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**A RELATION BETWEEN PERFECT SEPARABILITY,
COMPLETENESS, AND NORMALITY IN SEMI-METRIC
SPACES**

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1. Introduction. This paper proves that a regular semi-metric¹ topological space S may have such properties as hereditary separability, collectionwise normality [1], paracompactness [10], and weak completeness without being either a developable space [1] or a metric space. However, if S is strongly complete, then hereditary separability implies perfect separability [12] and consequently metrizability. It has been proved [1; 12] that a regular developable topological space (Moore space) is metrizable provided that it is perfectly separable. Thus, a regular semi-metric topological space may be far removed from a Moore space contrary to a result announced by C. W. Vickery [11]. The notion of p -separability due to Frechet is generalized and a question raised by W. A. Wilson [14, p. 336] is answered in the affirmative. Throughout this paper, S denotes a regular semi-metric topological space.

2. Weak and strong completeness.

DEFINITION 2.1. A space S is said to be $\left\{ \begin{array}{l} \text{weakly complete} \\ \text{strongly complete} \end{array} \right\}$ provided there exists a distance function d such that (1) the topology of S is invariant with respect to d and (2) if $\{M_i\}$ is a monotonic decending sequence of closed subsets of S such that, for each i , there exists a $1/i$ -neighborhood of a point p_i $\left\{ \begin{array}{l} \text{in } M_i \\ \text{in } S \end{array} \right\}$ which contains M_i , then $\cap M_i$ contains a point.

It is now shown that strong completeness is sufficient to bridge a gap between a hereditarily separable space S and a developable space.

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¹ A topological space S is said to be a semi-metric topological space provided there is a distance function d defined for S such that (1) if each of the letters x and y denotes a point of S , then $d(x, y) = d(y, x)$ denotes a non-negative number, (2) $d(x, y) = 0$ if and only if $x = y$, and (3) the topology of S is invariant with respect to the distance function d , that is, if p is a limit point of a subset M of S , then p is a distance limit point of M and conversely. As usual, S is said to be regular provided that if R is an open set containing a point p of S , then there exists an open set D such that $R \supset \bar{D} \supset p$. A topological space (T_1) is defined as in [9].

THEOREM 2.2. *Every hereditarily separable and strongly complete space S is perfectly separable.*

Proof. Let d denote a semi-metric for the space S . For each pair of natural numbers h and k , let M_{hk} denote the set of all points p such that for some open set R , the spherical neighborhoods $U_{1/h}(p)$ and $U_{1/k}(p)$ satisfy $U_{1/h}(p) \supset \bar{R} \supset R \supset U_{1/k}(p)$. It should be noted that the spherical neighborhoods defined by d may fail to be open sets. Since S is hereditarily separable, there exists a countable dense subset N_{hk} of M_{hk} . Let G_{hk} denote a countable collection of open sets such that for each point p in N_{hk} , there exists an open set R in G_{hk} such that $U_{1/h}(p) \supset \bar{R} \supset R \supset U_{1/k}(p)$. Clearly, G_{hk} covers M_{hk} . Furthermore, each point of S lies in M_{hk} for some h and k .

Let G denote a countable collection of open sets covering S such that (1) the intersection of two elements of G is an element of G and (2) if Q is an element of G_{hk} for some h and k , then $Q \in G$. The collection G is a basis for S . For, suppose that there exists an open set R containing a point p such that there exists no element of G that contains p and lies in R . Then, for each i , there exists an integer k_i and an element R_i of G_{ik_i} which contains p such that $\prod_{j=1}^i R_j$ fails to lie in R . Now, there exists a point p_i such that $U_{1/i}(p_i) \supset \bar{R}_i \supset \prod_{j=1}^i \bar{R}_j \cdot (S - R) = M_i$. Since M is strongly complete, ΠM_i contains a point $q \neq p$. Thus, $d(p_i, q) < 1/i$ and $d(p_i, p) < 1/i$ for each i . This is impossible. Hence, S is perfectly separable.

It is an interesting fact that Cauchy completeness, when defined in a natural way for a space S (see [9] and footnote 2), is equivalent to weak completeness in S .

