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ENTIRE FUNCTIONS

A. C. SCHAEFFER

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1. Introduction. The object of this note is to prove several results, which are related in that each is concerned with entire functions of exponential type. An entire function $f(z)$ is of exponential type τ if for every $\epsilon > 0$ there is a number $A = A(\epsilon)$ such that

$$(1) \quad |f(z)| \leq A e^{(\tau + \epsilon)|z|}.$$

The function is of precise type τ if (1) does not hold for any $\epsilon < 0$.

The first result is concerned with entire functions which are bounded at a sequence of points. Miss Cartwright's theorem states that if $f(z)$ is an entire function of exponential type τ , $\tau < \pi$, and is bounded by 1 at the integer points,

$$(2) \quad |f(n)| \leq 1, \quad n = 0, \pm 1, \pm 2, \dots,$$

then $f(z)$ is bounded on the entire real axis by a number which depends only on τ ,

$$(3) \quad |f(x)| < M(\tau), \quad -\infty < x < \infty.$$

Proofs of this and stronger results have been given by Cartwright, Pflunger, Macintyre, Boas, Korevaar, Duffin and Schaeffer, Levin, Ahiezer, Agmon, and others. These results are discussed in [2, Chapter 10] where further references are given.

Let N be a sequence of integers. The first question to be considered in the present note is: what conditions must N satisfy in order that for every entire function of exponential type less than π the condition

$$(4) \quad |f(n)| \leq 1, \quad n \in N,$$

will imply that $f(z)$ is bounded on the real axis? To answer this question we define a function $\lambda(t)$ for $t > 0$ by means of the given sequence N . Let $\lambda(t)$ be the greatest integer μ such that every interval of the real axis of length t has μ or more elements of N . For any positive t there is at least one interval, which may be supposed open, of length t which contains precisely $\lambda(t)$ elements of N , and every interval of length t whether open or closed contains $\lambda(t)$ or more elements of N . The following result is to be proved.

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THEOREM 1. *If N is a sequence of integers, the condition*

$$(5) \quad \lim_{t \rightarrow \infty} \frac{\lambda(t)}{t} = 1$$

is necessary and sufficient in order that every entire function of exponential type less than π which is bounded on N be bounded on the real axis.

Slightly more is true than stated in the theorem, for there is a uniform bound in the following sense. If (5) is true then any entire function $f(z)$ of exponential type τ , $\tau < \pi$, which satisfies (4) must also satisfy

$$(6) \quad |f(x+iy)| \leq M(\tau, N) e^{\tau|y|}.$$

Thus the bound for $|f(x)|$ depends only on N and the exponential type of f . Relation (6) also gives a dominant for $f(z)$ over the entire plane.¹ As τ approaches π the value of $M(\tau, N)$ must approach infinity. This was shown by Boas and the author, [2] where further references are given, in case N consists of all the integers, and is *a fortiori* true for any sequence of integers.

The remaining topics to be considered in the present work center around research problem number 4 in the Bulletin of the American Mathematical Society [3]. This problem, which is due to Boas, reads as follows.

“Let $f(z)$ be an entire function of exponential type. It is well known, and easily proved by Phragmén-Lindelöf arguments, that if $f(x)$ is bounded or approaches a limit as $x \rightarrow \infty$, then $f(x+iy)$ is bounded or approaches a limit, for each y , as $x \rightarrow \infty$. If $|f(x)|$ approaches a non-zero limit, does $|f(x+iy)|$ necessarily approach a limit?” To answer this question an entire function $f(z)$ of exponential type will be defined such that

$$(7) \quad \lim_{x \rightarrow \infty} |f(x)|$$

exists and is finite, but

$$(8) \quad \lim_{x \rightarrow \infty} |f(x+iy)|$$

exists only for $y=0$.

In view of this example it is natural to ask what hypothesis in addition to (7) will imply that (8) exists for all y . It is to be shown that if (7) exists both for f and one of its derivatives then (8) exists

¹ A similar dominant was obtained by Miss Cartwright which strengthens inequality (3).

for f' and all its derivatives. More precisely, we have the following theorem where $f^{(\nu)}(z)$ denotes the derivative of $f(z)$ of order ν , with $f(z) = f^{(0)}(z)$.

