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1. Introduction. Let G be a separable Lie group and let V be a complete, metrizable, topological vector space. The underlying space of G is a separable real analytic manifold so that we can define, by the methods of L. Schwartz (see [7], [12], [13]), the spaces $\mathcal{E}(V)$ of indefinitely differentiable maps of G into V , and $\mathcal{D}(V)$ which consists of those maps in $\mathcal{E}(V)$ which are of compact carrier. Their duals are $\mathcal{D}'(V)$, the space of distributions on G with values in V' (the dual of V), and $\mathcal{E}'(V)$ which is the space of distributions of compact carrier with values in V' .

By using the group structure in G , we can define the convolution $S * f \in \mathcal{E}(C)$ for any $S \in \mathcal{D}'(V)$, $f \in \mathcal{D}(V)$, where C is the complex plane. The main result of this paper is: Let $S \in \mathcal{D}'(V)$ have the property that $S * f \in \mathcal{D}(C)$ whenever $f \in \mathcal{D}(V)$; then $S \in \mathcal{E}'(V)$. Moreover, the topology of $\mathcal{E}'(V)$ is that obtained by considering each $S \in \mathcal{E}'(V)$ as defining the continuous linear transformation $f \rightarrow S * f$ of $\mathcal{D}(V) \rightarrow \mathcal{D}(C)$ and then giving this set of transformations the compact-open topology (see [6]). This generalizes the result of [6] in case G is a vector group and $V=C$.

This result is generalized to double coset spaces $L \backslash G / K$ where L and K are compact subgroups of G . In this form, the result will be used by the author and F. I. Mautner to generalize the Paley-Wiener theorem and the theory of mean-periodic functions of Schwartz (see [8]).

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2. Distributions on G . Instead of using the usual method of defining distributions on G , as for example in de Rham and Kodaira [12], we shall follow another approach which is more akin to the author's thesis [5]. We shall show that the two methods are equivalent.

By "function" we shall mean "complex-valued function" unless the contrary is specifically stated. "Linear" will mean "linear over the complex numbers" always. By 1 we denote the identity in G , and by \mathfrak{g} we denote the Lie algebra of G . For any $Y \in \mathfrak{g}$, we denote by $t \rightarrow \exp(tY)$ the unique one parameter subgroup in G whose direction

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at 1 is Y . Let V be a complete metrizable locally convex topological vector space.

The map f of G into V is said to be differentiable in the direction $Y \in \mathfrak{g}$ at $x \in G$ if $\left\{ \left(\frac{df}{dt} \right) [(\exp tY)x] \right\}_{t=0}$ exists; if this is the case, we set

$$(1) \quad (D_Y f)(x) = \left\{ \left(\frac{df}{dt} \right) [(\exp tY)x] \right\}_{t=0}.$$

If f is a continuous map of G into V , we say that f is in the domain of D_Y if, for any $x \in G$, f is continuously differentiable in the direction Y at x . $D_Y f$ is then defined as the (continuous) map $x \rightarrow (D_Y f)(x)$.

By \mathcal{E}^0 we denote the space of continuous maps of G into V with the topology of uniform convergence in V on the compact sets of G . By the *carrier* of an $f \in \mathcal{E}^0$ we mean the closure of the set of points where $f \neq 0$. An *operator* on G is a linear mapping of a subspace of \mathcal{E}^0 into \mathcal{E}^0 . The operator D is said to be *closed* if the conditions: $\{f_\alpha\}$ in the domain of D , $f_\alpha \rightarrow f$ and $Df_\alpha \rightarrow h$ in \mathcal{E}^0 , imply f is in the domain of D and $Df = h$.

PROPOSITION 1. *For any Y in \mathfrak{g} , D_Y is a closed operator.*

Proof. It is clear that D_Y is an operator.

It remains to show that D_Y is closed. Let $\{f_i\}$ be a sequence of functions in the domain of D_Y such that $\{f_i\}$ and $\{D_Y f_i\}$ are Cauchy sequences in \mathcal{E}^0 ; call $f = \lim f_i$, $h = \lim D_Y f_i$, the limits being taken in \mathcal{E}^0 . Let Y, X_2, X_3, \dots, X_n be a basis for \mathfrak{g} and N an open neighborhood of 1 in G in which $\exp(t_1 Y) \exp(t_2 X_2) \cdots \exp(t_n X_n)$ form a coordinate system. It is clearly sufficient to prove that f is in the domain of D_Y at 1 and that $(D_Y f)(x) = h(x)$ for any $x \in N$.

Now, $\theta: (t_1, t_2, \dots, t_n) \rightarrow (\exp(t_1 Y), \exp(t_2 X_2), \dots, \exp(t_n X_n))$ maps a circular neighborhood M of 0 in real Euclidean n -space homeomorphically onto N . It is immediate from the definitions that a continuous map p of G into V is differentiable in the direction Y at 1 if and only if $p\theta$ has a continuous partial derivative in the direction t_1 at 0, and then

$$(D_Y p)(x) = \left(\frac{\partial p\theta}{\partial t_1} \right) (x)$$

for all x in a suitable neighborhood of 1. From this and the known closure of $\partial/\partial t_1$ on Euclidean space, our assertion follows.

