

Pacific Journal of Mathematics

NOTE ON A THEOREM OF HADWIGER

ROBERT STEINBERG

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Throughout this paper, H denotes a Hilbert space over the real or complex numbers and (x, y) denotes the inner product of the vectors x, y of H . The only projections we consider are orthogonal ones.

Our starting point is the basic fact that, if $\{u_\alpha\}$ is an orthonormal basis of H , then the Parseval relation

$$(1) \quad (x, y) = \sum (x, u_\alpha)(u_\alpha, y)$$

is valid for each pair of vectors x, y of H . It is easy to see that (1) is also valid if $\{u_\alpha\}$ is the projection of an orthonormal basis $\{w_\alpha\}$ and if we restrict x and y to the range of the projection. Indeed, if E is the projection, so that $w_\alpha E = u_\alpha$ for each α , then

$$\begin{aligned} (x, y) &= \sum (x, w_\alpha)(w_\alpha, y) = \sum (xE, w_\alpha)(w_\alpha, yE) = \sum (x, w_\alpha E)(w_\alpha E, y) \\ &= \sum (x, u_\alpha)(u_\alpha, y). \end{aligned}$$

The theorem referred to in the title deals with this result and also with the converse question:

THEOREM 1. *If the Parseval relation (1) is valid for each pair of vectors x and y of H , then the set $\{u_\alpha\}$ is the projection of an orthonormal basis of a superspace K of H .*

This result was first proved by Hadwiger [1], and, then, by Julia [2]. We first give a simple proof of Theorem 1 that depends on a simple imbedding procedure, and then consider some related questions concerning projections of orthogonal sets of vectors.

Proof of Theorem 1. We choose as K coordinate Hilbert space [4, p. 120] of dimension equal to the cardinality of the set $\{u_\alpha\}$. We see from (1), with $x = u_\beta, y = u_\gamma$, that the matrix $U = ((u_\alpha, u_\beta))$ is idempotent. Since U is also Hermitian, it may be interpreted as a projection acting on K . We now imbed H in K by making correspond to x in H the (row) coordinate vector $x' = \{(x, u_\alpha)\}$ in K . In particular, to the vector u_β there corresponds the β th row of U which is manifestly the image, under the projection U , of the β th coordinate basis vector. Finally, if $x' = \{(x, u_\alpha)\}$ and $y' = \{(y, u_\alpha)\}$, then $(x', y') = \sum (x, u_\alpha)(y, u_\alpha) = \sum (x, u_\alpha)(u_\alpha, y) = (x, y)$; thus the imbedding is isometric and we are done.

We next prove a related result which is due to Julia [2, (c)].

THEOREM 2. *If the Parseval relation (1) is valid relative to the set $\{u_\alpha\}$ of H , and, if no u_α is in the closed subspace spanned by the others, then $\{u_\alpha\}$ is an orthonormal basis.*

Proof. The second assumption implies the existence of a dual set $\{v_\alpha\}$ in H such that $(u_\alpha, v_\beta) = \delta_{\alpha\beta}$ [see 3, p. 264]. Then, using (1), we get $\delta_{\alpha\beta} = (u_\alpha, v_\beta) = \sum_\gamma (u_\alpha, u_\gamma)(u_\gamma, v_\beta) = \sum_\gamma (u_\alpha, u_\gamma)\delta_{\gamma\beta} = (u_\alpha, u_\beta)$.

We remark at this point that the methods of proof of Theorems 1 and 2 can be used to give proofs of the corresponding results about projections of biorthonormal bases of vectors $\{u_\alpha; v_\alpha\}$ for which $(u_\alpha, v_\beta) = \delta_{\alpha\beta}$. These methods are also used in our next proof [see 2, (b)].

THEOREM 3. *A necessary and sufficient condition that a set of vectors $\{u_\alpha\}$ of H be the projection of an orthonormal set (not necessarily a basis) in some superspace K is that, for each x in H ,*

$$(2) \quad \Sigma |(x, u_\alpha)|^2 \leq (x, x) .$$

Proof. By the remarks preceding Theorem 1, the necessity is clear. In proving sufficiency, we may suppose $\{u_\alpha\}$ is complete in H , since, otherwise, by adding to $\{u_\alpha\}$ an orthonormal basis of the orthogonal complement of $\{u_\alpha\}$ in H , we get a larger set which is complete, and for which the condition (2) is still valid. Next we show that, if U is the matrix $((u_\alpha, u_\beta))$, then $0 \leq U \leq 1$, in the sense that both U and $1 - U$ are nonnegative [4, p. 213]. Let ξ_α be any set of scalars of which all but a finite number are zero. Then, using Schwarz' inequality and (2), we get

$$0 \leq (\Sigma_\alpha \xi_\alpha u_\alpha, \Sigma_\beta \xi_\beta u_\beta) = \Sigma_{\alpha, \beta} \xi_\alpha \bar{\xi}_\beta (u_\alpha, u_\beta) \\ \leq (\Sigma_\beta |\xi_\beta|^2)^{1/2} (\Sigma_\alpha |\xi_\alpha (u_\alpha, u_\beta)|^2)^{1/2} \leq (\Sigma_\beta |\xi_\beta|^2)^{1/2} (\Sigma_\alpha \xi_\alpha u_\alpha, \Sigma_\gamma \xi_\gamma u_\gamma)^{1/2} .$$