THEOREM 2. *If $f(z)$ is an entire function of exponential type such that for some $m, m \geq 1$, both $|f(x)|$ and $|f^{(m)}(x)|$ tend to finite limits as $x \rightarrow \infty$ then*

$$(9) \quad \lim_{x \rightarrow \infty} |f^{(\nu)}(x + iy)|$$

exists and is finite for all y and for $\nu = 0, 1, 2, \dots$.

2. Proof of Theorem 1. The proof of Theorem 1 will depend in part on the following result of R. J. Duffin and the author [1], [4], and [2] where further references are given.

THEOREM 3. *Let $\{\lambda_n\}, n = 0, \pm 1, \pm 2, \dots$, be a sequence such that*

$$\begin{aligned} |\lambda_n - \lambda_m| &\geq \delta > 0, & n \neq m, \\ |\lambda_n - n| &\leq A \end{aligned}$$

for some δ, A . If $f(z)$ is an entire function of exponential type $\tau, \tau < \pi$, such that

$$|f(\lambda_n)| \leq 1$$

then

$$|f(x + iy)| \leq R e^{\tau|y|}$$

where R depends exclusively on τ, δ, A .

It is first to be shown that (4), (5) imply the boundedness of $f(x)$ for any entire function of exponential type less than π . Thus suppose that N is a sequence of integers for which (5) is true, and consider any fixed number τ in the range $0 \leq \tau < \pi$. If $f(z)$ is an entire function of exponential type τ which satisfies (4) and

$$\alpha = \frac{\pi - \tau}{\pi + \tau}$$

then

$$F(z) = f\left(\frac{z}{1 - \alpha}\right)$$

is an entire function of exponential type τ' .

$$\tau' = \frac{\pi + \tau}{2}.$$

Now $F(z)$ is bounded by 1 at a sequence of points $\{\mu_n\}$ defined for

$n \in N$, where

$$\mu_n = n(1 - \alpha), \quad n \in N.$$

Using relation (5) it then follows that if T is sufficiently large, each interval of the real axis of length T contains at least T element of $\{\mu_n\}$. Let T be an integer. In each of the intervals

$$\nu T \leq x < (\nu + 1)T$$

choose T elements of $\{\mu_n\}$ to form a new sequence $\{\lambda_m\}$. Then $\{\lambda_m\}$ is defined for $m=0, \pm 1, \pm 2, \dots$, and

$$\begin{aligned} |\lambda_m - m| &< T, & m = 0, \pm 0, \pm 2, \dots, \\ |\lambda_m - \lambda_n| &\geq 1 - \delta, & m \neq n. \end{aligned}$$

Since $\tau' < \pi$, Theorem 3 shows that $F'(z)$ is bounded on the real axis, and indeed

$$|F'(x + iy)| \leq R e^{\tau'|y|}.$$

This proves that condition (5) is sufficient in Theorem 1, and since the sequence $\{\lambda_m\}$ depends only on τ , N we see that R is a function of τ , N only, which gives inequality (6).

In the proof of the necessity of condition (5) in Theorem 1 the following lemma will be used.

LEMMA. *If h is an integer, $h \geq 5$, and in the closed interval $-h \leq x \leq h$ there is a set N' of integers whose number is ρ , $\rho \geq 2$, then there is an entire function $F(z)$ of exponential type μ ,*

$$\mu = \pi - \frac{\pi\rho}{32h},$$

whose maximum modulus on the real axis occurs at some point x_0 satisfying $|x_0| < 4h + 1$, and $F(x_0) = 1$, but

$$\begin{aligned} |F(x)| &\leq 2^{1-\rho}, & |x| &\geq 14h - 1, \\ |F(n)| &\leq 2^{1-\rho}, & n &\notin N'. \end{aligned}$$

Proof. Since the object is to define a function $F(z)$ which is of exponential type μ and is small on the integer points except at the set N' we begin with the function $\sin \pi z$, which is of exponential type π and vanishes at all the integers.