Now, let Y_1, Y_2, \dots, Y_n be a basis for \mathfrak{g} . We set

$$D_1 = D_{Y_1}, D_2 = D_{Y_2}, \dots, D_n = D_{Y_n}$$

and we call \mathfrak{D} the family (D_1, D_2, \dots, D_n) so \mathfrak{D} is a family of closed operators. By means of D we can now define, by the methods of [5], the complete, locally convex, Hausdorff, topological vector spaces \mathcal{D} (or $\mathcal{D}(V)$) of indefinitely differentiable maps of compact carrier of G into V , and \mathcal{E} (or $\mathcal{E}(V)$) of all indefinitely differentiable maps of G into V . \mathcal{E} is a metrizable space; a sequence $\{f_i\}$ converges to zero in \mathcal{E} if and only if for any operator $D^* = D_{j_1} D_{j_2} \dots D_{j_r}$, $D_{j_m} \in \mathfrak{D}$, $Df \rightarrow 0$ uniformly in V on every compact set of G . The topology of \mathcal{D} may be described as follows: For each compact set K , let \mathcal{D}_K be the subspace of \mathcal{D} consisting of those maps of \mathcal{D} which have their carriers in K ; the topology of \mathcal{D}_K is that induced by \mathcal{E} . Then of all possible locally convex topologies which induce on each \mathcal{D}_K the topology of \mathcal{D}_K that may be given to the set of functions of \mathcal{D} , \mathcal{D} is given the strongest (see [4]).

PROPOSITION 2. *The spaces \mathcal{D} and \mathcal{E} are the same as those we would have obtained by considering G as an indefinitely differentiable manifold.¹*

Proof. Let N be a neighborhood of 1 in G in which $(\exp t_1 Y_1 \exp t_2 Y_2 \dots \exp t_n Y_n)$ form a coordinate system. Then it is clearly sufficient to prove the theorem for the restrictions of the functions of \mathcal{E} and \mathcal{D} to N . The result now follows by the method of the proof of Proposition 1.

PROPOSITION 3. *\mathcal{D} and \mathcal{E} are reflexive topological spaces.²*

Proof. We prove the theorem first for \mathcal{E} . Since \mathcal{E} is metrizable, it is sufficient to prove that \mathcal{E} is a Montel space, that is, that the bounded sets of \mathcal{E} are relatively compact (of compact closure). Let then B be a bounded set in \mathcal{E} . Let N be a compact neighborhood of 1 in G in which $(\exp t_1 Y_1 \exp t_2 Y_2 \dots \exp t_n Y_n)$ form a coordinate system. Since G is separable, we can find a sequence of points $a_i \in G$ such that $G = \bigcup (\text{interior } Na_i)$.

It is easily seen that it is sufficient to show that, for any i , and for any integers r_1, r_2, \dots, r_m , if we set $D^* = D_{r_1} D_{r_2} \dots D_{r_m}$, then the set $\{D^* f\}_{f \in B}$ is equicontinuous on $a_i N$. It follows immediately as in the proof of Proposition 1 that the restrictions of the maps $D^* f$ have the property that (if we identify them with maps on a circular neighborhood of zero in Euclidean n -space) their partial derivatives in all direc-

¹ That is, by applying the method of de Rham and Kodaira [12].

² See [3].

tions are uniformly bounded for $f \in B$. As is well-known, this implies the equicontinuity of $\{D^*f\}_{f \in B}$ on $a_i N$; hence Proposition 3 is established as regards the space \mathcal{E} .

If L is a bounded set in \mathcal{D} , then all the maps of L have their carriers in a fixed compact set K of G , that is, $L \subset \mathcal{D}_K$. Since the topology induced by \mathcal{D} on \mathcal{D}_K is also the topology induced by \mathcal{E} on \mathcal{D}_K , L is bounded in \mathcal{E} . Thus, L is relatively compact in \mathcal{E} , hence in \mathcal{D}_K , hence also in \mathcal{D} which concludes the proof of Proposition 3.

A sequence of open, relatively compact (that is, of compact closure) sets $K_i \subset G$ will be called a *scattered resolution* of G (see [5]) if $\bigcup K_i = G$ and if, given any compact set $K \subset G$, only a finite number of the K_i meet K . Given any scattered resolution $\{K_i\}$ of G , there exists a *partition of unity* $\{h_i\}$ relative to it; by this is meant that the indefinitely differentiable functions h_i have the properties that:

1. For each i , *carrier* $h_i \subset K_i$.
2. For any $x \in G$, $\sum h_i(x) = 1$.

(This sum has meaning because all but a finite number of terms are zero.) To establish the existence of the partition of unity $\{h_i\}$, we have only to note that the scattered resolution $\{K_i\}$ can be "refined" to a scattered resolution $\{L_i\}$ by coordinate neighborhoods (that is, each K_i is contained in a union of a finite number of L_j). The existence of a partition of unity relative to $\{L_i\}$ is readily verified and, in turn, implies immediately the existence of a partition of unity relative to $\{K_i\}$.

