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QUOTIENT ALGEBRA OF A FINITE AW^* -ALGEBRA

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1. Introduction. In a recent paper [5] Wright proves that if A is an AW^* -algebra [2] having a trace and if M is a maximal ideal of A , then A/M is an AW^* -factor (that is, an AW^* -algebra whose center consists of complex numbers) having a trace. The trace enters into his argument in the characterization [5, Theorem 3.1] of the one-to-one correspondence between maximal ideals of A and those of its center Z . This is, in turn, used to verify that A/M satisfies the countable chain condition, namely: every set of mutually orthogonal projections is at most countable, which is crucial to prove that every set of mutually orthogonal projections has a least upper bound (LUB). It is the purpose of this paper to prove the following.

THEOREM. *Let A be a finite AW^* -algebra, and M a maximal ideal of A . Then A/M is a finite AW^* -factor.*

It is not known whether a finite AW^* -factor always has a trace. Since [3] a finite AW^* -algebra of type I always has a trace, our result adds nothing new in this case, and we shall be solely concerned with algebras of type II_1 .

Our terminology is that of [2]. We assume familiarity with [2] and [1] (especially [1, pp. 234-242]).

2. Maximal ideal M . We begin with a slightly sharpened version of [5, Theorem 2.5] on p -ideals. A set P of projections is called a p -ideal if

- (1) P contains $e \vee f$ whenever it contains e and f
- (2) P contains f whenever it contains an $e \succ f$.

It follows from (1) that $e_1 \vee \dots \vee e_n$ is in P if e_1, \dots, e_n are in P . For any set S of A let S_p denote the set of projections contained in S .

LEMMA 1. *Let A be an AW^* -algebra. The closed linear subspace M generated by a p -ideal P is an ideal with $M_p = P$. Conversely an ideal M of A is the closed linear subspace generated by the p -ideal M_p .*

Proof. Let P be a p -ideal and M the closed linear subspace generated by P . For M to be an ideal we need to prove that M contains xe for any $x \in A$ and $e \in P$. The left projection [2, p. 244] f of xe , being $\prec e$, is contained in P . Hence P contains $g = e \vee f$. $xe \in gAg \subset M$,

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as gAg is the closed linear subspace generated by all projections $\leq g$.

Let M_0 denote the linear subspace algebraically generated by P ; the elements of M_0 are of the form $x = \sum_{i=1}^n \lambda_i e_i$ (λ_i complex numbers, $e_i \in P$). As P contains $e_1 \vee \dots \vee e_n$, the left and right projections of x are in P . Take an f in M_p , and an $\epsilon > 0$. There is an $x \in M_0$ with $\|f - x\| < \epsilon$. The left projection h of fx , being $<$ the right projection of x , is in P . We have $h \leq f$ and $\|f - h\| = \|(f - h)(f - fx)\| \leq \|f - fx\| < \epsilon$. Hence $f = h$. This proves that $M_p = P$.

Assume now that M is an ideal. M_p is [5, Lemma 2.1] a p -ideal. Let M' denote the closed linear subspace generated by M_p . We wish to prove that $M = M'$. Take $x \in M$ and $\epsilon > 0$. There is¹ [2, Lemma 2.1] a projection e , which is a multiple of x , such that $\|x - ex\| < \epsilon$. Since $ex \in M'$ and M' is closed, $x \in M'$.

Let now A be an AW^* -algebra of type II_1 , Z its center. Then [2, p. 247] A admits a dimension function D defined on A_p with values in Z . D has the following properties :

- (1) $0 \leq D(e) \leq 1$ for every e ,
- (2) $D(e) = e$ if $e \in Z$,
- (3) $D(e) = D(f)$ if and only if $e \sim f$,
- (4) $D(\sum e_i) = \sum D(e_i)$ if the e_i 's are mutually orthogonal [1, Lemma 6.13].