If $n_1, n_2, \dots, n_\alpha$ are integers which belong to N' then the function

$$(10) \quad f_1(z) = \frac{\sin \pi z}{\prod_{\nu=1}^{\alpha} \sin \frac{\pi(z - n_\nu)}{16h}}$$

is of exponential type

$$(11) \quad \mu' = \pi - \frac{\alpha\pi}{16h} .$$

However, this function satisfies the functional relation $f_1(x+16h) = \pm f_1(x)$ so it attains its maximum modulus over the real axis at points in each interval of length $16h$. If m_1, m_2, \dots, m_β are the remaining integers which belong to N' then the function

$$(12) \quad f_2(z) = \frac{\sin \pi z}{\prod_{\nu=1}^{\beta} (z - m_\nu) \prod_{\nu=1}^{\alpha} \sin \frac{\pi}{16h} (z - n_\nu)}$$

remains of exponential type μ' where μ' is defined by (11), but it is small when $|x|$ is large. The function $f_2(z)$ vanishes at all integers in the range $|x| \leq 15h - 1$ except those that belong to the set N' . It is to be shown that if α, β are suitably chosen then a suitable constant times the function $f_2(z)$ is the function $F(z)$ whose existence is asserted in the lemma.

Let $\alpha = \rho/2$ if ρ is even and $\alpha = (\rho + 1)/2$ if ρ is odd; and define

$$\phi_1(z) = \prod_{\nu=1}^{\alpha} \sin \frac{\pi}{16h} (z - n_\nu)$$

$$\phi_2(z) = \prod_{\nu=1}^{\beta} (z - m_\nu) .$$

Here m_ν, n_ν are together all the integers which belong to N' so we have

$$\beta = \rho - \alpha \geq \frac{\rho - 1}{2} .$$

By considering the individual terms of $\phi_1'(x)$ separately it is clear that

$$|\phi_1(1/2)| < |\phi_1(x)| , \quad 2h + 1 \leq |x| \leq 14h - 1 .$$

The function $f_1(z)$ defined by (10) may be written

$$f_1(z) = \frac{\sin \pi z}{\phi_1(z)} ,$$

and it therefore satisfies the inequality

$$(13) \quad |f_1(1/2)| > |f_1(x)| , \quad 2h + 1 \leq |x| \leq 14h - 1 .$$

Since $|f_1(x)|$ is periodic with period $16h$ the maximum modulus of $f_1(x)$ on the real axis is attained at some point x_1 in the interval $|x_1| \leq 8h$. But then (13) shows that

$$(14) \quad |x_1| < 2h + 1 .$$

Also, $f_1(z)$ is of exponential type ρ' , and since $\alpha \geq \rho/2$ we have $\rho' \leq \rho$.

The function $f_2(z)$ defined by (12) may be written

$$f_2(z) = \frac{f_1(z)}{\phi_2(z)} .$$

Now $f_2(x)$ attains its maximum modulus on the real axis at some point x_0 satisfying $|x_0| < 4h + 1$. For if $|x| \geq 4h + 1$ then using (14), we have

$$|\phi_2(x)| \geq (3h + 1)^\beta > |\phi_2(x_1)| .$$

Hence

$$|f_2(x)| = \left| \frac{f_1(x)}{\phi_2(x)} \right| < \frac{|f_1(x)|}{|\phi_2(x_1)|} \leq |f_2(x_1)| , \quad |x| \geq 4h + 1 .$$

We also have

$$|\phi_2(x)| \geq 4^\beta (3h + 1)^\beta > 4^\beta |\phi_2(x_1)| , \quad |x| \geq 14h - 1 ,$$

where we have used the supposition $h \geq 5$. Thus for $|x| \geq 14h - 1$ we have

$$|f_2(x)| = \frac{|f_1(x)|}{|\phi_2(x)|} < \frac{|f_1(x_1)|}{4^\beta |\phi_2(x_1)|} = \frac{|f_2(x_1)|}{4^\beta} < \frac{|f_2(x_0)|}{4^\beta} .$$

Since $\beta \geq (\rho - 1)/2$ the function

$$F(z) = \frac{f_2(z)}{f_2(x_0)}$$

has all the properties stated in the lemma.

