

# Pacific Journal of Mathematics

**ON SOME SPECIAL SYSTEMS OF EQUATIONS**

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# ON SOME SPECIAL SYSTEMS OF EQUATIONS

H. H. CORSON

1. Let  $F$  be an arbitrary field. Let  $S$  be a system of equations which, when solved for two of its variables, takes the following form:

$$(1) \quad \begin{aligned} x_1^{k_1} &= f(x_3, \dots, x_n), \\ x_2^{k_2} &= g(x_3, \dots, x_n), \end{aligned}$$

where  $f$  and  $g$  are arbitrary functions of the indicated variables. Consider also the equation

$$(2) \quad y^{k_1 k_2} = f^{s k_2}(y_3, \dots, y_n) g^{r k_1}(y_3, \dots, y_n).$$

**THEOREM 1.** *If  $(k_1, k_2) = 1$  and  $r k_1 + s k_2 = 1$ , then the distinct solutions of (1) in  $F$  with  $x_1 x_2 \neq 0$  may be put in one-to-one correspondence with the distinct solutions of (2) in  $F$  with  $y \neq 0$ . Moreover, these solutions of (1),  $x_1 x_2 \neq 0$ , may be determined from the solutions of (2),  $y \neq 0$ , and conversely, by means of transformations (3) and (4) below.*

*Proof.* Assuming for the rest of this section that  $x_1 x_2 \neq 0$ ,  $y \neq 0$ , we put

$$(3) \quad \begin{aligned} x_1 &= y^{k_2} \left\{ \frac{f(y_3, \dots, y_n)}{g(y_3, \dots, y_n)} \right\}^r, \\ x_2 &= y^{k_1} \left\{ \frac{g(y_3, \dots, y_n)}{f(y_3, \dots, y_n)} \right\}^s, \\ x_i &= y_i \qquad \qquad \qquad (i=3, \dots, n) \end{aligned}$$

and notice that if  $(y, y_3, \dots, y_n)$  is a solution of (2) then (3) determines a solution of (1). Now let

$$(4) \quad \begin{aligned} y &= x_1^s x_2^r, \\ y_i &= x_i \qquad \qquad \qquad (i=3, \dots, n). \end{aligned}$$

It may be verified directly that if  $(x_1, x_2, \dots, x_n)$  is a solution of (1) then (4) determines a solution of (2). Further, given a solution  $(x_1, x_2, \dots, x_n)$  of (1) and a solution  $(y, y_3, \dots, y_n)$  of (2) with  $x_i = y_i$  ( $i=3, \dots, n$ ), then (3) implies (4) and conversely—which may be verified with the use of the relation  $r k_1 + s k_2 = 1$ .

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We note that Theorem 1 may be extended by induction to apply to a system like (1) with an arbitrary number of equations, with  $z_1^{k_1}, z_2^{k_2}, \dots, z_m^{k_m}$  as left members, and with arbitrary functions of  $z_{m+1}, \dots, z_n$  as right members if  $(k_i, k_j)=1, i \neq j$ . The argument is the same in going from  $n$  to  $n+1$  equations, and transformations corresponding to (3) and (4) may be constructed.

Use will also be made of the fact that Theorem 1 is still valid if  $x_3, \dots, x_n$  are restricted to values in  $A$ , a subset of  $F$ , as long as  $y_3, \dots, y_n$  are similarly restricted.

2. Let  $F$  now be a finite field  $GF(q), q=p^t$ . Assume  $f$  and  $g$  to be homogeneous polynomials of degrees  $m_1$  and  $m_2$  respectively, where  $(m_1, k_1)=1$  and  $(m_2, k_2)=1$ . The solutions of (2) can be determined by the following method used by Hua and Vandiver [1] and Morgan Ward [2].

As  $(k_1k_2, sk_2m_1 + rk_1m_2)=1$ , there are integers  $a, b$ , and  $c$  such that  $ak_1k_2 + b(sk_2m_1 + rk_1m_2) + c(q-1)=1$  with  $(a, q-1)=1$ . First assuming that  $y \neq 0$ , set

$$(5) \quad \begin{aligned} y &= \lambda^a \\ y_i &= \lambda^{-b} z_i \quad (i=3, \dots, n). \end{aligned}$$

Equation (2) then assumes the following form:

$$(6) \quad \lambda = f^{sk_2}(z_3, \dots, z_n) g^{rk_1}(z_3, \dots, z_n).$$

Thus every choice of  $z_3, \dots, z_n$  such that  $f \neq 0, g \neq 0$  determines a solution of (2).

Now consider the system (1). Determine as above integers  $u, v$ , and  $w$  such that  $uk_2 + vm_2 + w(q-1)=1, (u, q-1)=1$ . Assuming  $x_2 \neq 0$ , set

$$(7) \quad \begin{aligned} x_2 &= \gamma^u \\ x_i &= \gamma^{-v} t_i \quad (i=3, \dots, n). \end{aligned}$$

It is readily seen that all values of  $t_3, \dots, t_n$  such that  $f(t_3, \dots, t_n)=0$  determine solutions of the system (1) whether  $g(t_3, \dots, t_n)=0$  or not.