By \mathcal{D}' (or $\mathcal{D}'(V)$) we denote the dual of \mathcal{D} with the topology of uniform convergence on the bounded (compact) sets of \mathcal{D} . It can be shown (see [7]) that, \mathcal{D}' can also be described as the space of continuous linear maps of $\mathcal{D}(C) \rightarrow V'$, this space of maps being given the compact-open topology. For this reason, \mathcal{D}' is usually called the space of *distributions on G with values in V'* . In this paper, we shall call the elements of \mathcal{D}' *distributions*.

For any distribution S , and any open set O in G , we say that S vanishes on O if $S \cdot f = 0$ for any $f \in \mathcal{D}$ whose carrier is contained in O . Because of the existence of partitions of unity, we can easily show that if S vanishes on O_α where O_α are open sets, then S vanishes also on $\bigcup O_\alpha$. Thus there is a largest open set on which S vanishes. The *carrier* of S is defined as the complement of this set.

\mathcal{E}' (or $\mathcal{E}'(V)$) is the dual of \mathcal{E} . It can be shown, as in [13], that \mathcal{E}' consists of all distributions of compact carrier.

For any $S \in \mathcal{D}'$, by \bar{S} is meant the distribution $f \rightarrow \overline{S \cdot f}$ for $f \in \mathcal{D}$,

where $\overline{f(x)} = \overline{f(x)}$ for any $x \in G$.

By $G \times G$ we denote the direct product of G with itself; $G \times G$ is again a Lie group whose underlying manifold is the Cartesian product of the underlying manifold of G with itself. By ${}_2\mathcal{D}, {}_2\mathcal{E}, {}_2\mathcal{D}', {}_2\mathcal{E}'$ we denote the spaces on $G \times G$ corresponding to $\mathcal{D}, \mathcal{E}, \mathcal{D}', \mathcal{E}'$ respectively.

Let k be a continuous map on $G \times G$ and $x \in G$. Then by $k_{x_1=x}$ we mean the map on $G: y \rightarrow k(x, y)$. Suppose that, for all $x \in G$, $k_{x_1=x}$ is in a space U of mappings on G . Then by k_1 we mean the mapping $x \rightarrow k_{x_1=x}$ of $G \rightarrow U$. Let L be a map defined on U ; then we say that k is in the domain of L , and we denote by L_2k the map

$$x \rightarrow Lk_{x_1=x}$$

for $x \in G$. If the range of L is again a space of mappings on G , then we say also that k is in the domain of L_{2l} and we shall denote by $L_{2l}k$ the mapping on $G \times G$:

$$(x, y) \rightarrow Lk_{x_1=x}(y).$$

L_{2l} is called the *lift* of L to $G \times G$. We define $k_{x_2=x}, k_2, L_1, L_{1l}$ similarly.

We can now define, as in [5], two products involving distributions and functions:

For any $S \in \mathcal{D}', k \in {}_2\mathcal{D}$, then we have two *inner products*: S_1k and S_2k which are both in \mathcal{D} .

For any $S, U \in \mathcal{D}'$ we define the *direct products* $S_1 \times U_2$ and $S_2 \times U_1 \in {}_2\mathcal{D}'$ by

$$S_1 \times U_2 \cdot k = S \cdot U_2k, \quad S_2 \times U_1 \cdot k = S \cdot U_1k$$

for any $k \in {}_2\mathcal{D}$.

The direct products define continuous bilinear maps which are commutative, while the inner products are only separately continuous bilinear maps. (If V, W, X are topological vector spaces and $t: V \times W \rightarrow X$ is a bilinear map, then t is called separately continuous (see [4], [5]) if, for B, B' any bounded sets in V, W respectively, the maps

$$w \rightarrow t(b, w), \quad v \rightarrow t(v, b')$$

are, for $b \in B, b' \in B'$, equicontinuous linear maps of $W \rightarrow X$ and $V \rightarrow X$ respectively.)

By $\{Q_i\}$ we shall denote an enumeration of the operators $D_{r_1}D_{r_2} \cdots D_{r_m}$ with $Q_1 = \text{identity}$.

For f a continuous map defined on G , \check{f} is the map $x \rightarrow f(x^{-1})$.

We shall denote by η the function on G defined by $dxg = \eta(g)dx$, where dx is a left invariant Haar measure. It is known that $\eta \in \mathcal{E}(C)$

and, moreover, η is a homomorphism on G . By ω we denote the function on G defined by $dx^{-1}=\omega(x)dx$. Again, $\omega \in \mathcal{E}(C)$ and ω is a homomorphism on G . It is readily verified that $\omega(y)=\eta(y^{-1})$ for any $y \in G$. For any $S \in \mathcal{D}'$, we write $\check{S} \cdot f = S \cdot \omega \check{f}$ for any $f \in \mathcal{D}$.

3. Convolution on G . For any continuous map f of G into V and any $x \in G$ we define the translations

$$(\mathfrak{L}(x)f)(y) = f(x^{-1}y) \quad (\mathfrak{R}(x)f)(y) = f(yx)$$

for any $y \in G$.