We now complete the proof of Theorem 1 by showing that condition (5) is necessary. It follows from the definition of $\lambda(t)$ that for any sequence of integers we automatically have

$$\limsup_{t \rightarrow \infty} \frac{\lambda(t)}{t} \leq 1 .$$

Thus suppose N is a sequence of integers such that

$$(15) \quad \liminf_{t \rightarrow \infty} \frac{\lambda(t)}{t} = \gamma < 1 .$$

It is to be shown under this supposition that there is an entire function $G(z)$ of exponential type less than π which is bounded on N , but not on the real axis. For this purpose the first objective is naturally to find arbitrarily long intervals $[a - h, a + h]$ in which we can use translations $F_n(z - a)$ of the functions defined in the lemma. The required function $G(z)$ will then be a suitable linear combination $\sum c_n F_n$ of these functions.

Let

$$\gamma_1 = \frac{1 + \gamma}{2} .$$

If h_0 is sufficiently large the following statement is true. Given a positive number r and a positive integer h , $h \geq h_0$, there is a closed interval I of length $2h$ lying entirely in $|x| > r$ and with integer end-points such that the number of elements of N lying in I is less than

$$2h\gamma_1 .$$

This can be proved by means of (15). The number of integer points in I which do not belong to N is therefore greater than

$$(16) \quad 2h + 1 - 2h\gamma_1 .$$

Let H be an integer such that $H \geq h_0$, $H \geq 5$; and, for later purposes, we also choose H so large that

$$(17) \quad \sum_{h=H}^{\infty} h 4^{-h(1-\gamma_1)} < 1 .$$

For each integer h satisfying $h > H$ there is a closed interval I_h of length $2h$ with integer end-points and center at x_h where

$$(18) \quad |x_{h+1}| > |x_h| + 28h + 14 , \quad h > H,$$

and the number of integer points in I_h which do not belong to N is ρ_h , where by (16),

$$(19) \quad \rho_h > 2h(1 - \gamma_1) + 1 .$$

Take

$$(20) \quad |x_H| > 28H .$$

We now use the lemma, where we can clearly translate the interval. The set N' of the lemma is the set of integers in I_h which do not belong to N . There is an entire function $F'_h(z)$ of exponential type λ ,

$$\lambda = \pi - \frac{\pi(1 - \gamma_1)}{16} ,$$

which satisfies

$$(21) \quad |F'_h(x)| \leq 4^{-h(1-\gamma_1)} , \quad |x - x_h| \geq 14h - 1 .$$

In the last inequality we have made use of (19). Also,

$$(22) \quad |F'_h(n)| \leq 4^{-h(1-\gamma_1)} , \quad n \in N .$$

We note also that because of (18) the intervals $|x - x_h| \leq 4h - 1$ are disjoint; hence at each point x of the real axis the inequality

$$(23) \quad |F'_h(x)| \leq 4^{-h(1-\gamma_1)}$$

is satisfied except possibly for one value of h . Each function $F'_h(x)$ has maximum modulus 1 on the real axis, and this maximum is attained at some point x'_h satisfying $|x'_h - x_h| < 4h + 1$.

