z)]$$

$$= [E^{(s+1+n)}(z)F_0^{(s+n)} - E^{(s+n)}(z)F_{-1}^{(s+1+n)}(z)]z^{2n}\prod_{i=1}^{n}\lambda_{s+i}$$

$$= E^{(s+1+n)}(z)z^{2n}\prod_{i=1}^{n}\lambda_{s+i}\ .$$

From this we may form the expression

(21)
$$z^{-n-p-1}F_n^{(s)}(z)\left(\frac{E^{(s+1)}(z)}{E^{(s)}(z)}\right)$$

$$= z^{-n-p-1}F_{n-1}^{(s+1)}(z) + \left(\frac{E^{(s+n+1)}(z)}{E^{(s)}(z)}\right)z^{n-p-1}\prod_{i=1}^{n}\lambda_{s+i}\ .$$

Let us consider the residues at the origin of the right member of (21). The residue of the first term of this right member is zero for $p \geq 0$, (p integral). The quotient of the entire functions that appears in the second term is unity at the origin. Hence, the residue of this second term is zero for $n-p>0$ and is not zero for $p=n$. Thus we have

(22)
$$\frac{1}{2\pi i}\int_{c'}z^{-n-p-1}F_n^{(s)}(z)\left(\frac{E^{(s+1)}(z)}{E^{(s)}(z)}\right)dz = \begin{cases}\prod_{i=1}^{n}\lambda_{s+i} & \text{for } p=n, \\ 0 & \text{for } p<n,\end{cases}$$

where again the product on the right is taken to be unity when $n=0$, and where the integration is taken counterclockwise around a circle c' that includes the origin but does not include any other singularities of the integrand.

If now in (22) we set $x \to z^{-1}$ and use (17), we have

(23)
$$\frac{1}{2\pi i}\int_c x^p\varphi_n^{(s)}(x)\left(\frac{E^{(s+1)}(1/x)}{xE^{(s)}(1/x)}\right)dx = \begin{cases}\prod_{i=1}^{n}\lambda_{s+i} & \text{for } p=n, \\ 0 & \text{for } p<n,\end{cases}$$

where the integration now is taken counterclockwise around a curve c that includes all of the singularities of the integrand.

It follows from the Sturmian properties of the polynomials of (18) and from the limit (19) that the weight function in the brackets of (23) has positive residues at all its poles. When (23) is now expressed as the sum of the residues of the integrand, we have

(24)
$$\int_{-a(s)}^{+a(s)}x^p\varphi_n^{(s)}(x)d\alpha^{(s)}(x) = \begin{cases}\prod_{i=1}^{n}\lambda_{i+s} & \text{for } n=p, \\ 0 & \text{for } p<n,\end{cases}$$

where $\alpha^{(s)}(x)$ is a monotonic, nondecreasing function whose increase at the point x is equal to the residue of

$$\frac{E^{(s+1)}(1/x)}{xE^{(s)}(1/x)}$$

at the point x. The limits of integration are the bounds of an interval large enough to contain the zeros of $E^{(s)}(1/x)$. Since (12) is bounded away from zero, these integration limits may be taken as finite. Again, when $n=0$, we define the product on the right to be unity.

We have thus proved the following

THEOREM 5. *Let*

$$\text{(25)} \qquad \varphi_{n+1}^{(s)}(x)=x\varphi_n^{(s)}(x)-\lambda_{n+s}\varphi_{n-1}^{(s)}(x)\ , \qquad (n\geqq 0),\ s\geqq 0,$$

where $\qquad \lambda_n>0,\ \sum\limits_{n=1}^{\infty}\lambda_n=B<\infty \qquad$ *and*

$$\text{(26)} \qquad \varphi_1^{(s)}(x)=x,\ \varphi_0^{(s)}(x)=1,\ \varphi_{-1}^{(s)}(x)=0.$$

Let, in addition,

$$\text{(27)} \qquad \lim_{n\to\infty}\frac{\varphi_n^{(s)}(x)}{x^n}=E^{(s)}\!\left(\frac{1}{x}\right).$$

Then

$$\text{(28)} \qquad \int_{-a(s)}^{+a(s)} x^p\varphi_n^{(s)}(x)\,d\alpha^{(s)}(x)=\begin{cases} \prod\limits_{i=1}^{n}\lambda_{i+s} & \text{for } n=p, \\ 0 & \text{for } p<n, \end{cases}$$

where $\alpha^{(s)}(x)$ *is a monotonic non-decreasing function whose increase at the point* x *is equal to the residue of*

$$\frac{E^{(s+1)}(1/x)}{xE^{(s)}(1/x)}$$

at the point x. *The limits of integration are the bounds of an interval large enough to contain the zeros of* $E^{(s)}(1/x)$.

COROLLARY. *Under the same conditions as Theorem 5,*

$$\text{(29)} \qquad \int_{-a(s)}^{a(s)} \varphi_m^{(s)}(x)\varphi_n^{(s)}(x)\,d\alpha^{(s)}(x)=\begin{cases} \prod\limits_{i=1}^{n}\lambda_{i+s} & \text{for } n=m, \\ 0 & \text{for } n\neq m. \end{cases}$$

4. An example. An illustration of our theory is given by a set of modified Lommel polynomials. They may be defined by the recurrence relation

$$L_{n-1}^{(s)}(x) = xL_n^{(s)}(x) - \frac{1}{4(s+n)(s+n+1)}L_{n-1}^{(s)}(x) \qquad n \geq 0, \ s > 0$$

together with

$$L_0^{(s)} = 1, \ L_{-1}^{(s)}(x) = 0 \ .$$

In terms of Watson's [5] definition of the Lommel polynomial $R_{n,\nu}(x)$, we have

$$L_n^{(s)}(x) = \frac{R_{n,s+1}(1/x)s!}{2^n(s+n)!} \ .$$

The special case of the limit (19),

$$\lim_{n \to \infty} x^n L_n^{(s)}(1/x) = \frac{(x/2)^s}{s!}J_s(x)$$

may be found in Watson [5], § 9.65.

The residue of

$$\frac{J_{s+1}(1/x)}{2x^2(s+1)J_s(1/x)}$$

at x_z, a zero of $J_s(1/x)$ is $2(s+1)x_z^2$.

Thus we have that

$$(25) \qquad 2(s+1)\sum x^p L_n^{(s)}(x)x^2 = \begin{cases} \dfrac{s!(s+1)!}{4^n(s+n)!(s+n+1)!} & \text{for } p=n, \\[2mm] 0 & \text{for } p < n, \end{cases}$$

where the summation is taken over all x that are finite zeros of $J_s(1/x)$.

The result (25) has also been obtained using a somewhat different approach [1].

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PACIFIC JOURNAL OF MATHEMATICS

Pacific Journal of Mathematics

Vol. 6, No. 2　　　December, 1956