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**A RELATION BETWEEN PERFECT SEPARABILITY,  
COMPLETENESS, AND NORMALITY IN SEMI-METRIC  
SPACES**

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# A RELATION BETWEEN PERFECT SEPARABILITY, COMPLETENESS, AND NORMALITY IN SEMI-METRIC SPACES

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**1. Introduction.** This paper proves that a regular semi-metric<sup>1</sup> topological space  $S$  may have such properties as hereditary separability, collectionwise normality [1], paracompactness [10], and weak completeness without being either a developable space [1] or a metric space. However, if  $S$  is strongly complete, then hereditary separability implies perfect separability [12] and consequently metrizability. It has been proved [1; 12] that a regular developable topological space (Moore space) is metrizable provided that it is perfectly separable. Thus, a regular semi-metric topological space may be far removed from a Moore space contrary to a result announced by C. W. Vickery [11]. The notion of  $p$ -separability due to Frechet is generalized and a question raised by W. A. Wilson [14, p. 336] is answered in the affirmative. Throughout this paper,  $S$  denotes a regular semi-metric topological space.

## 2. Weak and strong completeness.

**DEFINITION 2.1.** A space  $S$  is said to be  $\left\{ \begin{array}{l} \text{weakly complete} \\ \text{strongly complete} \end{array} \right\}$  provided there exists a distance function  $d$  such that (1) the topology of  $S$  is invariant with respect to  $d$  and (2) if  $\{M_i\}$  is a monotonic decending sequence of closed subsets of  $S$  such that, for each  $i$ , there exists a  $1/i$ -neighborhood of a point  $p_i \left\{ \begin{array}{l} \text{in } M_i \\ \text{in } S \end{array} \right\}$  which contains  $M_i$ , then  $\cap M_i$  contains a point.

It is now shown that strong completeness is sufficient to bridge a gap between a hereditarily separable space  $S$  and a developable space.

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<sup>1</sup> A topological space  $S$  is said to be a semi-metric topological space provided there is a distance function  $d$  defined for  $S$  such that (1) if each of the letters  $x$  and  $y$  denotes a point of  $S$ , then  $d(x, y) = d(y, x)$  denotes a non-negative number, (2)  $d(x, y) = 0$  if and only if  $x = y$ , and (3) the topology of  $S$  is invariant with respect to the distance function  $d$ , that is, if  $p$  is a limit point of a subset  $M$  of  $S$ , then  $p$  is a distance limit point of  $M$  and conversely. As usual,  $S$  is said to be regular provided that if  $R$  is an open set containing a point  $p$  of  $S$ , then there exists an open set  $D$  such that  $R \supset \bar{D} \supset p$ . A topological space  $(T_1)$  is defined as in [9].

**THEOREM 2.2.** *Every hereditarily separable and strongly complete space  $S$  is perfectly separable.*

*Proof.* Let  $d$  denote a semi-metric for the space  $S$ . For each pair of natural numbers  $h$  and  $k$ , let  $M_{hk}$  denote the set of all points  $p$  such that for some open set  $R$ , the spherical neighborhoods  $U_{1/h}(p)$  and  $U_{1/k}(p)$  satisfy  $U_{1/h}(p) \supset \bar{R} \supset R \supset U_{1/k}(p)$ . It should be noted that the spherical neighborhoods defined by  $d$  may fail to be open sets. Since  $S$  is hereditarily separable, there exists a countable dense subset  $N_{hk}$  of  $M_{hk}$ . Let  $G_{hk}$  denote a countable collection of open sets such that for each point  $p$  in  $N_{hk}$ , there exists an open set  $R$  in  $G_{hk}$  such that  $U_{1/h}(p) \supset \bar{R} \supset R \supset U_{1/k}(p)$ . Clearly,  $G_{hk}$  covers  $M_{hk}$ . Furthermore, each point of  $S$  lies in  $M_{hk}$  for some  $h$  and  $k$ .

Let  $G$  denote a countable collection of open sets covering  $S$  such that (1) the intersection of two elements of  $G$  is an element of  $G$  and (2) if  $Q$  is an element of  $G_{hk}$  for some  $h$  and  $k$ , then  $Q \in G$ . The collection  $G$  is a basis for  $S$ . For, suppose that there exists an open set  $R$  containing a point  $p$  such that there exists no element of  $G$  that contains  $p$  and lies in  $R$ . Then, for each  $i$ , there exists an integer  $k_i$  and an element  $R_i$  of  $G_{ik_i}$  which contains  $p$  such that  $\prod_{j=1}^i R_j$  fails to lie in  $R$ . Now, there exists a point  $p_i$  such that  $U_{1/i}(p_i) \supset \bar{R}_i \supset \prod_{j=1}^i \bar{R}_j \cdot (S - R) = M_i$ . Since  $M$  is strongly complete,  $\cap M_i$  contains a point  $q \neq p$ . Thus,  $d(p_i, q) < 1/i$  and  $d(p_i, p) < 1/i$  for each  $i$ . This is impossible. Hence,  $S$  is perfectly separable.