THEOREM 2.3. *A necessary and sufficient condition² that a semi-metric space S be weakly complete is that every Cauchy sequence³ of points of S have a limit point in S .*

Proof. The condition is necessary. Suppose that there exists a Cauchy sequence $\{p_i\}$ of points of S which has no limit point in S . Thus, there exists a subsequence $\{p_{n_i}\}$ of $\{p_i\}$ such that for each i ,

² This theorem was proved independently by my classmate Wyman Richardson in one of F. B. Jones' classes.

³ A Cauchy sequence $\{p_i\}$ of points is said to have a limit point p provided that there exists a subsequence $\{p_{n_i}\}$ of $\{p_i\}$ which converges to p . There exists a Cauchy sequence of points in a space S which has a limit point but which has no sequential limit point.

$U_{1/i}(p_{n_i}) \supset p_{n_j}$ for $j \geq i$. Let $M_i = \sum_{j=i}^{\infty} p_{n_j}$. Since $\Pi M_i = 0$, there is a contradiction to the hypothesis that S is weakly complete.

The condition is sufficient. Suppose that $\{M_i\}$ denotes a monotonic descending sequence of closed subsets of S such that for each i , there exists a point p_i such that $p_i \in M_i \subset U_{1/i}(p_i)$. Since $\{p_i\}$ is a Cauchy sequence, the set ΠM_i contains a point p .

3. Non-equivalence of regular semi-metric topological spaces and regular developable (Moore) spaces. Many theorems which are true for Moore spaces have analogues which hold for regular semi-metric topological spaces⁴. However, the fact that a regular semi-metric topological space S is far removed from a Moore space is stressed by the following examples and theorems. From these, it follows that the condition of either separability or screenability for the metrization of a normal Moore space due to Jones [5] and Bing [1], respectively, has no analogue which holds in a normal space S .

Consider the following example of a regular semi-metric space which is not a Moore space. Some additional properties of this space are given in Theorem 3.2.

EXAMPLE 3.1. Let X denote the x -axis of the Cartesian plane E^2 . A semi-metric $D(p, q)$ will be defined for E^2 in the following way. Suppose that each of the letters p and q denotes a point of E^2 . If X contains both or neither of the points p and q , then define $D(p, q)$ to be the Cartesian distance $d(p, q)$. If $p \in X$ and $q \notin X$, then define $D(p, q)$ to be $d(p, q) + \alpha$ where α is a non-obtuse angle (measured in radians) between X and the line L determined by p and q . If $p \notin X$ and $q \in X$, define $D(p, q)$ to be $D(q, p)$. Clearly, D is a semi-metric for E^2 . For each positive integer n and each point p , $U_{1/n}(p)$ is defined to be an open set provided that either $p \in X$ or $U_{1/n}(p)$ lies in one of the two components of $E^2 - X$. Considering the open sets defined in this way as the elements of basis for a topology, E^2 becomes a regular connected and locally connected semi-metric topological space S which is not a Moore space. It should be noted that S is hereditarily separable since it is the sum of two hereditarily separable sets $S - X$ and X .

THEOREM 3.2. *There exists a connected and locally connected regular semi-metric topological space S which is hereditarily separable, weakly complete, strongly screenable [1], collectionwise normal, completely normal, and paracompact but which is neither perfectly separable nor a Moore space nor metrizable.*

⁴ This is included in unpublished work of F. B. Jones.

Proof. Let S be the space E^2 with the topology defined in Example 3.1. The space S is not metrizable since it is not a Moore space.

Suppose that S is perfectly separable. Then there exists a countable collection H of spherical neighborhoods in S that defines the topology of S . For each number $e > 0$ and each point p in X , $U_e(p)$ contains an element $h(e, p)$ of H . By the definition of $U_e(p)$, it follows that the center of the spherical neighborhood $h(e, p)$ is p . This is impossible since H is countable and X is uncountable.

