

Pacific Journal of Mathematics

**A REAL INVERSION FORMULA FOR A CLASS OF
BILATERAL LAPLACE TRANSFORMS**

WILLIAM ROBERT GAFFEY

A REAL INVERSION FORMULA FOR A CLASS OF BILATERAL LAPLACE TRANSFORMS

WILLIAM R. GAFFEY

1. **Introduction.** The Post-Widder inversion formula for unilateral Laplace transformations [1] states that, under certain weak restrictions on $\phi(u)$,

$$\lim_{k \rightarrow \infty} \left(\frac{k}{c} \right)^{k+1} \frac{1}{k!} \int_0^{\infty} \phi(u) u^k \exp \left(-k \frac{u}{c} \right) du = \phi(c) ,$$

for any continuity point c of $\phi(u)$.

This formula applies when $\phi(u)$ is defined only for $u \geq 0$. A similar formula may be deduced if $\phi(u)$ is defined for $u \geq -a$, for some positive a . In such a case, we may let $\phi^*(u) = \phi(u-a)$, and we may then use the Post-Widder formula to determine $\phi^*(u)$ at the point $u=c+a$. The inversion formula then becomes

$$\lim_{k \rightarrow \infty} \left(\frac{k}{c+a} \right)^{k+1} \frac{1}{k!} \int_0^{\infty} \phi(u-a) u^k \exp \left(-k \frac{u}{c+a} \right) du = \phi(c) ,$$

or, if we make the transformation $z = u/(c+a)$,

$$(1) \quad \lim_{k \rightarrow \infty} \frac{k^{k+1}}{k!} \int_0^{\infty} \phi[(c+a)z-a] z^k \exp(-kz) dz = \phi(c) .$$

This suggests that, if $\phi(u)$ is defined for $-\infty < u < \infty$, some sort of limiting form of (1) applies. We shall prove that under suitable restrictions on ϵ and on the behavior of $\phi(u)$,

$$(2) \quad \lim_{k \rightarrow \infty} \frac{k^{k+1}}{k!} \int_{-\infty}^{\infty} \phi[(c+k^\epsilon)z-k^\epsilon] z^k \exp(-kz) dz = \phi(c) .$$

2. **Remarks.** In the following sections $\phi(u)$ will be assumed to be integrable over the interval from $-\infty$ to ∞ , and c will be assumed to be a continuity point of $\phi(u)$. All limits should be understood to be for increasing values of k .

The expression $\delta/(c+k^\epsilon)$, where δ and ϵ are positive numbers, occurs frequently. It will be denoted by $\delta(k, \epsilon)$.

Finally, it may be noted that in terms of the Laplace transform of $\phi(u)$ for real t ,

Received December 7, 1955, and in revised form April 13, 1956.

$$f(t) = \int_{-\infty}^{\infty} \phi(u) \exp(-tu) du,$$

the inversion formula (2) may be written in the form

$$\lim \frac{(-1)^k}{k!} \left(\frac{k}{c + k^\varepsilon} \right)^{k+1} \frac{d^k}{dt^k} [f(t) \exp(-tk^\varepsilon)]_{t=k/(c+k^\varepsilon)} = \phi(c).$$

3. Preliminary proofs. The results of the following four lemmas will be needed below. Proofs are given for the first two. The second two are proved in a similar way.

LEMMA 1. *If n is any fixed number and $1/3 < \varepsilon < 1/2$, then*

$$\lim k^n [1 + \delta(k, \varepsilon)]^k \exp[-k\delta(k, \varepsilon)] = 0.$$

Proof. If the logarithm of the expression under the limit sign is expanded in powers of $\delta(k, \varepsilon)$, the sum of two of the terms in the expansion approaches $-\infty$ as $k \rightarrow \infty$, while the sum of the rest of the terms is bounded.

LEMMA 2. *If $1/3 < \varepsilon < 1/2$, then*

$$\lim \frac{k^{k+1}}{k!} \int_1^{1+\delta(k, \varepsilon)} z^k \exp(-kz) dz = \frac{1}{2}.$$

Proof. It is well known [1] that

$$\lim \frac{k^{k+1}}{k!} \int_1^{\infty} z^k \exp(-kz) dz = \frac{1}{2}.$$

Therefore, it is sufficient to show that

$$\lim \frac{k^{k+1}}{k!} \int_{1+\delta(k, \varepsilon)}^{\infty} z^k \exp(-kz) dz = 0.$$

Since $z \exp(-z)$ is a decreasing function of z for $z > 1$, the above expression is, for fixed k , no larger than

$$\frac{k^{k+1}}{k!} [1 + \delta(k, \varepsilon)]^{k-1} \exp[-(k-1)(1 + \delta(k, \varepsilon))] \int_{1+\delta(k, \varepsilon)}^{\infty} z \exp(-z) dz.$$

By applying Stirling's formula and Lemma 1, we see that the upper bound approaches zero as k increases.