Moreover, D is uniquely determined by these properties. It is an immediate consequence of (4) that given $0 < \lambda < 1$ there is a projection e with $D(e) = \lambda$.

Let C be a commutative AW^* -subalgebra [3] of A . C is the closed linear subspace generated by C_p . We shall extend D to a linear transformation T_c of C into Z . First define T_c on the linear combinations of projections by setting

$$T_c(\sum_{i=1}^n \lambda_i e_i) = \sum_{i=1}^n \lambda_i D(e_i).$$

We must show that T_c is uniquely defined, i.e., if $x = y$ then $T_c(x) = T_c(y)$. If $x = \sum_{i=1}^n \lambda_i e_i$, there are orthogonal projections f_1, \dots, f_m such that each e_i is a sum of the f 's :

$$e_i = \sum_{j=1}^m \alpha_{ij} f_j \quad \text{where} \quad \alpha_{ij} = \begin{cases} 1 & \text{if } e_i f_j = f_j \\ 0 & \text{if } e_i f_j = 0. \end{cases}$$

$$x = \sum_{i=1}^n \lambda_i e_i = \sum_{j=1}^m (\sum_{i=1}^n \lambda_i \alpha_{ij}) f_j.$$

¹ To use [2, Lemma 2.1] we first imbed ax^* in a maximal commutative self-adjoint subalgebra of A . Working in this subalgebra we get a projection e with $\|ax^* - eax^*\| < \epsilon^2$. Then $\|x - ex\| = \|(x - ex)(x - ex)^*\|^{1/2} = \|ax^* - eax^*\|^{1/2} < \epsilon$.

It follows from $D(e_i) = \sum_{j=1}^m \alpha_{ij} D(f_j)$ that

$$T_c(\sum_{i=1}^n \lambda_i e_i) = T_c(\sum_{i,j} \lambda_i \alpha_{ij} f_j).$$

Hence to prove the uniqueness of T_c we may restrict ourselves to the linear combinations of mutually orthogonal projection, $\sum_{i=1}^n \lambda_i e_i$. Moreover, as D is additive on orthogonal projections, we may assume that all the coefficients λ_i are unequal. Suppose therefore

$$x = \sum_{i=1}^n \lambda_i e_i = \sum_{j=1}^m \mu_j f_j,$$

where the e 's and f 's are mutually orthogonal and the λ 's and μ 's are all different. Then $x f_j = \mu_j f_j = (\sum_{i=1}^n \lambda_i e_i) f_j$. Since the λ 's are all different, to each j there is exactly one i such that $e_i f_j = f_j$ and $\lambda_i = \mu_j$. By symmetry $e_i f_j = e_i$. Hence $\sum_{j=1}^m \mu_j f_j$ is merely a rearrangement of $\sum_{i=1}^n \lambda_i e_i$. This proves the uniqueness of T_c . If $x = \sum_{i=1}^n \lambda_i e_i$ where the e 's are mutually orthogonal and $\lambda_i > 0$, then

$$\begin{aligned} \|T_c(x)\| &= \|\sum_{i=1}^n \lambda_i D(e_i)\| \leq \max \lambda_i \|\sum_{i=1}^n D(e_i)\| \\ &\leq \max \lambda_i = \|x\|. \end{aligned}$$

Hence T_c is a bounded linear operator defined on a dense subset of C , therefore can be extended to all of C . T_c is positive because D is. If $x \in A$ is normal, x can be imbedded in a maximal commutative self-adjoint subalgebra C' of A . Let C be the intersection of all such C' , C and C' are AW^* -subalgebras of A . As x can be approximated with in both C and C' , $T_{c'}(x) = T_c(x)$. Let $T(x)$ denote their common value. T is unitarily invariant (i.e. $T(uxu^{-1}) = T(x)$ for every unitary u), because if $\sum \lambda_i e_i$ is an approximation of x then $\sum \lambda_i u e_i u^{-1}$ is one of uxu^{-1} and D is unitarily invariant. T is also linear on each commutative AW^* -subalgebra of A . We shall use this T to play the role of trace.