In order to show that S is weakly complete, a distance function E different from that given in Example 3.1 will be introduced. Let L_1 and L_2 be two distinct lines parallel to and at a unit distance from X , and denote by $C(X)$ the component of $S - (L_1 + L_2)$ that contains X . For any pair of points p and q of $C(X) - X$, define $E(p, q)$ to be $d(p, q) / d(X, p)d(X, q)$, where d is the ordinary Cartesian distance function. If either of two points p and q fails to lie in $C(X) - X$, then define $E(p, q)$ to be $D(p, q)$ as given in Example 3.1. It follows that the topology of S is unchanged by E . In the remainder of this paragraph, the spherical neighborhoods considered will be those defined by E . Now, suppose that $\{M_i\}$ is a monotonic descending sequence of closed point sets and $\{p_i\}$ is a sequence of points such that for each i , $p_i \in M_i \subset U_{1/i}(p_i)$. If there exists a subscript n such that $X \cdot M_n = 0$, then $X \cdot M_i = 0$ for $i > n$. From this it follows that there exists $m > n$ such that $U_{1/m}(p_m) \cdot X = 0$. For, suppose that this is not the case. Then there must exist a subsequence $\{p_{n_i}\}$ of $\{p_i\}$ such that $\{d(X, p_{n_i})\}$ converges to 0. Consequently, by the definition of E , the sequence $\{E(p_{n_i}, p_{n_i})\}$ of real numbers is unbounded. This is contrary to the assumption that $U_{1/n}(p_n) \supset M_n$. Thus, the existence of the required integer m is established. It follows that $\Pi M_i \neq 0$ in this case. For the remaining case, suppose that for each i , $X \cdot M_i \neq 0$. Since $\{X \cdot M_i\}$ is a bounded monotonic descending sequence of non-empty closed subsets of X , it follows that $\Pi M_i \neq 0$. Hence, S is weakly complete.

The space S is strongly screenable. Consider the metric subspaces $S - X$ and X of S . These are strongly screenable by theorems due to Bing [1]. Let G denote an open covering of S . Denote by H and K open coverings of X and $S - X$, respectively, such that for g in G , $g \cdot X \in H$ and $g \cdot (S - X) \in K$. There exists a sequence $\{H_i\}$ of discrete collections [1] of open intervals of X such that $\sum H_i$ covers X and for each i , H_i is a refinement of H . Let I denote an interval in H_i for some i . Since \bar{I} contains no point of the closure of $(H_i - I)^*$ [the logical sum of the elements of $H_i - I$] and I lies in some element g of G , it follows that there exist discrete collections P and Q of $1/n$ -neighborhoods of points in X such that (1) each element of P and each element of Q lies in g , (2) the closure of no element of either P or Q intersects the closure of

$(H_i - I)^*$, and (3) $P + Q$ covers I . It follows that there exists a sequence $\{X_i\}$ of discrete collections each of which is a refinement of G and such that $\Sigma X_i \supset X$. Similarly, there exists a sequence $\{K_i\}$ of discrete collections each of which is a refinement of K and such that $\Sigma K_i \supset S - X$. For each natural number i , let $G_{2i} = X_i$ and $G_{2i-1} = K_i$. Thus, $\{G_i\}$ is a sequence of discrete collections of open subsets of S such that ΣG_i covers S and G_i refines G for each i . Hence, S is strongly screenable.

Now S , being a regular strongly screenable topological space, is collectionwise normal [1]. It also follows that S is paracompact by a theorem due to Ernest Michael [6].

To complete the proof of theorem 3.2, it must be shown that S is completely normal. It has been proved by F. B. Jones [5] that every normal Moore space is completely normal⁵. A simple modification of his argument shows that every normal semi-metric topological space is completely normal. This completes the proof.

Mary E. Estill [3] has considered complete Moore spaces in any one of three definitions of completeness. Perhaps intuition would lead one to suspect that a complete Moore space, in one of these senses, would be strongly complete. The following example and theorem shows that this is not the case. As a matter of fact, in a Moore space, the concept of strong completeness is more restrictive than that of completeness.

EXAMPLE 3.3. Let X denote the x -axis of the Cartesian plane E^2 . A semi-metric $D(p, q)$ will be defined for E^2 in the following way. Suppose that p and q are two distinct points of E^2 . If neither p nor q lies in X , then define $D(p, q)$ to be $d(p, q)$ where d is the ordinary Cartesian metric. If $p \in X$, then let $D(p, q) = d(p, q) + \alpha$ where α is an angle (measured in radians) between a line L_1 containing $p + q$ and a vertical line L_2 containing p such that $0 \leq \alpha \leq \pi/2$. If $D(q, p)$ is not defined above, then let $D(q, p) = D(p, q)$. For p in X , let $D(p, p) = 0$. Clearly, D is a semi-metric for E^2 . For each point p in E^2 and each natural number n , $U_{1/n}(p)$ is defined to be an open set. With this definition of open sets, E^2 becomes a regular connected and locally connected semi-metric topological space S . It should be noted that S is separable but not hereditarily separable.