LEMMA 3. *If n is any fixed number and $0 < \varepsilon < 1/2$, then*

$$\lim k^n [1 - \delta(k, \epsilon)]^k \exp [k\delta(k, \epsilon)] = 0 ,$$

LEMMA 4. *If* $0 < \epsilon < 1/2$, *then*

$$\lim \frac{k^{k+1}}{k!} \int_{1-\delta(k, \epsilon)}^1 z^k \exp(-kz) dz = \frac{1}{2} .$$

4. The inversion formula.

THEOREM. *If*

$$(a) \quad \left| \int_{-\infty}^{-d} \phi(z) dz \right| \leq A \exp(-d\alpha)$$

for some positive quantities A, d , *and* α , *and if*

$$(b) \quad \max(1/3, 1/(2+\alpha)) < \epsilon < 1/2,$$

then

$$\lim I_k = \lim \frac{k^{k+1}}{k!} \int_{-\infty}^{\infty} \phi[(c+k^\epsilon)z - k^\epsilon] z^k \exp(-kz) dz = \phi(c) .$$

Proof. For any $\delta > 0$, the infinite interval may be partitioned into the four subintervals $(-\infty, 1 - \delta(k, \epsilon))$, $(1 - \delta(k, \epsilon), 1)$, $(1, 1 + \delta(k, z))$, and $(1 + \delta(k, \epsilon), \infty)$. I_k may be considered as the sum of four integrals over these intervals, so that we may write

$$I_k = I_k^{(1)} + I_k^{(2)} + I_k^{(3)} + I_k^{(4)} .$$

$I_k^{(1)}$ is understood to represent the integral over $(-\infty, 1 - \delta(k, \epsilon))$ etc.

$$|I_k - \phi(c)| \leq |I_k^{(1)}| + \left| I_k^{(2)} - \frac{\phi(c)}{2} \right| + \left| I_k^{(3)} - \frac{\phi(c)}{2} \right| + |I_k^{(4)}| .$$

We prove first that $I_k^{(1)}$ and $I_k^{(4)}$ approach zero as $k \rightarrow \infty$. For $I_k^{(1)}$, consider first the integral over the interval from 0 to $1 - \delta(k, \epsilon)$. The function $z \exp(-z)$ attains its maximum at the upper endpoint. Therefore an upper bound for the absolute value of this portion of the expression is

$$\frac{k^{k+1}}{k!} [1 - \delta(k, \epsilon)]^k \exp[-k + k\delta(k, \epsilon)] \int_0^{1-\delta(k, \epsilon)} |\phi[(c+k^\epsilon)z - k^\epsilon]| dz ,$$

which approaches zero by Stirling's formula and Lemma 3.

Consider now the integral over the interval from $-\infty$ to 0. Integrating by parts, we find that it is equal to

$$-\frac{1}{c+k^\varepsilon} \frac{k^{k+2}}{k!} \int_{-\infty}^0 F[(c+k^\varepsilon)z-k^\varepsilon] z^{k+1} (1-z) \exp(-kz) dz,$$

where $F(z) = \int_{-\infty}^z \phi(u) du$. Note that, by the assumption on $F(z)$,

$$|F[(c+k^\varepsilon)z-k^\varepsilon]| \leq A \exp[-d\{-(c+k^\varepsilon)z+k^\varepsilon\}^{2+\alpha}],$$

which is in turn equal to or less than

$$A \exp[dz(c+k^\varepsilon)k^{\varepsilon(1+\alpha)}].$$

The result of the integration by parts may be written as the difference between two integrals, the first containing z^{k-1} and the second containing z^k . The first integral is no greater in absolute value than

$$\frac{A}{(c+k^\varepsilon)} \frac{k^{k+2}}{k!} \int_{-\infty}^0 |z^{k-1}| \exp[z\{d(c+k^\varepsilon)k^{\varepsilon(1+\alpha)}-k\}] dz.$$

Since $\varepsilon(2+\alpha) > 1$, the coefficient of z in the exponent above is positive for sufficiently large k . Therefore, after some manipulation, this upper bound can be shown to be equal to

$$\frac{A}{(c+k^\varepsilon)} \frac{k^{k+2}}{k!} \cdot \frac{\Gamma(k)}{[d(c+k^\varepsilon)k^{\varepsilon(1+\alpha)}-k]^k},$$

which approaches zero as $k \rightarrow \infty$.

By the same argument, the second integral approaches zero, so that $\lim I_k^{(3)} = 0$.

For $I_k^{(4)}$, observe that since $z \exp(-z)$ is a decreasing function of z for $z > 1$, the expression has the following upper bound for its absolute value:

$$\frac{k^{k+1}}{k!} [1 + \delta(k, \varepsilon)]^k \exp[-k - k\delta(k, \varepsilon)] \int_{1+\delta(k, \varepsilon)}^\infty |\phi[(c+k^\varepsilon)z-k^\varepsilon]| dz.$$

Since the integral is bounded, the whole upper bound approaches zero by virtue of Stirling's formula and Lemma 1.