How consider the functions

$$G_m(z) = \sum_{h=H}^m h F'_h(z), \quad m = H, H+1, \dots$$

If ν lies in the range $H \leq \nu \leq m$ then by (21),

$$|G_m(x'_\nu)| \geq \nu - \sum_{h \neq \nu} h |F'_h(x'_\nu)| > \nu - \sum_{h=H}^{\infty} h 4^{-h(1-\gamma_1)},$$

so (17) yields

$$(24) \quad |G_m(x'_\nu)| > \nu - 1.$$

To obtain a dominant for $G_m(z)$ on the real axis which is independent of m we note first that if x lies outside all the intervals $|x - x_h| \leq 14h - 1$ then (23) is valid for all h , and

$$|G_m(x)| < 1.$$

If, on the other hand, x lies in one of these intervals, say $|x - x_k| < 14k - 1$, then we shall have

$$|G_m(x)| < k + \sum_{h=H}^{\infty} h 4^{-h(1-\gamma_1)} < k + 1.$$

But (18), (20) imply that $|x_h| > 28h$ for all h , so the inequality $|x - x_k| < 14k - 1$ shows $|x| > 14k + 1$. Thus

$$(25) \quad |G_m(x)| \leq 1 + \frac{|x|}{14}.$$

Now $G_m(z)$ is an entire function of exponential type λ , so a dominant which is independent of m can be obtained for $G_m(z)$ over the entire plane by use of (25). One argument makes use of Theorem 3. The function $\{G_m(z) - G_m(0)\}/z$ is bounded by 3 on the part of the real axis for which $|x| \geq 1$, and it is an entire function of exponential type λ , $\lambda < \pi$. Thus the function is bounded by 3 at a sequence $\{\lambda_n\}$ which satisfies the conditions of Theorem 3. Hence

$$|G'_m(x + iy) - G'_m(0)| \leq R' |z| e^{\lambda |y|}$$

where R' is independent of m .

Since $|G_m(0)| \leq 1$ the functions $G_m(z)$, $m=H, H+1, \dots$, are uniformly bounded in each bounded domain, hence for some subsequence of m tending to infinity the functions $G_m(z)$ tend to a limit $G(z)$. Moreover

$$|G(x+iy)| \leq 1 + R'|z| e^{\lambda|y|},$$

so $G(z)$ is an entire function of exponential type λ , $\lambda < \pi$. Inequalities (22), (17) show that

$$|G(n)| \leq 1, \quad n \in N,$$

but inequality (24) shows that

$$|G(x'_\nu)| \geq \nu - 1, \quad \nu = H, H+1, \dots,$$

and $G(z)$ is not bounded on the real axis.

3. The question of Boas and proof of Theorem 2. An entire function $f(z)$ of exponential type is to be defined for which

$$\lim_{x \rightarrow \infty} |f(x+iy)|$$

exists if and only if $y=0$. This function will be

$$(26) \quad f(z) = z e^{-\pi z} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2} \right)$$

where

$$\lambda_n = \frac{n}{\sqrt{2}} \exp\left(\frac{i\pi}{4} \delta_n\right).$$

Here

$$\delta_n = \pm 1$$

is to be determined. Pairs of zeros of $f(z)$ are replaced by their conjugates on changing the sign of δ_n . Thus the modulus of $f(x)$ is independent of the choice of δ_n , but $|f(x+iy)|$ does in general depend on δ_n .

In case $\delta_n=1$ for all n let the function be designated $f_0(z)$, that is, define

$$f_0(z) = z e^{-\pi z} \prod_1^{\infty} \left(1 - \frac{2z^2}{in^2} \right) = e^{-\pi z} \frac{\sin \pi(1-i)z}{\pi(1-i)}.$$

Likewise, if $\delta=-1$ for all n call the function $f_1(z)$,

$$f_1(z) = z e^{-\pi z} \prod_1^{\infty} \left(1 + \frac{2z^2}{in^2} \right) = e^{-\pi z} \frac{\sin \pi(1+i)z}{\pi(1+i)}.$$

Thus we have

$$(27) \quad \lim_{x \rightarrow \infty} |f_0(x + iy)| = \frac{e^{-\pi y}}{2\pi\sqrt{2}}$$

$$(28) \quad \lim_{x \rightarrow \infty} |f_1(x + iy)| = \frac{e^{\pi y}}{2\pi\sqrt{2}}$$

The δ_n are to be chosen so that in some neighborhoods $f(z)$ will have approximately the same modulus as $f_0(z)$, while in other neighborhoods $f(z)$ will have approximately the same modulus as $f_1(z)$.