THEOREM 3.4. *There exists a complete Moore space S which is not strongly complete.*

Proof. Let S be the space E^2 with the topology defined in Example

⁵ A space S is said to be completely normal provided that for two mutually separate subsets H and K of S there exists mutually exclusive open coverings of H and K . See [5].

3.3. It will first be shown that S is not strongly complete.

Suppose that S is strongly complete. Then there exists a semi-metric E defined for S such that (1) the topology of S is unchanged by E and (2) if $\{M_i\}$ is a monotonic descending sequence of closed subsets of S such that for each i and some point p_i in S , $U_{1/i}(p_i) \supset M_i$, then $\prod M_i \neq \emptyset$. It should be noted that the spherical neighborhoods defined by E may fail to be open sets.

Consider an interval A of X . For each pair of natural numbers h and k , let M_{hk} denote the subset of A of all points p such that for some open set R , $U_{1/h}(p) \supset R \supset U_{1/k}(p)$. For some natural number h_1 , the set M_{1h_1} is uncountable. Now, M_{1h_1} contains an uncountable subset N_{1h_1} such that

- (1) there exists a line L_1 parallel to X where $d(L_1, X) \leq 1$ and
- (2) for each point p in N_{1h_1} , there exists an open set $R(p)$ where $U_1(p) \supset R(p) \supset U_{1/h_1}(p)$ such that $R(p)$ contains an interval I of L_1 whose length (in the Cartesian sense) is greater than a positive number e_1 and which has as its center a point q whose projection on X is p .

Now there exists an integer $h_2 > h_1$ such that N_{1h_1} contains an uncountable subset N_{2h_2} such that

- (1) there exists a line L_2 parallel to X where $d(L_2, X) \leq 1/2$ and
- (2) for each point p in N_{2h_2} , there exists an open set $R(p)$ where $U_{1/2}(p) \supset R(p) \supset U_{1/h_2}(p)$ such that $R(p)$ contains an interval I of L_2 whose length is greater than a positive number e_2 and which has as its center a point q whose projection on X is p .

It follows that there exists a monotonic descending sequence $\{N_{ih_i}\}$ of subsets of A and a sequence $\{L_i\}$ of lines parallel to X and converging to it such that for each i , if p_i is a point of N_{ih_i} , there exists an open set $R(p_i)$ where $U_{1/i}(p_i) \supset R(p_i) \supset U_{1/h_i}(p_i)$ such that $R(p_i)$ contains an interval I_i of L_i whose length is greater than a positive number e_i and which has as its center a point q_i whose projection on X is p_i . Since A is a compact subset of E^2 there exists a monotone sequence $\{p_i\}$ of points converging to a point p in A such that for each i , $p_i \in N_{ih_i}$. Let L be a vertical line containing p , and for each i , define $x_i = L \cdot L_i$. It follows that there exists a monotonic increasing sequence $\{k_i\}$ of natural numbers such that for each i , $E(x_i, p_j) < 1/i$ for all $j > k_i$. The set $M_i = \sum_{k=k_i}^{\infty} p_k$ is closed in S for each i and $U_{1/i}(x_i) \supset M_i$. It follows that $\prod M_i = \emptyset$.

This is contrary to the assumption that S is strongly complete.

It now remains to be shown that S is a complete Moore space. For a point p in X , there exists a sequence $\{R_i\}$ of open sets closing down⁶

⁶ A sequence of open sets $\{R_i\}$ is said to close down on a point p if for each i , $R_i \supset \bar{R}_{i+1}$ and $\prod R_i = p$.

on p . On the other hand, if p denotes a point of $S-X$, there exists a sequence $\{R_i\}$ of open sets closing down on p such that for each i , $R_i \cdot X = 0$. With each point p of S , associate exactly one such sequence $\{R_i\}$. For each i , let G_i denote the collection of all open sets R such that for some point p of S , R is the j th member of the sequence associated with p , and $j \geq i$. It follows that S is a complete Moore space.