We now prove that

$$\left| \lim I_k^{(3)} - \frac{1}{2} \phi(c) \right| < \frac{1}{2} \eta$$

for any $\eta > 0$. By Lemma 2, it is sufficient to show that

$$\left| \lim \frac{k^{k+1}}{k!} \int_1^{1+\delta(k, \varepsilon)} \{\phi[(c+k^\varepsilon)z-k^\varepsilon] - \phi(c)\} z^k \exp(-kz) dz \right| < \frac{\eta}{2}.$$

Since c is a continuity point of $\phi(u)$, there is a $\delta > 0$ such that if $|(c+k^\varepsilon)z-k^\varepsilon-c| < \delta$, that is, if $|z-1| < \delta(k, \varepsilon)$, then

$$|\phi[(c+k^\varepsilon)z-k^\varepsilon]-\phi(c)| < \eta .$$

For such a δ , the absolute value of the expression above is equal to or less than

$$\eta \lim \frac{k^{k+1}}{k!} \int_1^{1+\delta(k, \varepsilon)} z^k \exp(-kz) dz = \frac{\eta}{2} .$$

By the use of Lemma 4, it may be shown in a similar way that

$$\left| \lim I_k^{(2)} - \frac{1}{2} \phi(c) \right| < \frac{1}{2} \eta .$$

Putting together these results, we have $|\lim I_k - \phi(c)| < \eta$ for any $\eta > 0$, which proves the theorem.

REFERENCE

1. C. V. Widder, *Inversion of the Laplace transform and the related moment problem*, Trans. Amer. Math. Soc. **36** (1934), 107-200.

UNIVERSITY OF CALIFORNIA, BERKELEY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. L. ROYDEN
Stanford University
Stanford, California

R. A. BEAUMONT
University of Washington
Seattle 5, Washington

A. R. WHITEMAN
University of Southern California
Los Angeles 7, California

E. G. STRAUS
University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH
C. E. BURGESS
M. HALL
E. HEWITT

A. HORN
V. GANAPATHY IYER
R. D. JAMES
M. S. KNEBELMAN

L. NACHBIN
I. NIVEN
T. G. OSTROM
M. M. SCHIFFER

G. SZEKERES
F. WOLF
K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
OREGON STATE COLLEGE
UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF UTAH
WASHINGTON STATE COLLEGE
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
HUGHES AIRCRAFT COMPANY
THE RAMO-WOOLDRIDGE CORPORATION

Printed in Japan by Kokusai Bunken Insatsusha
(International Academic Printing Co., Ltd.), Tokyo, Japan

Pacific Journal of Mathematics

Vol. 7, No. 1

January, 1957

Richard Davis Anderson, <i>Zero-dimensional compact groups of homeomorphisms</i>	797
Hans-Joachim Bremermann, <i>Holomorphic functionals and complex convexity in Banach spaces</i>	811
Hugh D. Brunk, G. M. Ewing and W. R. Utz, <i>Minimizing integrals in certain classes of monotone functions</i>	833
Philip David, <i>Uniqueness theory for asymptotic expansions in general regions</i>	849
Paul Erdős and Harold Nathaniel Shapiro, <i>On the least primitive root of a prime</i>	861
Watson Bryan Fulks, <i>Regular regions for the heat equation</i>	867
William Robert Gaffey, <i>A real inversion formula for a class of bilateral Laplace transforms</i>	879
Ronald Kay Getoor, <i>On characteristic functions of Banach space valued random variables</i>	885
Louis Guttman, <i>Some inequalities between latent roots and minimax (maximin) elements of real matrices</i>	897
Frank Harary, <i>The number of dissimilar supergraphs of a linear graph</i>	903
Edwin Hewitt and Herbert S. Zuckerman, <i>Structure theory for a class of convolution algebras</i>	913
Amnon Jakimovski, <i>Some Tauberian theorems</i>	943
C. T. Rajagopal, <i>Simplified proofs of "Some Tauberian theorems" of Jakimovski</i>	955
Paul Joseph Kelly, <i>A congruence theorem for trees</i>	961
Robert Forbes McNaughton, Jr., <i>On the measure of normal formulas</i>	969
Richard Scott Pierce, <i>Distributivity in Boolean algebras</i>	983
Calvin R. Putnam, <i>Continuous spectra and unitary equivalence</i>	993
Marvin Rosenblum, <i>Perturbation of the continuous spectrum and unitary equivalence</i>	997
V. N. Singh, <i>Certain generalized hypergeometric identities of the Rogers-Ramanujan type</i>	1011
Peter Swerling, <i>Families of transformations in the function spaces H^p</i>	1015