Let N_1, N_2, \dots be a sequence of integers such that

$$N_{\nu+1} \geq 4N_\nu^2,$$

and choose $N_1=1$. Define

$$\delta_n = (-1)^\nu, \quad N_\nu \leq n < N_{\nu+1}.$$

This completes the definition of the function $f(z)$. If ν is even then for z in the ring $N_\nu^2 \leq |z| \leq 2N_\nu^2$ we have

$$\left| \Re \log \frac{f(z)}{f_0(z)} \right| \leq \sum_{n < N_\nu} \left| \log \frac{1 - \lambda_n^2/z^2}{1 - n^2 i/2z^2} \right| + \sum_{n > N_{\nu+1}} \left| \log \frac{1 - z^2/\lambda_n^2}{1 - 2z^2/in^2} \right|.$$

Using the estimates

$$|\log(1-u)| \leq 2|u|, \quad |u| \leq 1/2,$$

and $\sum_1^k n^2 \leq k^3, \sum_k^\infty n^{-2} < 2/k$ we have

$$\left| \Re \log \frac{f(z)}{f_0(z)} \right| \leq \frac{18}{N_\nu}, \quad N_\nu^2 \leq |z| \leq 2N_\nu^2.$$

Likewise, if ν is odd we have

$$\left| \Re \log \frac{f(z)}{f_1(z)} \right| \leq \frac{18}{N_\nu}, \quad N_\nu^2 \leq |z| \leq 2N_\nu^2.$$

These inequalities together with (27), (28) show that

$$\limsup_{x \rightarrow \infty} |f(x + iy)| = \frac{e^{\pi|y|}}{2\pi\sqrt{2}}$$

and

$$\liminf_{x \rightarrow \infty} |f(x + iy)| = \frac{e^{-\pi|y|}}{2\pi\sqrt{2}}.$$

It is also clear that $f(z)$ is an entire function of exponential type. Indeed,

$$|f(re^{i\theta})| \leq r e^{\pi r} \prod_{n=1}^{\infty} \left(1 + \frac{2r^{n^2}}{n^2}\right) < e^{\pi(1+\sqrt{2})r}.$$

We now turn to proof of Theorem 2. Thus suppose that $f(z)$ satisfies the conditions of Theorem 2, so

$$|f(z)| \leq A e^{\tau|z|}$$

for some A, τ . Since $|f(x)|$ has a finite limit it follows that the functions $f(z)e^{i\tau z}$ is bounded on the positive halves of the real and imaginary axes. Then the Phragmén-Lindelöf principal shows that since this function is of order one it is bounded throughout the first quadrant. Likewise, $f(z)e^{-i\tau z}$ is bounded in the fourth quadrant, and we have

$$(29) \quad |f(x+iy)| \leq B e^{\tau|y|}, \quad x \geq 0.$$

write

$$L = \lim_{x \rightarrow \infty} |f(x)|, \quad L_m = \lim_{\gamma \rightarrow \infty} |f^{(m)}(x)|.$$

Now let a_1, a_2, a_3, \dots be any sequence of real numbers such that $a_n \rightarrow \infty$ as $n \rightarrow \infty$. Because of (29) the functions $f(z+a_n)$ are uniformly bounded in each bounded domain for large n . Thus there is a subsequence b_1, b_2, b_3, \dots of the a_n such that the functions $f(z+b_n)$ tend to a limit $F(z)$ as n becomes large,

$$F(z) = \lim_{n \rightarrow \infty} f(z+b_n).$$

Then $F(z)$ is an entire function of exponential type, indeed it satisfies

$$|F(x+iy)| \leq B e^{\tau|y|}$$

for all real x, y . We also have

$$F^{(\nu)}(z) = \lim_{n \rightarrow \infty} f^{(\nu)}(z+b_n), \quad \nu = 0, 1, 2, \dots$$

The theorem will follow if it is shown that the value of $|F^{(\nu)}(z)|$ is a function of y only and is independent of the particular sequence a_1, a_2, \dots .