It is an interesting fact that Cauchy completeness, when defined in a natural way for a space  $S$  (see [9] and footnote 2), is equivalent to weak completeness in  $S$ .

**THEOREM 2.3.** *A necessary and sufficient condition<sup>2</sup> that a semi-metric space  $S$  be weakly complete is that every Cauchy sequence<sup>3</sup> of points of  $S$  have a limit point in  $S$ .*

*Proof.* The condition is necessary. Suppose that there exists a Cauchy sequence  $\{p_i\}$  of points of  $S$  which has no limit point in  $S$ . Thus, there exists a subsequence  $\{p_{n_i}\}$  of  $\{p_i\}$  such that for each  $i$ ,

<sup>2</sup> This theorem was proved independently by my classmate Wyman Richardson in one of F. B. Jones' classes.

<sup>3</sup> A Cauchy sequence  $\{p_i\}$  of points is said to have a limit point  $p$  provided that there exists a subsequence  $\{p_{n_i}\}$  of  $\{p_i\}$  which converges to  $p$ . There exists a Cauchy sequence of points in a space  $S$  which has a limit point but which has no sequential limit point.

$U_{1/i}(p_{n_i}) \supset p_{n_j}$  for  $j \geq i$ . Let  $M_i = \sum_{j=i}^{\infty} p_{n_j}$ . Since  $\cap M_i = \emptyset$ , there is a contradiction to the hypothesis that  $S$  is weakly complete.

The condition is sufficient. Suppose that  $\{M_i\}$  denotes a monotonic descending sequence of closed subsets of  $S$  such that for each  $i$ , there exists a point  $p_i$  such that  $p_i \in M_i \subset U_{1/i}(p_i)$ . Since  $\{p_i\}$  is a Cauchy sequence, the set  $\cap M_i$  contains a point  $p$ .

**3. Non-equivalence of regular semi-metric topological spaces and regular developable (Moore) spaces.** Many theorems which are true for Moore spaces have analogues which hold for regular semi-metric topological spaces<sup>4</sup>. However, the fact that a regular semi-metric topological space  $S$  is far removed from a Moore space is stressed by the following examples and theorems. From these, it follows that the condition of either separability or screenability for the metrization of a normal Moore space due to Jones [5] and Bing [1], respectively, has no analogue which holds in a normal space  $S$ .

Consider the following example of a regular semi-metric space which is not a Moore space. Some additional properties of this space are given in Theorem 3.2.

**EXAMPLE 3.1.** Let  $X$  denote the  $x$ -axis of the Cartesian plane  $E^2$ . A semi-metric  $D(p, q)$  will be defined for  $E^2$  in the following way. Suppose that each of the letters  $p$  and  $q$  denotes a point of  $E^2$ . If  $X$  contains both or neither of the points  $p$  and  $q$ , then define  $D(p, q)$  to be the Cartesian distance  $d(p, q)$ . If  $p \in X$  and  $q \notin X$ , then define  $D(p, q)$  to be  $d(p, q) + \alpha$  where  $\alpha$  is a non-obtuse angle (measured in radians) between  $X$  and the line  $L$  determined by  $p$  and  $q$ . If  $p \notin X$  and  $q \in X$ , define  $D(p, q)$  to be  $D(q, p)$ . Clearly,  $D$  is a semi-metric for  $E^2$ . For each positive integer  $n$  and each point  $p$ ,  $U_{1/n}(p)$  is defined to be an open set provided that either  $p \in X$  or  $U_{1/n}(p)$  lies in one of the two components of  $E^2 - X$ . Considering the open sets defined in this way as the elements of basis for a topology,  $E^2$  becomes a regular connected and locally connected semi-metric topological space  $S$  which is not a Moore space. It should be noted that  $S$  is hereditarily separable since it is the sum of two hereditarily separable sets  $S - X$  and  $X$ .

**THEOREM 3.2.** *There exists a connected and locally connected regular semi-metric topological space  $S$  which is hereditarily separable, weakly complete, strongly screenable [1], collectionwise normal, completely normal, and paracompact but which is neither perfectly separable nor a Moore space nor metrizable.*

<sup>4</sup> This is included in unpublished work of F. B. Jones.