The same argument is valid if  $g$  is assumed zero, which proves the following.

**THEOREM 2.** *If  $f$  and  $g$  are homogeneous polynomials of degrees  $m_1$  and  $m_2$  respectively,  $(m_1, k_1)=1$  and  $(m_2, k_2)=1$ , then the total number of solutions of the system (1) in  $GF(q)$  is  $q^{n-2}$*

A similar application of Theorem 1 is the following. First let  $S$  be

$$(8) \quad \begin{aligned} x_1^{k_1} &= a_3 x_3^{em_3} + a_4 x_4^{em_4} + \dots + a_n x_n^{em_n} \\ x_2^{k_2} &= b_3 x_3^{am_3} + b_4 x_4^{am_4} + \dots + b_n x_n^{am_n} \end{aligned}$$

where  $(k_1, k_2)=1$ . Also if  $M$  is the least common multiple of  $m_3, \dots, m_n$ , assume  $(eM, k_1)=1$  and  $(dM, k_2)=1$ . In place of (5) we employ the following transformation in (2), following Carlitz [3]:

$$(9) \quad \begin{aligned} y &= \lambda^a \\ y_i &= \lambda^{-bM/m_i} z_i \quad (i=3, \dots, n), \end{aligned}$$

where  $ak_1k_2 + bM(sk_2e + rk_1d) + c(q-1)=1$ ,  $(a, q-1)=1$ . Exactly as above follows the next theorem.

**THEOREM 3.** *The total number of solutions of (8) subject to the conditions stated above is  $q^{n-2}$ .*

Also [3] suggests the following generalization of Theorem 2. Let  $f_3(x_3), f_4(x_4), \dots, f_n(x_n)$  and  $g_3(x_3), g_4(x_4), \dots, g_n(x_n)$  be homogeneous polynomials of degrees  $em_3, em_4, \dots, em_n$  and  $dm_3, dm_4, \dots, dm_n$  respectively, where now  $(x_i)=(x_{i1}, x_{i2}, \dots, x_{i s_i})$  ( $i=3, \dots, n$ ). Thus by the same argument follows the next theorem.

**THEOREM 4.** *Replacing in (8)  $x_i^{em_i}$  by  $f_i(x_i)$  and  $x_i^{am_i}$  by  $g_i(x_i)$ , ( $i=3, \dots, n$ ), then the total number of solutions of the resulting system is  $q^{s_3+\dots+s_n}$ .*

3. Now let  $F$  be the rational field and let  $f$  and  $g$  in (1) be polynomials with integral coefficients. If  $x_3, \dots, x_n$  are restricted to be integers, then  $x_1$  and  $x_2$  in any solution must be integers.

In the equation  $rk_1 + sk_2 = 1$  we may assume that  $r > 0, s < 0$ . In place of system (1) write

$$(10) \quad \begin{aligned} x_1^{k_1} &= \frac{1}{x_1^{k_1}} = \frac{1}{f(x_3, \dots, x_n)} = f'(x_3, \dots, x_n) \\ x_2^{k_2} &= g(x_3, \dots, x_n). \end{aligned}$$

we assume as in Theorem 2 that  $f$  and  $g$  are homogeneous of degrees  $m_1$  and  $m_2$  respectively,  $(m_1, k_1)=1$  and  $(m_2, k_2)=1$ . Let  $a, b$  and  $c$  satisfy  $ak_1k_2 + b(rk_1m_2 - sk_2m_1) + c(q-1)=1$ ,  $(a, q-1)=1$ ; then (5) determines a family of solutions in integers of

$$(11) \quad y^{k_1k_2} = f'^{sk_2}(y_3, \dots, y_n) g^{rk_1}(y_3, \dots, y_n),$$

$y \neq 0$ . By Theorem 1, (3) determines a family of solutions of (10) with

$x_3, \dots, x_n$  integers, and by the remark at the first of this section, a family of solutions of equations (1) with  $x_1, x_2, \dots, x_n$  integers,  $x_1 x_2 \neq 0$ . The cases where  $f$  or  $g$  is zero may be treated as in § 2, which proves the following.

**THEOREM 5.** *If  $f$  and  $g$  are homogeneous polynomials with integral coefficients of degrees  $m_1$  and  $m_2$  respectively,  $(m_1, k_1)=1$  and  $(m_2, k_2)=1$  then a family of solutions in integers may be found for equations (1) by the method above.*

See [2] for remarks on the solution of equation (11) under the above hypotheses. Note especially the above method does not in general give all solutions.

I should like to thank Professor L. Carlitz for his very helpful interest in this material.

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