THEOREM 1. *Let A be an AW^* -algebra of type II_1 , Z its center. Let N be a maximal ideal of Z . Then the unique maximal ideal M of A containing N is that generated by the p -ideal P consisting of all projections e with $T(e) \in N$. Or, equivalently, M is the set of elements x with $T(x^*x) \in N$.*

Proof. Consider Z as functions on its structure space of maximal ideals. Then N contains $b \geq 0$ whenever it contains $a \geq b$; therefore P

satisfies (2) of a p -ideal. (1) follows from $T(e \vee f) = T(e) + T(f) - T(e \wedge f)$ because [2, Theorem 5.4] $e \vee f - e \sim f - e \wedge f$. Thus M is an ideal by Lemma 1. Moreover $M \neq A$ as $1 \notin P$. Let M' be a maximal ideal containing M . Then M is maximal if and only if $M_p = M'_p$. Take an $e \in M'_p$. If $e \notin P$ then $T(e) \equiv \lambda \pmod{N}$ with $\lambda > 0$. Choose an integer n and a projection f such that $T(f) = 1/n < \lambda$. f is a simple projection with central carrier 1, that is, there exist mutually orthogonal projections $f = f_1, \dots, f_n$ with $f_1 + \dots + f_n = 1$. Compare e and f ; there exists [2, Theorem 5.6] a central projection g with $ge \succ gf$ and $(1-g)e \prec (1-g)f$. Then gf and, therefore, g are in M' . As

$$0 \leq T((1-g)f) - T((1-g)e) \equiv (1/n - \lambda)(1-g) \pmod{N}$$

and $1/n - \lambda < 0$, $1-g$ is also in M' . Hence $1 \in M'$, contradicting the choice of M' . Hence $e \in P$ and $M = M'$ is maximal. The uniqueness follows from [5, Theorem 2.5].

Finally we assert that $x \in M$ if and only if $T(x^*x) \in N$. It is well known that $x \in M$ if and only if $x^*x \in M$. Thus we need only to prove that $0 < x \in M$ if and only if $T(x) \in N$. Suppose $0 < x \in M$. Given $\varepsilon > 0$ there is a projection e , which is a multiple of x , such that $\|x - ex\| < \varepsilon$. $T(e) \in N$ because $e \in M$. Then $T(ex) \leq \|x\|T(e)$ is also in N . Therefore $T(x) \in N$. Conversely, assume $T(x) \in N$, $x > 0$. Imbed x in a maximal commutative subalgebra C . Given $\varepsilon > 0$ there are projections e_1, \dots, e_n in C and positive real numbers $\lambda_1, \dots, \lambda_n$ such that

$$0 \leq x - \sum_{i=1}^n \lambda_i e_i < \varepsilon.$$

$T(e_i) \in N$ ($i=1, \dots, n$) for $\lambda_i T(e_i) \leq T(x)$. Hence $e_i \in M$ ($i=1, \dots, n$), and $x \in M$.

3. The quotient algebra A/M .

LEMMA 2. Let $\bar{e}_1, \bar{e}_2, \dots$ be a countable set of mutually orthogonal projections in A/M . There exist mutually orthogonal projections e_1, e_2, \dots in A such that $\bar{e}_n = e_n + M$, ($n=1, 2, \dots$).

Proof. By [5, Theorem 3.2] we can find a projection e_1 representing \bar{e}_1 . If x is a representative of \bar{e}_2 , so is $(1-e_1)x(1-e_1)$. Hence the proof of [5, Theorem 3.2] shows that \bar{e}_2 admits a projection representative e_2 orthogonal to e_1 . A straight forward induction yields Lemma 2.