*Proof.* Let  $S$  be the space  $E^2$  with the topology defined in Example 3.1. The space  $S$  is not metrizable since it is not a Moore space.

Suppose that  $S$  is perfectly separable. Then there exists a countable collection  $H$  of spherical neighborhoods in  $S$  that defines the topology of  $S$ . For each number  $\epsilon > 0$  and each point  $p$  in  $X$ ,  $U_\epsilon(p)$  contains an element  $h(\epsilon, p)$  of  $H$ . By the definition of  $U_\epsilon(p)$ , it follows that the center of the spherical neighborhood  $h(\epsilon, p)$  is  $p$ . This is impossible since  $H$  is countable and  $X$  is uncountable.

In order to show that  $S$  is weakly complete, a distance function  $E$  different from that given in Example 3.1 will be introduced. Let  $L_1$  and  $L_2$  be two distinct lines parallel to and at a unit distance from  $X$ , and denote by  $C(X)$  the component of  $S - (L_1 + L_2)$  that contains  $X$ . For any pair of points  $p$  and  $q$  of  $C(X) - X$ , define  $E(p, q)$  to be  $d(p, q) / d(X, p)d(X, q)$ , where  $d$  is the ordinary Cartesian distance function. If either of two points  $p$  and  $q$  fails to lie in  $C(X) - X$ , then define  $E(p, q)$  to be  $D(p, q)$  as given in Example 3.1. It follows that the topology of  $S$  is unchanged by  $E$ . In the remainder of this paragraph, the spherical neighborhoods considered will be those defined by  $E$ . Now, suppose that  $\{M_i\}$  is a monotonic descending sequence of closed point sets and  $\{p_i\}$  is a sequence of points such that for each  $i$ ,  $p_i \in M_i \subset U_{1/i}(p_i)$ . If there exists a subscript  $n$  such that  $X \cdot M_n = 0$ , then  $X \cdot M_i = 0$  for  $i > n$ . From this it follows that there exists  $m > n$  such that  $U_{1/m}(p_m) \cdot X = 0$ . For, suppose that this is not the case. Then there must exist a subsequence  $\{p_{n_i}\}$  of  $\{p_i\}$  such that  $\{d(X, p_{n_i})\}$  converges to 0. Consequently, by the definition of  $E$ , the sequence  $\{E(p_n, p_{n_i})\}$  of real numbers is unbounded. This is contrary to the assumption that  $U_{1/n}(p_n) \supset M_n$ . Thus, the existence of the required integer  $m$  is established. It follows that  $HM_i \neq 0$  in this case. For the remaining case, suppose that for each  $i$ ,  $X \cdot M_i \neq 0$ . Since  $\{X \cdot M_i\}$  is a bounded monotonic descending sequence of non-empty closed subsets of  $X$ , it follows that  $HM_i \neq 0$ . Hence,  $S$  is weakly complete.

The space  $S$  is strongly screenable. Consider the metric subspaces  $S - X$  and  $X$  of  $S$ . These are strongly screenable by theorems due to Bing [1]. Let  $G$  denote an open covering of  $S$ . Denote by  $H$  and  $K$  open coverings of  $X$  and  $S - X$ , respectively, such that for  $g$  in  $G$ ,  $g \cdot X \in H$  and  $g \cdot (S - X) \in K$ . There exists a sequence  $\{H_i\}$  of discrete collections [1] of open intervals of  $X$  such that  $\Sigma H_i$  covers  $X$  and for each  $i$ ,  $H_i$  is a refinement of  $H$ . Let  $I$  denote an interval in  $H_i$  for some  $i$ . Since  $\bar{I}$  contains no point of the closure of  $(H_i - I)^*$  [the logical sum of the elements of  $H_i - I$ ] and  $I$  lies in some element  $g$  of  $G$ , it follows that there exist discrete collections  $P$  and  $Q$  of  $1/n$ -neighborhoods of points in  $X$  such that (1) each element of  $P$  and each element of  $Q$  lies in  $g$ , (2) the closure of no element of either  $P$  or  $Q$  intersects the closure of

$(H_i - I)^*$ , and (3)  $P + Q$  covers  $I$ . It follows that there exists a sequence  $\{X_i\}$  of discrete collections each of which is a refinement of  $G$  and such that  $\Sigma X_i \supset X$ . Similarly, there exists a sequence  $\{K_i\}$  of discrete collections each of which is a refinement of  $K$  and such that  $\Sigma K_i \supset S - X$ . For each natural number  $i$ , let  $G_{2i} = X_i$  and  $G_{2i-1} = K_i$ . Thus,  $\{G_i\}$  is a sequence of discrete collections of open subsets of  $S$  such that  $\Sigma G_i$  covers  $S$  and  $G_i$  refines  $G$  for each  $i$ . Hence,  $S$  is strongly screenable.