Now $F(z)$ has the constant modulus L on the real axis. If $L=0$ then $F(z)$ and all its derivatives vanish identically so (9) is true. Thus suppose $L \neq 0$. Then $F(z)$ can have no zero, for it would then have a pole at the conjugate point. Thus

$$(30) \quad F(z) = L e^{i\gamma} e^{i\alpha z}$$

where γ, α are real numbers. Now

$$F^{(m)}(z) = L e^{i\gamma} (i\alpha)^m e^{i\alpha z},$$

and $F^{(m)}(z)$ has the constant modulus L_m on the real axis. This gives the relation

$$(31) \quad \alpha = \pm (L_m/L)^{1/m}.$$

If $L_m=0$ then $\alpha=0$ for all sequences and (9) is true. Thus suppose $L_m>0$. Now consider the values of $|f(x+i)|$. If one sequence a_1, a_2, \dots leads to a function $F(z)$ of the form (30) with $\alpha=(L_m/L)^{1/m}$, and another sequence leads to a function with $\alpha=-(L_m/L)^{1/m}$ then

$$\limsup_{x \rightarrow \infty} |f(x+i)| = Le^{|\alpha|}, \quad \liminf_{x \rightarrow \infty} |f(x+i)| = Le^{-|\alpha|}.$$

Now $|f(x+i)|$ varies continuously as x increases, so there will be a sequence x_1, x_2, \dots with $x_n \rightarrow \infty$ as $n \rightarrow \infty$ such that $|f(x_n+i)|=L$. A subsequence of the functions $f(z+x_n)$ will tend to a limit $F(z)$ satisfying $|F(i)|=L$. But $F(z)$ will be of the form (30), so $\alpha=0$. This contradicts (31) and proves the theorem.

Indeed, the foregoing argument proves slightly more than the theorem states. First, under the conditions of Theorem 2 we have

$$\lim_{x \rightarrow \infty} |f^{(\nu)}(x+iy)| = L|\alpha|^\nu e^{-\alpha y}, \quad \nu=0, 1, 2, \dots,$$

where L, α are independent of ν, y . Secondly, the requirement that $f(z)$ is an entire function of exponential type can be replaced by the supposition that $f(z)$ is regular in some half-plane $x \geq c$, where it is of exponential growth.

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December, 1956

Louis Auslander, <i>Remark on the use of forms in variational calculations</i>	209
Hubert Spence Butts, Jr. and Henry B. Mann, <i>Corresponding residue systems in algebraic number fields</i>	211
L. Carlitz and John Herbert Hodges, <i>Distribution of matrices in a finite field</i>	225
Paul Civin and Bertram Yood, <i>Invariant functionals</i>	231
David James Dickinson, Henry Pollak and G. H. Wannier, <i>On a class of polynomials orthogonal over a denumerable set</i>	239
Bernard Friedman and Luna Mishoe, <i>Eigenfunction expansions associated with a non-self-adjoint differential equation</i>	249
Luna Mishoe and G. C. Ford, <i>On the uniform convergence of a certain eigenfunction series</i>	271
John W. Green, <i>Mean values of harmonic functions on homothetic curves</i>	279
Charles John August Halberg, Jr. and Angus E. Taylor, <i>On the spectra of linked operators</i>	283
Chuan Chih Hsiung, <i>Some integral formulas for closed hypersurfaces in Riemannian space</i>	291
Norman D. Lane, <i>Differentiable points of arcs in conformal n-space</i>	301
Louis F. McAuley, <i>A relation between perfect separability, completeness, and normality in semi-metric spaces</i>	315
G. Power and D. L. Scott-Hutton, <i>The slow shearing motion of a liquid past a semi-infinite plane</i>	327
A. C. Schaeffer, <i>Entire functions</i>	351
Edward Silverman, <i>An intrinsic inequality for Lebesgue area</i>	363
Choy-Tak Taam, <i>Asymptotic relations between systems of differential equations</i>	373
Ti Yen, <i>Quotient algebra of a finite AW^*-algebra</i>	389