LEMMA 3. $e \equiv f \pmod{M}$ if and only if $T(e) \equiv T(f) \equiv T(efe) \equiv T(fef) \pmod{N = M \cap Z}$. Consequently A/M satisfies the countable chain condition.

Proof. If $e \equiv f \pmod{M}$ then $0 \leq e - efe \in M$. Hence $T(e) \equiv T(efe)$

(mod N). Similarly $T(f) \equiv T(fef) \pmod{N}$. But [6, Corollary to Lemma 2.1] $efe \equiv u(fe'f)u^*$ for some unitary u . Hence $T(e) \equiv T(f) \equiv T(efe) \equiv T(fe'f) \pmod{N}$. Conversely, if $T(e) \equiv T(f) \equiv T(efe) \equiv T(fe'f)$, then $e \equiv efe$ and $f \equiv fef \pmod{M}$. As $(e-f)^*(e-f) = e-efe \equiv 0 \pmod{M}$ we have, $e \equiv fe$ and $e \equiv f \pmod{M}$. The above result permits us to define an "additive" function \bar{D} on the projections of A/M by setting $\bar{D}(\bar{e})$ to be the common value of $T(e)$ at N where $e+M=\bar{e}$. $\bar{D}(\bar{e}) \neq 0$ if $\bar{e} \neq 0$. Hence A/M satisfies the countable chain condition.

LEMMA 4. Any set of mutually orthogonal projections in A/M has a least upper bound.

Proof. By Lemma 3 such a set is countable. Let $\bar{e}_1, \bar{e}_2, \dots$ be mutually orthogonal projections. We first prove a sharpened version of Lemma 2:

(*) there exist mutually orthogonal projections e_1, e_2, \dots representing $\bar{e}_1, \bar{e}_2, \dots$, respectively, such that $T(e_n) = \bar{D}(\bar{e}_n)$ for $n=1, 2, \dots$.

Let f_1 be a projection representing \bar{e}_1 and g_1 a projection with $T(g_1) = \bar{D}(\bar{e}_1)$. Compare f_1 and g_1 ; there is a central projection h_1 such that $h_1g_1 > h_1f_1$ and $(1-h_1)g_1 < (1-h_1)f_1$. There are projections e'_1 and e''_1 such that $h_1g_1 \sim e'_1 \geq h_1f_1$ and $(1-h_1)g_1 \sim e''_1 \leq (1-h_1)f_1$. From

$$0 \leq T(e'_1 - h_1f_1) = T(e'_1) - T(h_1f_1) = h_1(T(g_1) - T(f_1)) \equiv 0 \pmod{M \cap Z}$$

it follows that $e'_1 \equiv h_1f_1 \pmod{M}$. Similarly $e''_1 \equiv (1-h_1)f_1 \pmod{M}$. Hence $e_1 = e'_1 + e''_1 \equiv f_1 \pmod{M}$ and $T(e_1) = \bar{D}(\bar{e}_1)$. Next let f_2, g_2 be projections $< 1 - e_1$ and be such that $\bar{e}_2 = f_2 + M$ and $T(g_2) = \bar{D}(\bar{e}_2)$. Repeat the argument applied to f_1 and g_1 we can find the desired $e_2 = e'_2 + e''_2$; (Since $1 - e_1 > h_2g_2$ and h_2g_2, e'_2 can be taken inside $1 - e_1$. So is e''_2 , therefore $e_1e_2 = 0$). A simple induction yields (*).