Now  $S$ , being a regular strongly screenable topological space, is collectionwise normal [1]. It also follows that  $S$  is paracompact by a theorem due to Ernest Michael [6].

To complete the proof of theorem 3.2, it must be shown that  $S$  is completely normal. It has been proved by F. B. Jones [5] that every normal Moore space is completely normal<sup>5</sup>. A simple modification of his argument shows that every normal semi-metric topological space is completely normal. This completes the proof.

Mary E. Estill [3] has considered complete Moore spaces in any one of three definitions of completeness. Perhaps intuition would lead one to suspect that a complete Moore space, in one of these senses, would be strongly complete. The following example and theorem shows that this is not the case. As a matter of fact, in a Moore space, the concept of strong completeness is more restrictive than that of completeness.

**EXAMPLE 3.3.** Let  $X$  denote the  $x$ -axis of the Cartesian plane  $E^2$ . A semi-metric  $D(p, q)$  will be defined for  $E^2$  in the following way. Suppose that  $p$  and  $q$  are two distinct points of  $E^2$ . If neither  $p$  nor  $q$  lies in  $X$ , then define  $D(p, q)$  to be  $d(p, q)$  where  $d$  is the ordinary Cartesian metric. If  $p \in X$ , then let  $D(p, q) = d(p, q) + \alpha$  where  $\alpha$  is an angle (measured in radians) between a line  $L_1$  containing  $p + q$  and a vertical line  $L_2$  containing  $p$  such that  $0 \leq \alpha \leq \pi/2$ . If  $D(q, p)$  is not defined above, then let  $D(q, p) = D(p, q)$ . For  $p$  in  $X$ , let  $D(p, p) = 0$ . Clearly,  $D$  is a semi-metric for  $E^2$ . For each point  $p$  in  $E^2$  and each natural number  $n$ ,  $U_{1/n}(p)$  is defined to be an open set. With this definition of open sets,  $E^2$  becomes a regular connected and locally connected semi-metric topological space  $S$ . It should be noted that  $S$  is separable but not hereditarily separable.

**THEOREM 3.4.** *There exists a complete Moore space  $S$  which is not strongly complete.*

*Proof.* Let  $S$  be the space  $E^2$  with the topology defined in Example

<sup>5</sup> A space  $S$  is said to be completely normal provided that for two mutually separate subsets  $H$  and  $K$  of  $S$  there exists mutually exclusive open coverings of  $H$  and  $K$ . See [5].

3.3. It will first be shown that  $S$  is not strongly complete.

Suppose that  $S$  is strongly complete. Then there exists a semi-metric  $E$  defined for  $S$  such that (1) the topology of  $S$  is unchanged by  $E$  and (2) if  $\{M_i\}$  is a monotonic descending sequence of closed subsets of  $S$  such that for each  $i$  and some point  $p_i$  in  $S$ ,  $U_{1/i}(p_i) \supset M_i$ , then  $\Pi M_i \neq 0$ . It should be noted that the spherical neighborhoods defined by  $E$  may fail to be open sets.

Consider an interval  $A$  of  $X$ . For each pair of natural numbers  $h$  and  $k$ , let  $M_{hk}$  denote the subset of  $A$  of all points  $p$  such that for some open set  $R$ ,  $U_{1/h}(p) \supset R \supset U_{1/k}(p)$ . For some natural number  $h_1$ , the set  $M_{1h_1}$  is uncountable. Now,  $M_{1h_1}$  contains an uncountable subset  $N_{1h_1}$  such that

- (1) there exists a line  $L_1$  parallel to  $X$  where  $d(L_1, X) \leq 1$  and
- (2) for each point  $p$  in  $N_{1h_1}$ , there exists an open set  $R(p)$  where  $U_{1/h_1}(p) \supset R(p) \supset U_{1/h_1}(p)$  such that  $R(p)$  contains an interval  $I$  of  $L_1$  whose length (in the Cartesian sense) is greater than a positive number  $e_1$  and which has as its center a point  $q$  whose projection on  $X$  is  $p$ .