4. A question due to W. A. Wilson. An affirmative answer is given in this section to a question raised by Wilson [14, p. 366] in 1931. The following axioms and definitions [14] are listed for convenience.

A set Z is said to be a (Menger) semi-metric space provided that corresponding to each pair of points (a, b) of Z , there is a non-negative real number $d(a, b)$ satisfying the following axioms:

Axiom I. $d(a, b) = d(b, a)$.

Axiom II. $d(a, b) = 0$ if and only if $a = b$.

Wilson has introduced the following additional axiom:

Axiom W . For each point a and each positive number k , there is a positive number r such that if b is a point for which $d(a, b) \geq k$ and c is any point, then $d(a, c) + d(b, c) \geq r$.

Now, let $r = f(a, k)$ denote the largest r such that $d(a, c) + d(b, c) \geq r$ in Axiom W . For each point a and each positive number k , let $r = f(a, k)$, $r_1 = f(a, r)$, and r_2 denote a positive number such that $r_2 < r_1$. Wilson calls the set σ of points x such that $d(a, x) < r_2$ an inner sphere, with center a , corresponding to a and k .

THEOREM 4.1. *Suppose that Z denotes a separable semi-metric space satisfying Axiom W . If d denotes a distance function defined for Z which leaves limit points invariant, then there exists a countable dense subset $E = \Sigma p_i$ of Z such that for any positive number k , each point p of Z lies in an inner sphere σ corresponding to p_i and k for some natural number i .*

Proof. By a corollary due to Wilson [14], Z is homeomorphic to a metric space. Since a separable metric space is hereditarily separable, it follows that Z is hereditarily separable.

Let $S_\epsilon(p)$ denote a spherical neighborhood in Z . For each pair of natural numbers h and k , let M_{hk} denote the set of all points p such that there exists an inner sphere σ corresponding to $1/h$ and p such that $S_{1/h}(p) \supset \sigma \supset S_{1/k}(p)$. Since Z is hereditarily separable, M_{hk} contains a countable dense subset N_{hk} . Let K_{hk} be a countable collection of inner spheres such that if $p \in N_{hk}$, then there exists an inner sphere σ in K_{hk} corresponding to $1/h$ and p such that $S_{1/h}(p) \supset \sigma \supset S_{1/k}(p)$. It

follows that $K_{h,k}$ covers $M_{h,k}$. Denote by $E = \sum p_i$, the countable dense subset $\sum_{h,k=1}^{\infty} N_{h,k}$ of Z .

The set E satisfies the conclusion of Theorem 4.1. For, if c is any positive number, there exists a positive integer h such that $1/h < c$. Also, for p in Z , there exists k such that $p \in M_{h,k}$. Since $K_{h,k}$ covers $N_{h,k}$, there exists an inner sphere σ corresponding to p_i and $1/h$ for some i such that $\sigma \supset p$. Hence, the inner sphere σ_i which corresponds to p_i and c contains σ and p .

Now Wilson's question referred to above is answered.

5. Generalized Frechet p -separability. The following definition is a natural generalization of the notion of p -separability [4]. It is proved that in a space S this notion is equivalent to hereditary separability.

DEFINITION 5.1. A regular semi-metric topological space S (or semi-metric space Z) is said to be p -separable provided that

(1) given any distance function d which leaves limit points invariant and

(2) given any collection H of subsets of S which has the property that for each number $k > 0$ and each point p of S , there exists h in H such that $U_k(p) \supset h \supset U_c(p)$ for some positive number c ,

then there exists a countable dense subset $E = \sum p_i$ such that for each positive number f , each point p of S lies in an element h of H such that $U_f(p_i) \supset h \supset p_i$ for some i .

The following theorem may be proved in a manner analogous to that used in the proof of Theorem 4.1.

THEOREM 5.2. *Every hereditarily separable semi-metric space Z is p -separable.*

THEOREM 5.3. *A necessary and sufficient condition that a regular semi-metric topological space S be hereditarily separable is that S be p -separable.*

Proof. The necessity of the condition follows from Theorem 5.2.