PROPOSITION 4. $(x, f) \rightarrow \mathfrak{L}(x)f$ and $(x, f) \rightarrow \mathfrak{R}(x)f$ are continuous maps of $G \times \mathcal{D} \rightarrow \mathcal{D}$ and also of $G \times \mathcal{E} \rightarrow \mathcal{E}$.

Proof. We shall establish the theorem for the map $(x, f) \rightarrow \mathfrak{L}(x)f$ of $G \times \mathcal{D} \rightarrow \mathcal{D}$; the other parts of the proposition may be established by similar methods. By the results of Dieudonné and Schwartz (see [4], [5]) it is sufficient to prove that this is a continuous map of $G \times \mathcal{D}_K \rightarrow \mathcal{D}$ for any compact set K of G . Since the map is linear in f and a homomorphism in x , it is sufficient to prove continuity at $f=0$ and $x=1$. Let K be a given compact set in G and choose K' a compact set in G so large that K' contains the carriers of all $\mathfrak{L}(x)f$ for $x \in \mathcal{D}_K$. Let M be a neighborhood of zero in $\mathcal{D}_{K'}$. Then we can find operators Q_1, Q_2, \dots, Q_r , and continuous semi-norms $\rho_1, \rho_2, \dots, \rho_n$ on V , and a positive number a so that M contains the set of $h \in \mathcal{D}_{K'}$ which satisfy

$$\max_{y \in G, i} \rho_i[(Q_j h)(y)] \leq a$$

for $j=1, 2, \dots, r$.

For any $p \in \mathcal{D}$, any k , and $x, z \in G$,

$$\begin{aligned} (D_k \mathfrak{L}(z)p)(x) &= \left\{ \left[\left(\frac{d}{dt} \right) \mathfrak{L}(z)p \right] [(\exp tY_k)x] \right\}_{t=0} \\ &= \left\{ \left[\left(\frac{d}{dt} \right) p \right] [z^{-1}(\exp tY_k)x] \right\}_{t=0} \\ &= \left\{ \left(\frac{dp}{dt} \right) [z^{-1}(\exp tY_k)zz^{-1}x] \right\}_{t=0} \\ &= \left\{ \left(\frac{dp}{dt} \right) [(\exp tz^{-1}Y_k z)z^{-1}x] \right\}_{t=0}. \end{aligned}$$

Now, write $z^{-1}Y_k z = \Sigma c_{ki}(z)Y_i$ where (c_{ki}) is the matrix of the adjoint representation of G on \mathfrak{g} . Then we have

$$(D_z \mathfrak{L}(z)p)(x) = (D_z^{-1} \mathfrak{L}_z f)(z^{-1}x).$$

We also have

$$D_z^{-1} \mathfrak{L}_z p = \sum c_{kj}(z) D_j p.$$

The functions c_{kj} are continuous and even indefinitely differentiable on G . Hence, we can find an $A > 0$ so that

$$\max_{z \in K} |c_{kj}(z)| \leq A$$

for all k, j .

It follows immediately from this that we can be assured that, for $q \in \mathcal{D}_K, z \in K$,

$$\max_{x \in G, i} \rho_i[(D_K \mathfrak{L}(z)q)(x)]$$

will be small by making

$$\max_{x \in G, j, i} \rho_i[(D_j q)(x)]$$

sufficiently small. Proposition 4 now follows by iteration, since each Q_i is of the form $D_{r_1} D_{r_2} \cdots D_{r_m}$.

For any continuous map f on G , $\mathfrak{L}f$ is the map on $G \times G: (x, y) \rightarrow f(x^{-1}y)$; \mathfrak{L}^*f is the map on $G \times G: (x, y) \rightarrow f(xy)$. By the method of proof of Proposition 1, we can establish

PROPOSITION 5. $f \rightarrow \mathfrak{L}f$ and $f \rightarrow \mathfrak{L}^*f$ are continuous linear maps of $\mathcal{E} \rightarrow_2 \mathcal{E}$.

We are now in a position to define the convolution product involving distributions and functions. The definition differs slightly from that of Schwartz [13]: For any $S \in \mathcal{D}'$, $f \in \mathcal{D}$, $x \in G$, we set

$$(1) \quad (S * f)(x) = \bar{S} \cdot \mathfrak{L}(x)f$$

This formula can also be considered valid if $S \in \mathcal{E}'$ and $f \in \mathcal{E}$.

PROPOSITION 6. $(S, f) \rightarrow S * f$ is a separately continuous map of

- (a) $\mathcal{E}' \times \mathcal{E} \rightarrow \mathcal{E}(C)$
- (b) $\mathcal{E}' \times \mathcal{D} \rightarrow \mathcal{D}(C)$
- (c) $\mathcal{D}' \times \mathcal{D} \rightarrow \mathcal{E}(C)$.

which is antilinear in S and linear in f .