Now there exists an integer  $h_2 > h_1$  such that  $N_{1h_1}$  contains an uncountable subset  $N_{2h_2}$  such that

- (1) there exists a line  $L_2$  parallel to  $X$  where  $d(L_2, X) \leq 1/2$  and
- (2) for each point  $p$  in  $N_{2h_2}$ , there exists an open set  $R(p)$  where  $U_{1/2}(p) \supset R(p) \supset U_{1/h_2}(p)$  such that  $R(p)$  contains an interval  $I$  of  $L_2$  whose length is greater than a positive number  $e_2$  and which has as its center a point  $q$  whose projection on  $X$  is  $p$ .

It follows that there exists a monotonic descending sequence  $\{N_{ih_i}\}$  of subsets of  $A$  and a sequence  $\{L_i\}$  of lines parallel to  $X$  and converging to it such that for each  $i$ , if  $p_i$  is a point of  $N_{ih_i}$ , there exists an open set  $R(p_i)$  where  $U_{1/i}(p_i) \supset R(p_i) \supset U_{1/h_i}(p_i)$  such that  $R(p_i)$  contains an interval  $I_i$  of  $L_i$  whose length is greater than a positive number  $e_i$  and which has as its center a point  $q_i$  whose projection on  $X$  is  $p_i$ . Since  $A$  is a compact subset of  $E^2$  there exists a monotone sequence  $\{p_i\}$  of points converging to a point  $p$  in  $A$  such that for each  $i$ ,  $p_i \in N_{ih_i}$ . Let  $L$  be a vertical line containing  $p$ , and for each  $i$ , define  $x_i = L \cdot L_i$ . It follows that there exists a monotonic increasing sequence  $\{k_i\}$  of natural numbers such that for each  $i$ ,  $E(x_i, p_j) < 1/i$  for all  $j > k_i$ . The set  $M_i = \sum_{k=k_i}^{\infty} p_k$  is closed in  $S$  for each  $i$  and  $U_{1/i}(x_i) \supset M_i$ . It follows that  $\Pi M_i = 0$ .

This is contrary to the assumption that  $S$  is strongly complete.

It now remains to be shown that  $S$  is a complete Moore space. For a point  $p$  in  $X$ , there exists a sequence  $\{R_i\}$  of open sets closing down<sup>6</sup>

<sup>6</sup> A sequence of open sets  $\{R_i\}$  is said to close down on a point  $p$  if for each  $i$ ,  $R_i \supset \bar{R}_{i+1}$  and  $\Pi R_i = p$ .

on  $p$ . On the other hand, if  $p$  denotes a point of  $S-X$ , there exists a sequence  $\{R_i\}$  of open sets closing down on  $p$  such that for each  $i$ ,  $R_i \cdot X = 0$ . With each point  $p$  of  $S$ , associate exactly one such sequence  $\{R_i\}$ . For each  $i$ , let  $G_i$  denote the collection of all open sets  $R$  such that for some point  $p$  of  $S$ ,  $R$  is the  $j$ th member of the sequence associated with  $p$ , and  $j \geq i$ . It follows that  $S$  is a complete Moore space.

**4. A question due to W. A. Wilson.** An affirmative answer is given in this section to a question raised by Wilson [14, p. 366] in 1931. The following axioms and definitions [14] are listed for convenience.

A set  $Z$  is said to be a (Menger) semi-metric space provided that corresponding to each pair of points  $(a, b)$  of  $Z$ , there is a non-negative real number  $d(a, b)$  satisfying the following axioms:

Axiom I.  $d(a, b) = d(b, a)$ .

Axiom II.  $d(a, b) = 0$  if and only if  $a = b$ .

Wilson has introduced the following additional axiom:

Axiom  $W$ . For each point  $a$  and each positive number  $k$ , there is a positive number  $r$  such that if  $b$  is a point for which  $d(a, b) \geq k$  and  $c$  is any point, then  $d(a, c) + d(b, c) \geq r$ .

Now, let  $r = f(a, k)$  denote the largest  $r$  such that  $d(a, c) + d(b, c) \geq r$  in Axiom  $W$ . For each point  $a$  and each positive number  $k$ , let  $r = f(a, k)$ ,  $r_1 = f(a, r)$ , and  $r_2$  denote a positive number such that  $r_2 < r_1$ . Wilson calls the set  $\sigma$  of points  $x$  such that  $d(a, x) < r_2$  an inner sphere, with center  $a$ , corresponding to  $a$  and  $k$ .

**THEOREM 4.1.** *Suppose that  $Z$  denotes a separable semi-metric space satisfying Axiom  $W$ . If  $d$  denotes a distance function defined for  $Z$  which leaves limit points invariant, then there exists a countable dense subset  $E = \sum p_i$  of  $Z$  such that for any positive number  $k$ , each point  $p$  of  $Z$  lies in an inner sphere  $\sigma$  corresponding to  $p_i$  and  $k$  for some natural number  $i$ .*

*Proof.* By a corollary due to Wilson [14],  $Z$  is homeomorphic to a metric space. Since a separable metric space is hereditarily separable, it follows that  $Z$  is hereditarily separable.