It will now be shown that the condition is sufficient. Suppose that d denotes a semi-metric for S , and that S is not hereditarily separable. Then S contains an uncountable subset N which has no limit point in S . Now, consider a semi-metric D defined in the following way. For each i , let D_i denote the set of all points x of S such that for some point p in N , x lies in an open set $R \subset u_{1/i}(p)$ where $u_{1/i}(p)$ is a spherical neighborhood defined by d . Thus, $\{D_i\}$ is a monotonic descending sequ-

ence of open sets such that $\bigcap D_i = N$. For each i and each point p in $D_i - N$, associate exactly one open set $R_i(p)$ containing p and lying in D_i such that for some number e , $u_e(p) \supset \bar{R}(p)$ and $u_e(p) \cdot N = 0$. If x and y denote points of $S - N$ such that for some i , $D_i \supset x + y$ and $D_{i+1} \not\supset x + y$, then define $D(x, y)$ to be i provided that $R_i(x) \not\supset y$ and $R_i(y) \not\supset x$. For points x and y of S for which $D(x, y)$ is not defined above, let $D(x, y) = d(x, y)$. It follows that limit points are invariant with respect to D .

Next, let H denote a collection of open sets such that for each natural number i and each point p in S , there exists h in H such that $U_{1/i}(p) \supset h \supset p$ where $U_{1/i}(p)$ is a spherical neighborhood defined by D . Since S is p -separable, there exists a countable dense subset $E = \sum p_i$ of S such that for each positive number f , each point p of S lies in an element $h(p)$ of H such that for some i , $U_f(p_i) \supset h(p)$. There exists an uncountable subset M of $N - E$ and a natural number t such that if x is a point and $D(x, M) < 1/t$, then x lies in D_t . Let $p \in M$. Then there exist

- (1) a number $e > 0$ such that $D(p, N - p) = d(p, N - p) > e$,
- (2) a positive integer n such that $1/n < \text{smaller}[e, 1/t]$,
- (3) h in H ,
- (4) an integer i such that $U_{1/n}(p_i) \supset h \supset p$ [thus, $p_i \in D_t$],
- (5) an integer $m \geq t$ such that $p_i \in D_m - D_{m+1}$,
- (6) an open set $R_m(p_i)$ associated with p_i and D_m such that for

some number c , $u_c(p_i) \supset \bar{R}_m(p_i)$ and $u_c(p_i) \cdot N = 0$,

- (7) a positive number z such that for q in $S - R_m(p_i)$, $d(p_i, q) > z$,
- (8) $x \in h \cdot D_m - [R_m(p) + N]$ such that $D(p, x) < z$, and
- (9) an open set $R_m(x)$ associated with x and D_m such that for some

number b , $u_b(x) \supset \bar{R}_m(x)$ and $u_b(x) \cdot N = 0$.

Therefore, $b < z$. Consequently, $R_m(x) \not\supset p_i$. By definition, $D(x, p_i) = m > 1/n$. This is impossible since $U_{1/n}(p_i) \supset h \supset p + x$. Hence, S is hereditarily separable.

It follows from Theorem 3.2 that S may fail to be either perfectly separable or a metric space.

6. Conditions for semi-metric, regular developable (Moore), and metric spaces. Consider the following three conditions on a topological space T .

A. There exists a sequence $\{H_i\}$ such that (a) for each i , H_i is a collection of open subsets of T , (b) if p is a point and R is an open set containing p , then there exists an integer n such that H_n contains exactly one element $g(p)$ associated with p such that $R \supset g(p) \supset p$ and

(c) if n is an integer and $\{g_i(p_i)\}$ is a sequence such that for each i , $g_i(p_i)$ belongs to H_n and is associated with p_i , then Σp_i has no limit point in $T - \Sigma g_i(p_i)$.

B. If p is a point and R is an open set containing p , then there exists an integer n such that for $m > n$, each element g of H_m which contains p has the property that $R \supset \bar{g}$.