Let $e = \text{LUB}_n e_n$. We wish to prove that $\bar{e} = e + M$ is the LUB of \bar{e}_n . Or, equivalently, $\bar{f}\bar{e} = 0$ if $\bar{f}\bar{e}_n = 0$ for all n . Choose representatives f, f_1, f_2, \dots of \bar{f} so that $f_n e_n = 0$ for all $m \leq n$. Consider efe . We have

$$efe \equiv e f_n f e \equiv g_n f_n f e \equiv g_n f e \equiv g_n e f e \pmod{M}$$

where $g_n = e - e_1 - \dots - e_n$. Imbed efe in a maximal commutative self-adjoint subalgebra C and apply [2, Lemma 2.1] which states (in C): given $\varepsilon > 0$ there exists a projection h , which is a multiple of efe , such that $\|efe - hefe\| < \varepsilon$. efe is in M if all such h 's are.

$$h = e f e y \equiv g_n e f e y \equiv g_n h \pmod{M}.$$

Hence $h \equiv h g_n h \pmod{M}$ and $T(h) \equiv T(h g_n h) \pmod{M \cap Z}$. But $T(h g_n h) = T(g_n h g_n) \leq T(g_n)$ and $T(g_n)$ can be made arbitrarily small when n is

large enough. Hence $T(h) \equiv 0 \pmod{M \cap Z}$ and $h \in M$. This completes the proof.

THEOREM 2. *A/M is a finite AW^* -factor.*

Proof. To show that A/M is an AW^* -algebra we need to verify two things: (1) every set of mutually orthogonal projections has a LUB and (2) any maximal commutative self-adjoint subalgebra is generated by its projections. (1) is the context of Lemma 4. (2) is equivalent to that every element of A/M has a left and a right projection, or the left (right) annihilator of every element is a principal left (right) ideal generated by a projection. This last can be easily verified following the argument used in [2, Lemmas 2.1, 2.2, and Theorem 2.3]. As A/M is simple it must be factorial. It remains to prove the finiteness. This will be the case if we show that $\overline{D(\bar{e})} = \overline{D(\bar{f})}$ if $\bar{e} \sim \bar{f}$, since \overline{D} is non-zero on non-zero projections. This is a consequence of the following lemma, a special case of [4, Proposition 2] if A is a W^* -algebra.

LEMMA 5. *Suppose $\bar{e} \sim \bar{f}$. Then there exists equivalent projections e, f representing \bar{e}, \bar{f} respectively.*

Proof. Let $\bar{x}\bar{x} = \bar{e}$ and $\bar{x}\bar{x}^* = \bar{f}$. Let x, e_1 and f_1 respectively be the representative of \bar{x}, \bar{e} and \bar{f} . Then

$$e_1 \equiv x^*x \equiv e_1x^*xe_1 \equiv e_1x^*(xx^*)xe_1 \equiv e_1x^*f_1xe_1 \equiv (e_1x^*f_1)(f_1xe_1),$$

and

$$f_1 \equiv xx^* \equiv f_1xx^*f_1 \equiv (f_1xe_1)(e_1x^*f_1).$$

Let e be the left projection of $e_1x^*f_1$ and f the right projection of $e_1x^*f_1$. e and f are the desired projections, for

$$e = ee_1 \equiv e(e_1x^*f_1)(f_1xe_1) \equiv (e_1x^*f_1)(f_1xe_1) \equiv e_1$$

and, similarly, $f \equiv f_1$.

REMARK. If an AW^* -factor always possesses a trace, then any AW^* -algebra of type II_1 will admit a trace, for $T(x+y) - T(x) - T(y)$ takes the value 0 at every maximal ideal of Z .

REFERENCES

1. J. Dixmier, *Les anneaux d'opérateurs de classe finie*, Ann. Sci. École Norm. Sup., **66** (1949), 209-261.
2. I. Kaplansky, *Projections in Banach algebras*, Ann. of Math., **53** (1951), 235-249.

3. ———, *Algebras of type I*, Ann. of Math., **56** (1952), 460-472.
4. Y. Misonou, *On a weakly central operator algebra*, Tôhoku Math. J., **4** (1952), 194-202.
5. F. B. Wright, *A reduction for algebras of finite type*, Ann. of Math., **60** (1954), 560-570.
6. T. Yen, *Trace on finite AW*-algebras*, Duke Math. J., **22** (1955), 207-222.

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