Let  $S_\epsilon(p)$  denote a spherical neighborhood in  $Z$ . For each pair of natural numbers  $h$  and  $k$ , let  $M_{hk}$  denote the set of all points  $p$  such that there exists an inner sphere  $\sigma$  corresponding to  $1/h$  and  $p$  such that  $S_{1/h}(p) \supset \sigma \supset S_{1/k}(p)$ . Since  $Z$  is hereditarily separable,  $M_{hk}$  contains a countable dense subset  $N_{hk}$ . Let  $K_{hk}$  be a countable collection of inner spheres such that if  $p \in N_{hk}$ , then there exists an inner sphere  $\sigma$  in  $K_{hk}$  corresponding to  $1/h$  and  $p$  such that  $S_{1/h}(p) \supset \sigma \supset S_{1/k}(p)$ . It



follows that  $K_{h,k}$  covers  $M_{h,k}$ . Denote by  $E = \sum p_i$ , the countable dense subset  $\sum_{h,k=1}^{\infty} N_{h,k}$  of  $Z$ .

The set  $E$  satisfies the conclusion of Theorem 4.1. For, if  $c$  is any positive number, there exists a positive integer  $h$  such that  $1/h < c$ . Also, for  $p$  in  $Z$ , there exists  $k$  such that  $p \in M_{h,k}$ . Since  $K_{h,k}$  covers  $N_{h,k}$ , there exists an inner sphere  $\sigma$  corresponding to  $p_i$  and  $1/h$  for some  $i$  such that  $\sigma \supset p$ . Hence, the inner sphere  $\sigma_i$  which corresponds to  $p_i$  and  $c$  contains  $\sigma$  and  $p$ .

Now Wilson's question referred to above is answered.

**5. Generalized Frechet  $p$ -separability.** The following definition is a natural generalization of the notion of  $p$ -separability [4]. It is proved that in a space  $S$  this notion is equivalent to hereditary separability.

**DEFINITION 5.1.** A regular semi-metric topological space  $S$  (or semi-metric space  $Z$ ) is said to be  $p$ -separable provided that

(1) given any distance function  $d$  which leaves limit points invariant and

(2) given any collection  $H$  of subsets of  $S$  which has the property that for each number  $k > 0$  and each point  $p$  of  $S$ , there exists  $h$  in  $H$  such that  $U_k(p) \supset h \supset U_c(p)$  for some positive number  $e$ ,

then there exists a countable dense subset  $E = \sum p_i$  such that for each positive number  $f$ , each point  $p$  of  $S$  lies in an element  $h$  of  $H$  such that  $U_f(p_i) \supset h \supset p_i$  for some  $i$ .

The following theorem may be proved in a manner analogous to that used in the proof of Theorem 4.1.

**THEOREM 5.2.** *Every hereditarily separable semi-metric space  $Z$  is  $p$ -separable.*

**THEOREM 5.3.** *A necessary and sufficient condition that a regular semi-metric topological space  $S$  be hereditarily separable is that  $S$  be  $p$ -separable.*

*Proof.* The necessity of the condition follows from Theorem 5.2.

It will now be shown that the condition is sufficient. Suppose that  $d$  denotes a semi-metric for  $S$ , and that  $S$  is not hereditarily separable. Then  $S$  contains an uncountable subset  $N$  which has no limit point in  $S$ . Now, consider a semi-metric  $D$  defined in the following way. For each  $i$ , let  $D_i$  denote the set of all points  $x$  of  $S$  such that for some point  $p$  in  $N$ ,  $x$  lies in an open set  $R \subset u_{1/i}(p)$  where  $u_{1/i}(p)$  is a spherical neighborhood defined by  $d$ . Thus,  $\{D_i\}$  is a monotonic descending sequ-

ence of open sets such that  $\cup D_i = N$ . For each  $i$  and each point  $p$  in  $D_i - N$ , associate exactly one open set  $R_i(p)$  containing  $p$  and lying in  $D_i$  such that for some number  $e$ ,  $u_e(p) \supset \bar{R}(p)$  and  $u_e(p) \cdot N = 0$ . If  $x$  and  $y$  denote points of  $S - N$  such that for some  $i$ ,  $D_i \supset x + y$  and  $D_{i+1} \not\supset x + y$ , then define  $D(x, y)$  to be  $i$  provided that  $R_i(x) \not\supset y$  and  $R_i(y) \not\supset x$ . For points  $x$  and  $y$  of  $S$  for which  $D(x, y)$  is not defined above, let  $D(x, y) = d(x, y)$ . It follows that limit points are invariant with respect to  $D$ .