Proof. (a) Let j be fixed and write $A=D_j$. We find from the definitions that, for $S \in \mathcal{E}'$, $f \in \mathcal{E}$, $S * f \in \mathcal{E}(C)$ and, moreover,

$$(2) \quad [A(S * f)](x) = \bar{S} \cdot (A_{1/} \mathfrak{L} \check{f})_{x_1=x}.$$

From this it follows by iteration that, for any $Q=Q_s$, we have

$$(3) \quad [Q(S * f)](x) = \bar{S} \cdot (Q_{1/} \mathfrak{L} \check{f})_{x_1=x}.$$

Part (a) results immediately from (3) together with Proposition 5.

(b) By a result of Dieudonné and Schwartz (see [4]) it is sufficient to prove that, for K a compact set in G , $(S, f) \rightarrow S * f$ is a separately continuous map of $\mathcal{E}' \times \mathcal{D}_K \rightarrow \mathcal{D}$. Now, it is obvious that

$$(4) \quad (\text{carrier } S * f) \subset (\text{carrier } S)(\text{carrier } f).$$

Our assertion now follows from (a) above and the fact that \mathcal{D}_K has the topology induced by \mathcal{E} .

(c) This is proven by essentially the same reasoning as that employed in the proof of (a) above.

4. \mathcal{E} as a space of linear transformations. In this section we shall prove our main result.

THEOREM 1. *Let $S \in \mathcal{D}'$ have the property that $S * f \in \mathcal{D}(C)$ whenever $f \in \mathcal{D}$; then $S \in \mathcal{E}'$.*

Proof. Let us suppose that S satisfies the hypotheses of Theorem 1, and let K be a fixed compact set in G . We shall show first that there exists a compact set $K' \subset G$ such that $S * \mathcal{D}_K \subset \mathcal{D}_{K'}$. Assume this is not the case, and let $\{K_i\}$ be a compact exhaustion of G . (That is, each K_i is a compact set which is the closure of a nonempty open set. Moreover, $K_i \subset K_{i+1}$ and $\bigcup K_i = G$.) We shall produce a sequence $\{g_i\}$ with the following properties:

1. Each $g_i \in \mathcal{D}_K$.
2. $\sum g_i$ converges in \mathcal{D}_K .
3. There is a sequence of positive numbers m_i with $m_{i+1} - m_i \geq 1$ for all i such that

$$\begin{aligned} \text{carrier } (S * g_i) &\subset K_{m_i} \\ \text{carrier } (S * g_{i+1}) &\not\subset K_{m_i}. \end{aligned}$$

4. There is a sequence of points $a_i \in G$ such that a_i is a point of $K_{m_i} - K_{m_{i-1}}$ (where K_{m_0} is the empty set) for which $(S * g_i)(a_i) \neq 0$ and

$$|(S * g_{i+k})(a_i)| \leq \frac{1}{3^k} |(S * g_i)(a_i)|$$

for all $k > 0$.

Suppose that the sequences $\{g_i\}$, $\{m_i\}$, $\{a_i\}$ can be found. Then for any $i > 1$,

$$\begin{aligned} |(S * \Sigma g_j)(a_i)| &= |\Sigma(S * g_j)(a_i)| \\ &= |\Sigma_{j \geq i}(S * g_j)(a_i)| \\ &\geq |(S * g_i)(a_i)| - \sum_{j > i} |(S * g_j)(a_i)| \\ &\geq |(S * g_i)(a_i)| \left[1 - \sum_{j \geq 1} \frac{1}{3^j} \right] \\ &= \frac{1}{2} |(S * g_i)(a_i)| \\ &> 0. \end{aligned}$$

Since the set $\{a_i\}$ is clearly not contained in any compact set of G , we conclude that $S * \Sigma g_j$ is not of compact carrier, which contradicts our hypothesis.

It remains to define the sequences $\{g_i\}$, $\{m_i\}$, and $\{a_i\}$. Let $g_1 \in \mathcal{D}_K$ be chosen so that $S * g_1 \neq 0$. Let a_1 be any point in G for which $(S * g_1)(a_1) \neq 0$, and choose $m_1 > 0$ so that

$$\text{carrier}(S * g_1) \subset K_{m_1}.$$

Assume that $g_1, \dots, g_k, a_1, \dots, a_k, m_1, \dots, m_k$ have been defined with the required properties; we shall now define $g_{k+1}, a_{k+1}, m_{k+1}$. Now, by our assumption, there is an $f \in \mathcal{D}_K$ such that

$$\text{carrier}(S * f) \not\subset K_{m_k+1}.$$

Let m_{k+1} be chosen so that $\text{carrier}(S * f) \subset K_{m_{k+1}}$, and let a_{k+1} be some point in $K_{m_{k+1}} - K_{m_k}$ such that $(S * f)(a_{k+1}) \neq 0$. Define

$$g_{k+1} = \frac{f}{\max_{x \in G} (1, \max_{i \leq k} |f(x)| 3^{k+1}) \max_{j, i \leq k} (1, \max_{x \in G} [(Q_i f)(x)], [(S * g_i)(a_i)]^{-1})}$$

The sequences $\{g_i\}$, $\{m_i\}$, $\{a_i\}$ are thus defined. It is clear that conditions 1, 3, 4 are satisfied. Further, each $g_i \in \mathcal{D}_K$ and, for R any semi-norm on D_K of the kind used to define the topology of that space, it is clear that

$$\Sigma R(g_i) > \infty.$$

Thus Σg_i converges in \mathcal{D}_K .