C. For each i , the sum of the closures of any subcollection of H_i is closed.

THEOREM 6.1. *A necessary and sufficient condition that a topological space T be semi-metric is that T satisfy Condition A.*

Proof. It will first be shown that the condition is sufficient. It follows from Condition A that T satisfies the first axiom of countability. Consider a semi-metric d defined as follows. For two distinct points p and q of T , denote by i the least integer such that H_i contains an element $g(p)$ associated with p but not containing q . Similarly, let j denote the least integer such that H_j contains an element $g(q)$ associated with q but not containing p . Define $d(p, q)$ to be $1/\min(i, j)$. For each point p , define $d(p, p)$ to be 0.

Limit points are invariant with respect to d . For suppose that p is a limit point (defined by the open sets of T) of a subset M of T and that p is not a distance limit point of M . Then there exists a sequence $\{p_i\}$ of points of $M - p$ which converges to p such that for some integer n and each i , $d(p, p_i) > 1/n$. Thus, there exists an integer m , such that, for infinitely many integers i , either (1) H_m contains $g_m(p)$ and $g_m(p) \not\supset p_i$ or (2) H_m contains $g_m(p_i)$ and $g_m(p_i) \not\supset p$. Since $\{p_i\} \rightarrow p$, (1) is impossible. By Condition A, (2) is impossible. Hence, p is a distance limit point of M . It also follows easily that a distance limit point of a subset M of T is an open set limit point of M . This completes the proof of the sufficiency.

The condition is necessary. For each point p and each pair of natural numbers h and k , let $R_{hk}(p)$ denote an open set when it exists, such that $U_{1/h}(p) \supset R_{hk}(p) \supset U_{1/k}(p)$. With h , k , and p associate exactly one such open set, and let G_{hk} denote the corresponding collection of open sets for each point p in T . There exists a sequence $\{H_i\}$ such that there is a one to one correspondence between the elements of $\{H_i\}$ and the elements of $\{G_{nm}\}$. It follows that $\{H_i\}$ satisfies Condition A.

As Example 3.1 illustrates, a regular semi-metric topological space may fail to be a Moore space.

THEOREM 6.2. *A necessary and sufficient condition that a topological space T be a Moore space is that T satisfy Conditions A and B.*

Proof. The condition is sufficient. For each positive integer i , let $G_i = \sum_{j=i}^{\infty} H_j$. If the word "region" is interpreted as "open set," then it follows that Axioms 0 and 1 (1)–(3) due to Moore [7] are satisfied.

The condition is necessary. It will be shown first that T is a semi-metric topological space. Let p and q be distinct points of T . Denote by n the least positive integer such that if $g(p)$ and $g(q)$ are regions in G_n containing p and q , respectively, then $g(p) \cdot g(q) = 0$. Note that $\{G_i\}$ is given by Axiom 1 of [7]. Consequently, define $d(p, q)$ to $1/n$. It follows that d is a semi-metric distance function and that limit points are invariant with respect to d . By Theorem 6.1, T satisfies Condition A.

Now, define $\{H_i\}$ in a manner described in the proof of Theorem 6.1 with the additional requirement that $R_{hk}(p)$ lie in a region of G_h . It follows that $\{H_i\}$ satisfies Conditions A and B.

THEOREM 6.3. *A necessary and sufficient condition that a topological space T be metric is that it satisfy Conditions A, B, and C.*

A proof of Theorem 6.3 follows by use of Bing's Theorem 4 of [1] and Theorem 6.1 above.

Question. Is it possible to partition either Bing's Theorem 4 of [1] or Moore's metrization theorem [8; 13], stated below, into three or more parts which begins with a condition for a topological space and which ends with a condition for a metrizable space, but with necessary and sufficient conditions somewhere between these extremes for semi-metric spaces and Moore spaces?

THEOREM (Moore)⁷. *A necessary and sufficient condition that a space S satisfying Axiom 0 of [7] be metrizable is that there exist a sequence $\{K_i\}$ such that (1) for each natural number n , K_n is a collection of regions in S covering S and (2) if p is a point, q is a point distinct from p , and R is a region containing p , then there exists a natural number n such that if each of the letters h and k denotes an element of K_n , $g \supset p$, and $g \cdot h \neq 0$, then $R - q \supset h$.*

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⁷ The terms "point" and "region" are undefined. Axiom 0 states that every region is a point set.

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