Next, let  $H$  denote a collection of open sets such that for each natural number  $i$  and each point  $p$  in  $S$ , there exists  $h$  in  $H$  such that  $U_{1/i}(p) \supset h \supset p$  where  $U_{1/i}(p)$  is a spherical neighborhood defined by  $D$ . Since  $S$  is  $p$ -separable, there exists a countable dense subset  $E = \sum p_i$  of  $S$  such that for each positive number  $f$ , each point  $p$  of  $S$  lies in an element  $h(p)$  of  $H$  such that for some  $i$ ,  $U_f(p_i) \supset h(p)$ . There exists an uncountable subset  $M$  of  $N - E$  and a natural number  $t$  such that if  $x$  is a point and  $D(x, M) < 1/t$ , then  $x$  lies in  $D_t$ . Let  $p \in M$ . Then there exist

- (1) a number  $e > 0$  such that  $D(p, N - p) = d(p, N - p) > e$ ,
- (2) a positive integer  $n$  such that  $1/n < \text{smaller } [e, 1/t]$ ,
- (3)  $h$  in  $H$ ,
- (4) an integer  $i$  such that  $U_{1/n}(p_i) \supset h \supset p$  [thus,  $p_i \in D_i$ ],
- (5) an integer  $m \geq t$  such that  $p_i \in D_m - D_{m+1}$ ,
- (6) an open set  $R_m(p_i)$  associated with  $p_i$  and  $D_m$  such that for

some number  $c$ ,  $u_c(p_i) \supset \bar{R}_m(p_i)$  and  $u_c(p_i) \cdot N = 0$ ,

- (7) a positive number  $z$  such that for  $q$  in  $S - R_m(p_i)$ ,  $d(p_i, q) > z$ ,
- (8)  $x \in h \cdot D_m - [R_m(p) + N]$  such that  $D(p, x) < z$ , and
- (9) an open set  $R_m(x)$  associated with  $x$  and  $D_m$  such that for some

number  $b$ ,  $u_b(x) \supset \bar{R}_m(x)$  and  $u_b(x) \cdot N = 0$ .

Therefore,  $b < z$ . Consequently,  $R_m(x) \not\supset p_i$ . By definition,  $D(x, p_i) = m > 1/n$ . This is impossible since  $U_{1/n}(p_i) \supset h \supset p + x$ . Hence,  $S$  is hereditarily separable.

It follows from Theorem 3.2 that  $S$  may fail to be either perfectly separable or a metric space.

**6. Conditions for semi-metric, regular developable (Moore), and metric spaces.** Consider the following three conditions on a topological space  $T$ .

A. There exists a sequence  $\{H_i\}$  such that (a) for each  $i$ ,  $H_i$  is a collection of open subsets of  $T$ , (b) if  $p$  is a point and  $R$  is an open set containing  $p$ , then there exists an integer  $n$  such that  $H_n$  contains exactly one element  $g(p)$  associated with  $p$  such that  $R \supset g(p) \supset p$  and

(c) if  $n$  is an integer and  $\{g_i(p_i)\}$  is a sequence such that for each  $i$ ,  $g_i(p_i)$  belongs to  $H_n$  and is associated with  $p_i$ , then  $\Sigma p_i$  has no limit point in  $T - \Sigma g_i(p_i)$ .

B. If  $p$  is a point and  $R$  is an open set containing  $p$ , then there exists an integer  $n$  such that for  $m > n$ , each element  $g$  of  $H_m$  which contains  $p$  has the property that  $R \supset \bar{g}$ .

C. For each  $i$ , the sum of the closures of any subcollection of  $H_i$  is closed.

**THEOREM 6.1.** *A necessary and sufficient condition that a topological space  $T$  be semi-metric is that  $T$  satisfy Condition A.*

*Proof.* It will first be shown that the condition is sufficient. It follows from Condition A that  $T$  satisfies the first axiom of countability. Consider a semi-metric  $d$  defined as follows. For two distinct points  $p$  and  $q$  of  $T$ , denote by  $i$  the least integer such that  $H_i$  contains an element  $g(p)$  associated with  $p$  but not containing  $q$ . Similarly, let  $j$  denote the least integer such that  $H_j$  contains an element  $g(q)$  associated with  $q$  but not containing  $p$ . Define  $d(p, q)$  to be  $1/\min(i, j)$ . For each point  $p$ , define  $d(p, p)$  to be 0.