To complete the proof of Theorem 1, let us assume that S is not of compact carrier, and let K be a given compact symmetric neighborhood of 1 in G . It is clear that we can choose an open set U in G such that S does not vanish on U and such that $U \cap K'$ is empty, where K' is a compact symmetric set such that $\bar{S} * \mathcal{D}_K \subset \mathcal{D}_{K'}$. It follows easily that we can find a $g \in G$, and an $f \in \mathcal{D}$ such that *carrier* $f \subset Kg \subset U$, $S \cdot f \neq 0$.

On the other hand, by definition,

$$S \cdot \check{f} = (\bar{S} * \mathfrak{L}(g)f)(g^{-1}).$$

But, *carrier* $f \subset Kg$ implies *carrier* $\mathfrak{L}(g)f \subset K$ because K is symmetric. Also, $g \notin K'$ because $1 \in K$ and $K' \cap U$ is empty. Since K' is symmetric, also $g^{-1} \notin K'$. Thus, $S \cdot \check{f} = (\bar{S} * \mathfrak{L}(g)f)(g^{-1}) = 0$; this contradiction completes the proof of Theorem 1.

The set of distributions of \mathcal{E}' forms a vector space of continuous linear mappings of $\mathcal{D} \rightarrow \mathcal{D}$ under convolution; we give this space the compact-open topology (see [6]) and obtain a topological vector space J . A fundamental system of neighborhoods of zero in J consists of all sets N for which we can find a compact set K in \mathcal{D} and a neighborhood of zero M in \mathcal{D} so that N consists of those $S \in \mathcal{E}'$ with $S * h \in M$ for all $h \in K$. By Proposition 1 of § 5 of [6], we would have obtained the same topologies if we had considered the distributions of \mathcal{E}' as defining, under convolution, continuous linear maps of $\mathcal{D}' \rightarrow \mathcal{D}'$.

THEOREM 2. *The natural map $u: \mathcal{E}' \rightarrow J$ is a topological isomorphism onto.*

Proof. u is clearly one-to-one, linear, and onto. Moreover, J is given the weakest topology to make the maps

$$S \rightarrow (u^{-1}S) * f$$

of $J \rightarrow \mathcal{D}$ equicontinuous for f in any compact set of \mathcal{D} ; by Proposition 6 this implies that u is continuous.

Since u^{-1} is linear, we need verify continuity only at zero. Let T be a neighborhood of zero in \mathcal{E}' ; there is a bounded set $\beta \subset \mathcal{E}$ so that T contains the set of $S \in \mathcal{E}'$ which satisfy $|S \cdot b| \leq 1$ for all of $b \in \beta$.

Let K be an open symmetric neighborhood of 1 in G whose closure is compact. Then it is clear that we can find a sequence of points $a_i \in G$ such that $\{a_i K\}$ is a scattered resolution of G (see § 2). We can also insure that, if a is one of the a_i , so is a^{-1} . Let $\{h\}$ be a partition of unity relative to this scattered resolution (see § 2). It is readi-

ly verified by the method of proof of Proposition 1 of § 3 that, for each i , the set B_i of functions $\mathcal{L}(a_i)(h_i f)$ for $f \in \beta$ is bounded in \mathcal{D} . For each j there is a double sequence $s_j = M_{jik}$ of positive numbers so that B_j is contained in the bounded (in \mathcal{D}) set L_j of all $g \in D$ whose carriers are contained in K and which satisfy

$$\max_{x \in G} \rho_k[(Q_i g)(x)] \leq M_{jik}$$

for all i . From the denumerable number of double sequences s_j we construct a double sequence $s = \{M_{ik}\}$ of positive numbers such that, for each j , $M_{jik} \leq M_{ik}$ for all but a finite number of i, k . Hence, for each j , we can find an $e_j > 0$ so that $e_j M_{jik} \leq M_{ik}$ for all i, k ; we can clearly make $e_j = e_i$ if $a_j = a_i^{-1}$.

Let A be the set of $f \in \mathcal{D}$ for which

1. carrier $f \subset K$
2. $\max_{x \in G} \rho_k[(Q_i f)(x)] \leq M_{ik}$ for all i, k ,

so A is bounded in \mathcal{D} . Let M be the neighborhood of zero in \mathcal{D} consisting of those $h \in \mathcal{D}$ with

$$\max_{x \in a_j K} \rho_k h(x) \leq e_j d_j$$

for all j , where d_j are positive numbers which satisfy $\sum d_j = 1$. Call N the set of $S \in J$ with $S * f \in M$ for all $f \in A$, so N is a neighborhood of zero in J ; we claim that $u^{-1}(N) \subset T$.