Thus $0 \leq \Sigma_{\alpha, \beta} \xi_\alpha \bar{\xi}_\beta (u_\alpha, u_\beta) \leq \Sigma_\beta |\xi_\beta|^2$; so that $0 \leq U \leq 1$, U^2 exists and $0 \leq U - U^2$ [4, p. 217]. Consider now the matrix $E = \begin{pmatrix} U & \sqrt{U - U^2} \\ \sqrt{U - U^2} & 1 - U \end{pmatrix}$. [See

4, pp. 215, 224]. This is Hermitian and idempotent and hence represents a projection in coordinate Hilbert space K of the appropriate dimension. As in Theorem 1, the (row) vectors given by the upper half of E not only are the images, under E , of "half" of the coordinate basis vectors of K , but also constitute an isometric imbedding of the set $\{u_\alpha\}$ in K . Since $\{u_\alpha\}$ is complete in H , the imbedding can be extended to all of H ; and the proof is complete.

At this stage, we introduce the following *definition*: A set of vectors $\{u_\alpha\}$ in H has the property P if each x in H is orthogonal to all but a countable number of u_α .

LEMMA (1) Any orthogonal set has property P . (2) Property P is invariant under projection: if $\{u_\alpha\}$ has property P and E is a projection, then so does $\{u_\alpha E\}$.

Proof. The statement (1) is a classical result [4, p. 114]. To prove (2) we select any x in H . Then $(x, u_\alpha E) = (xE, u_\alpha)$ which is zero for all but countably many u_α .

This lemma leads us to the following conjecture: A necessary and sufficient condition that $\{u_\alpha\}$ be the projection of an orthogonal set (not necessarily normal) is that $\{u_\alpha\}$ has property P .

The lemma proves necessity. We have been unable to prove sufficiency. However, we can prove the following special case:

THEOREM 4. A necessary and sufficient condition for the set of non-zero vectors $\{u_\alpha\}$ in a separable Hilbert space H to be the projection of an orthogonal set is that the set be countable.

Proof. Suppose first that $\{u_\alpha\}$ is the projection of an orthogonal set. Then, by the lemma, it has property P . Let $\{x_i\}$ be a (countable) basis for H . Then all but a countable number of u_α are orthogonal to each x_i and hence to their union $\{x_i\}$. That is, all but countably many u_α are 0. This proves the necessity. To prove sufficiency, we suppose that $\{u_\alpha\}$ is countable and indexed by the positive integers. We then define $v_\alpha = 2^{-\alpha} u_\alpha / (u_\alpha, u_\alpha)^{1/2}$, for each α . Then, if x is any vector of H , it follows, by Schwarz' inequality, that $\sum |(x, v_\alpha)|^2 \leq (x, x) \sum (v_\alpha, v_\alpha) = (x, x) \sum 2^{-2\alpha} \leq (x, x)$. Thus, by Theorem 3, $\{v_\alpha\}$ is the projection of an orthogonal set and so is $\{u_\alpha\}$.

We close with an example of a set $\{u_\alpha\}$ which is not the projection of an orthogonal set. Let $\{x_\alpha\}$ be an uncountable orthonormal set in nonseparable Hilbert space and set $u_\alpha = x_1 + x_\alpha$, for each α . Then $\{u_\alpha\}$ does not have property P and hence, by the lemma, is not the projection of an orthogonal set. It is to be noted that Theorem 4 cannot be used to prove this result since every uncountable subset of $\{u_\alpha\}$ spans a nonseparable subspace of H .

REFERENCES

1. H. Hadwiger, *Über ausgezeichnete Vectorsterne und reguläre Polytope*, Comm. Math. Helv., **13** (1940-41), 90.
2. G. Julia, C. R. Acad. Sci. Paris, (a) **218** (1944), 892, (b) **219** (1944), 8, (c) **226** (1948), 1485, (d) **227** (1948), 168 and 317.
3. S. Kaczmark and H. Steinhaus, *Theorie der Orthogonalreihen*, Warsaw and Lwow, 1935.
4. A. C. Zaanan, *Linear analysis*, New York, 1953.

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