Limit points are invariant with respect to  $d$ . For suppose that  $p$  is a limit point (defined by the open sets of  $T$ ) of a subset  $M$  of  $T$  and that  $p$  is not a distance limit point of  $M$ . Then there exists a sequence  $\{p_i\}$  of points of  $M - p$  which converges to  $p$  such that for some integer  $n$  and each  $i$ ,  $d(p, p_i) > 1/n$ . Thus, there exists an integer  $m$ , such that, for infinitely many integers  $i$ , either (1)  $H_m$  contains  $g_m(p)$  and  $g_m(p) \not\supset p_i$  or (2)  $H_m$  contains  $g_m(p_i)$  and  $g_m(p_i) \not\supset p$ . Since  $\{p_i\} \rightarrow p$ , (1) is impossible. By Condition A, (2) is impossible. Hence,  $p$  is a distance limit point of  $M$ . It also follows easily that a distance limit point of a subset  $M$  of  $T$  is an open set limit point of  $M$ . This completes the proof of the sufficiency.

The condition is necessary. For each point  $p$  and each pair of natural numbers  $h$  and  $k$ , let  $R_{hk}(p)$  denote an open set when it exists, such that  $U_{1/h}(p) \supset R_{hk}(p) \supset U_{1/k}(p)$ . With  $h$ ,  $k$ , and  $p$  associate exactly one such open set, and let  $G_{hk}$  denote the corresponding collection of open sets for each point  $p$  in  $T$ . There exists a sequence  $\{H_i\}$  such that there is a one to one correspondence between the elements of  $\{H_i\}$  and the elements of  $\{G_{nm}\}$ . It follows that  $\{H_i\}$  satisfies Condition A.

As Example 3.1 illustrates, a regular semi-metric topological space may fail to be a Moore space.

**THEOREM 6.2.** *A necessary and sufficient condition that a topological space  $T$  be a Moore space is that  $T$  satisfy Conditions A and B.*

*Proof.* The condition is sufficient. For each positive integer  $i$ , let  $G_i = \sum_{j=i}^{\infty} H_j$ . If the word "region" is interpreted as "open set," then it follows that Axioms 0 and 1 (1)-(3) due to Moore [7] are satisfied.

The condition is necessary. It will be shown first that  $T$  is a semi-metric topological space. Let  $p$  and  $q$  be distinct points of  $T$ . Denote by  $n$  the least positive integer such that if  $g(p)$  and  $g(q)$  are regions in  $G_n$  containing  $p$  and  $q$ , respectively, then  $g(p) \cdot g(q) = 0$ . Note that  $\{G_i\}$  is given by Axiom 1 of [7]. Consequently, define  $d(p, q)$  to  $1/n$ . It follows that  $d$  is a semi-metric distance function and that limit points are invariant with respect to  $d$ . By Theorem 6.1,  $T$  satisfies Condition A.

Now, define  $\{H_i\}$  in a manner described in the proof of Theorem 6.1 with the additional requirement that  $R_{n,k}(p)$  lie in a region of  $G_n$ . It follows that  $\{H_i\}$  satisfies Conditions A and B.

**THEOREM 6.3.** *A necessary and sufficient condition that a topological space  $T$  be metric is that it satisfy Conditions A, B, and C.*

A proof of Theorem 6.3 follows by use of Bing's Theorem 4 of [1] and Theorem 6.1 above.

*Question.* Is it possible to partition either Bing's Theorem 4 of [1] or Moore's metrization theorem [8; 13], stated below, into three or more parts which begins with a condition for a topological space and which ends with a condition for a metrizable space, but with necessary and sufficient conditions somewhere between these extremes for semi-metric spaces and Moore spaces?

**THEOREM (Moore)**<sup>7</sup>. *A necessary and sufficient condition that a space  $S$  satisfying Axiom 0 of [7] be metrizable is that there exist a sequence  $\{K_i\}$  such that (1) for each natural number  $n$ ,  $K_n$  is a collection of regions in  $S$  covering  $S$  and (2) if  $p$  is a point,  $q$  is a point distinct from  $p$ , and  $R$  is a region containing  $p$ , then there exists a natural number  $n$  such that if each of the letters  $h$  and  $k$  denotes an element of  $K_n$ ,  $g \supset p$ , and  $g \cdot h \neq 0$ , then  $R - q \supset h$ .*

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<sup>7</sup> The terms "point" and "region" are undefined. Axiom 0 states that every region is a point set.

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