Let us assume this is not the case; then we can find an $S \in N$ with $u^{-1}S \notin T$, that is, $S \in N$ but

$$|u^{-1}S \cdot f| > 1$$

for some $f \in \beta$. Now, $u^{-1}S$ is of compact carrier; thus we can find an r such that

$$\sum_{k=1}^r h_k(x) = 1$$

for any $x \in \text{carrier}(u^{-1}S)$. Hence

$$(5) \quad |u^{-1}S \cdot f| \leq |u^{-1}S \cdot h_1 f| + |u^{-1}S \cdot h_2 f| + \dots + |u^{-1}S \cdot h_r f|.$$

It is clear from the definitions that, for each i ,

$$e_i \mathcal{L}(a_i)(h_i f) \in A.$$

Thus,

$$h_i f = \frac{1}{e_i} \mathfrak{L}(a_i^{-1})g$$

for some $g \in A$, which gives, for $i=1, 2, \dots, r$,

$$\begin{aligned} |u^{-1}S \cdot h_i f| &= \frac{1}{e_i} |u^{-1}S \cdot \mathfrak{L}(a_i^{-1})g| \\ &= \frac{1}{e_i} |(\overline{u^{-1}S} * g)(a_i^{-1})| \\ &\leq \frac{1}{e_i} e_j d_j, \end{aligned}$$

where $a_j = a_i^{-1}$, because $g \in A$ and $u^{-1}S \in N$. Now, since $e_i = e_j$, we have

$$|u^{-1}S \cdot h_i f| \leq d_j.$$

Applying this to equation (5) we obtain

$$|u^{-1}S \cdot f| \leq d_{1'} + d_{2'} + \dots + d_{r'} \leq 1$$

(where we set $a_{j'} = a_j^{-1}$). This contradiction proves the theorem.

5. Extension of the main result. We assumed in §§ 2, 3, 4 that V is metrizable. In case V is not metrizable, then the spaces \mathcal{E} and \mathcal{D} can be defined as before, but E is no longer metrizable, and \mathcal{D} is not an \mathcal{LF} space in the sence of Dieueonné and Schwartz [4]. However, there is no difficulty in extending the definition and continuity properties of the convolution product to this case. Theorem 1 can be extended to this case, but the proof of Theorem 2 does not extend to the case of V not metrizable. All that can be proven (and the proof is much simpler than the proof of Theorem 2 above) is that u is continuous and that u^{-1} is sequentially continuous and takes bounded sets into bounded sets. The continuity of u^{-1} is an open question.

We assume in the following that V is a complete, locally convex, Hausdorff, topological vector space. By V^* we denote the space of continuous linear maps of V into V with the compact-open topology, so V^* is again a complete, locally convex, Hausdorff, topological vector space.

Let K and L denote compact subgroups of G . By a *representation of K on V* we mean a continuous homomorphism U of K into V^* . Let U and W be representations of V of K and L respectively. By ${}_{UW}\mathcal{D}$ we denote the space of those $f \in \mathcal{D}(V^*)$ for which

$$(6) \quad \mathfrak{L}(k^{-1})\mathfrak{R}(l)f = U(k)fW(l)$$

for any $k \in K, l \in L$. We give ${}_{UW}\mathcal{D}$ the topology induced by \mathcal{D} . ${}_{UW}\mathcal{E}$

is defined similarly.

For any $T \in \mathcal{D}'(V^*)$, $g \in G$, we define $\mathfrak{L}(g)T$ and $\mathfrak{R}(g)T$ as the distributions

$$(7) \quad \mathfrak{L}(g)T \cdot f = T \cdot \mathfrak{L}(g^{-1})f, \quad \mathfrak{R}(g)T \cdot f = \eta(g)T \cdot \mathfrak{R}(g^{-1})f$$

for any $f \in \mathcal{D}(V^*)$. (η was defined in § 2.) Let us denote by ${}_{UW}\mathcal{D}'$ the space of all $S \in \mathcal{D}'(V^*)$ which satisfy

$$(8) \quad \mathfrak{L}(k)\mathfrak{R}(l)S \cdot f = S \cdot U(k^{-1})f W(l)$$

for any $f \in \mathcal{D}(V^*)$, $k \in K$, $l \in L$.³ We shall write $U(k)SW(l^{-1}) \cdot f$ for the right side of (8). We give ${}_{UW}\mathcal{D}'$ the topology induced by $\mathcal{D}'(V^*)$. ${}_{UW}\mathcal{E}'$ is defined similarly.

We can easily show

PROPOSITION 7.

$$f \rightarrow P_{UW}f = \int_{K \times L} U(k)\mathfrak{L}(h)\mathfrak{R}(l)f V^{-1}(l)dkdl$$

(where dk and dl are the respective Haar measures on K and L so normalized that $\int_K dk = \int_L dl = 1$) defines continuous open projections of $\mathcal{D}(V^*)$ onto ${}_{UW}\mathcal{D}$ and $\mathcal{E}(V^*)$ onto ${}_{UW}\mathcal{E}$. Also

$$S \rightarrow P_{UW}S = \int_{K \times L} \mathfrak{L}(k^{-1})\mathfrak{R}(l)[U^{-1}(k)SV(l)]dkdl$$

defines continuous open projections of $\mathcal{D}'(V^*)$ onto ${}_{UW}\mathcal{D}'$ and $\mathcal{E}'(V^*)$ onto ${}_{UW}\mathcal{E}'$.

COROLLARY. ${}_{UW}\mathcal{D}'$ is the dual of ${}_{UW}\mathcal{D}$ and ${}_{UW}\mathcal{E}'$ is the dual of ${}_{UW}\mathcal{E}$.

Proof. This is an immediate consequence of Proposition 6 and the fact that, for $S \in \mathcal{D}'$, $f \in \mathcal{D}$ (or for $S \in \mathcal{E}'$, $f \in \mathcal{E}$), we have $P_{UW}S \cdot f = S \cdot P_{UW}f$.

Suppose that $K=L$; then we see easily that the convolution defined in § 3 defines a separately continuous bilinear map of ${}_{UW}\mathcal{D}' \times {}_{WZ}\mathcal{D} \rightarrow {}_{UW}\mathcal{E}(C)$ (where U, W, Z are representations of K on V). The method of proof of Theorems 1 and 2 can be used to show.

THEOREM 3. ${}_{UW}\mathcal{E}'$ consists of all $S \in {}_{UW}\mathcal{D}'$ such that $S * f \in {}_{UW}\mathcal{D}$ for any $f \in {}_{WZ}\mathcal{D}$. The topology of ${}_{UW}\mathcal{E}'$ is sequentially the same as that obtained by considering the elements of ${}_{UW}\mathcal{E}'$ as defining (by convolution) continuous linear maps of ${}_{WZ}\mathcal{D} \rightarrow {}_{UW}\mathcal{D}$ and giving this set the

³ Note that since L is compact, the restriction of η to L is 1.

compact-open topology τ . Moreover, the bounded sets of ${}_{UW}\mathcal{E}'$ are the same as those of τ .

REMARK 1. We do not know whether the topologies τ and that of ${}_{UW}\mathcal{E}'$ are the same. The difficulty is that, for $f \in {}_{WW}\mathcal{D}$, $g \in G$, $\mathfrak{L}(g)f$ is no longer in ${}_{WW}\mathcal{D}$.

REMARK 2. In case that $K=L$, V is finite dimensional, and U, W, X, Z are irreducible unitary representations of K on V , then it follows easily from the Schur orthogonality relations that $S * f = 0$ for any $S \in {}_{UW}\mathcal{E}'$ $f \in {}_{XZ}\mathcal{D}$ if W is not equivalent to X .

REMARK 3. The conclusion of Theorem 3 does not necessarily hold if the space ${}_{WW}\mathcal{D}$ in the hypothesis of the theorem is replaced by ${}_{WZ}\mathcal{D}$ where Z is different from W , even if V is finite dimensional and U, W, Z are irreducible unitary representations of K . An example will be given in a forthcoming paper of the author and F. I. Mautner. (G can be taken as the complex unimodular group.)

6. **General remarks.** We have assumed that G is a separable Lie group. In the general case, the spaces \mathcal{E} and \mathcal{D} can be defined as before, but \mathcal{E} will not be metrizable and \mathcal{D} will not be an \mathcal{LF} space in the sense of Dieudonné and Schwartz [4] because \mathcal{D} will be the inductive limit of a non-denumerable number of spaces. For this reason, the topology of \mathcal{D} is best defined as follows: Let $\{f_{ij}\} = \sigma$ be a family of continuous functions on G such that

(a) For each i , only a finite number of j appear.

(b) Only a finite number of f_{ij} are different from zero on any compact set of G .

Then we define N_σ as the set of $h \in \mathcal{D}$ for which

$$\max_{x \in G} \rho_j[f_{ij}(x)Q_j h(x)] \leq 1$$

for all i, j , where the Q_j are as in § 2, and $\{\rho_j\}$ denotes an enumeration of semi-norms which are sufficient to define the topology of V . The sets N_σ are seen to form a fundamental system of neighborhoods of zero of a locally convex topological vector space which we shall call \mathcal{D} . In case \mathcal{D} is separable it is easily verified that the two definitions agree.

The advantage of the above definition is that it implies immediately the completeness of \mathcal{D} . For, the completion of \mathcal{D} obviously consists of indefinitely differentiable maps. Moreover, if h is any map in the completion of \mathcal{D} , then, for any continuous function f on G , and any

k , it is easily seen that $\rho_k(fh)$ is a bounded function. This implies immediately that h is of compact carrier, hence $h \in \mathcal{D}$.

The properties of convolution can be extended to the nonseparable case and there is no difficulty in extending part of our main results. We can, as in § 5, prove only that the topology of \mathcal{E}' is sequentially, and in regard to bounded sets, the same as the compact-open topology of the space of linear transformations of $\mathcal{D} \rightarrow \mathcal{D}$ (under convolution).

The results of § 5 on double coset spaces $K \backslash G / L$ can also be extended to functions invariant under a compact group of automorphisms of G (the group of automorphisms of G is given the compact-open topology).

In addition, the main results of this paper can be extended to locally compact groups. There \mathcal{E} is replaced by the space of continuous functions, \mathcal{D} the space of continuous functions of compact carrier, \mathcal{D}' the space of measures and \mathcal{E}' the space of measures of compact carrier.

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