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THE NUMBER OF DISSIMILAR SUPERGRAPHS OF A LINEAR GRAPH

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1. Introduction. A (p, q) *graph* is one with p vertices and q lines. A formula is obtained for the number of dissimilar occurrences of a given (α, β) graph H as a subgraph of all (p, q) graphs G, $\alpha \leqq p$, $\beta \leqq q$, that is, for the number of dissimilar (p, q) supergraphs of H. The enumeration methods are those of Pólya [7]. This result is then applied to obtain formulas for the number of dissimilar complete subgraphs (cliques) and cycles among all (p, q) graphs. The formula for the number of rooted graphs in [2] is a special case of the number of dissimilar cliques. This note complements [3] in which the number of dissimilar (p, k) subgraphs of a given (p, q) graph is found. We conclude with a discussion of two unsolved problems.

A (*linear*) *graph* G (see [5] as a general reference) consists of a finite set V of *vertices* together with a prescribed subset W of the collection of all unordered pairs of distinct vertices. The members of W are called *lines* and two vertices v_1, v_2 are *adjacent* if $\{v_1, v_2\} \in W$, that is, if there is a line joining them. By the *complement* G' of a graph G, we mean the graph whose vertex-set coincides with that of G, in which two vertices are adjacent if and only if they are not adjacent in G.

Two graphs are *isomorphic* if there is a one-to-one adjacency-preserving correspondence between their vertex sets. An *automorphism* of G is an isomorphism of G with itself. The *group of a graph* G, written $\Gamma_0(G)$, is the group of all automorphisms of G. A *subgraph* G_1 of G is given by subsets $V_1 \subseteqq V$ and $W_1 \subseteqq W$ which in turn form a graph. If H is a subgraph of G, we also say G is a *supergraph* of H. Two subgraphs H_1, H_2 of G are *similar* if there is an automorphism of G which maps H_1 onto H_2. Obviously similarity is an equivalence relation and by the number of *dissimilar* vertices, lines, $\cdots$ of G, we mean the number of similarity classes (as in [3, 4, 6]).

Two supergraphs G_1 and G_2 of H are *H-similar* if there exists an isomorphism between G_1 and G_2 which leaves H invariant. It is clear that the number of dissimilar (p, q) supergraphs of H is equal to the number of dissimilar occurrences of H as a subgraph of all (p, q) graphs.

2. Supergraphs. Let H be an arbitrary (α, β) graph. We wish to enumerate the dissimilar (p, q) supergraphs of H where $p \geqq \alpha$, $q \geqq \beta$.

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Let $s^H_{p,q}$ be the number of dissimilar supergraphs of H with p vertices and q lines. For given p, let

$$(1) \qquad s^H_p(x) = \sum_{q=\beta}^{p(p-1)/2} s^H_{p,q} x^q$$

be the *counting polynomial* for the numbers $s^H_{p,q}$. We shall develop a formula for $s^H_p(x)$ using Pólya's enumeration theorem.

In precisely the form in which we require it, Pólya's Theorem is reviewed briefly in § 2 of [2]. Therefore, we shall not repeat here the definitions leading up to the statement of Pólya's Theorem, but shall only restate the theorem itself.

PÓLYA'S THEOREM. The configuration counting series $F(x)$ is obtained by substituting the figure counting series $\phi(x)$ into the cycle index $Z(\Gamma)$ of the configuration group Γ. Symbolically,

$$(2) \qquad F(x) = Z(\Gamma,\ \phi(x)) .$$

This theorem reduces the problem of finding the configuration counting series to the determination of the figure counting series and the cycle index of the configuration group.

The observations needed to make our problem amenable to Pólya's Theorem are as follows: A (p, q) supergraph G of the given (α, β) graph H is a configuration of length $p(p-1)/2-\beta$ whose figures are precisely those vertex-pairs of G not adjacent in H. The content of a figure is one if the vertices are adjacent and is zero otherwise, so that the figure counting series $\phi(x)=1+x$. Hence the content of the configuration G is $q-\beta$. The desired configuration series is $s^H_p(x)$.

In order to apply Pólya's Theorem, we still need to know the cycle index of the configuration group $\Gamma_{H,p}$. The degree of this group is $p(p-1)/2-\beta$ since the objects acted on by its permutations are the lines of the complement of H in the complete graph of p vertices containing H. All permutations of these lines which are compatible with $\Gamma_0(H)$ are in $\Gamma_{H,p}$. Before obtaining the cycle index of $\Gamma_{H,p}$ we state the form of the result by applying (2) to the present situation:

$$(3) \qquad s^H_p(x) = x^\beta Z(\Gamma_{H,p},\ 1+x) .$$

We now turn to the development of the permutation group $\Gamma_{H,p}$ in a form which will yield its cycle index. Let F_p denote the complete graph of p vertices, that is, the graph with p vertices and all $p(p-1)/2$ possible lines. As in [3], let $\Gamma_1(G)$ be the *line-group* of the graph G, that is, the permutation group whose objects are the lines of G, and whose permutations are induced by those of $\Gamma_0(G)$, the group of automorphisms of G. If Γ is a permutation group of degree s, let $T(\Gamma)$

be the *pair-group* of Γ, that is, the permutation group of degree $s(s-1)/2$ which acts on the pairs of the object-set of Γ but is isomorphic to Γ as an abstract group. Then clearly $\Gamma_1(F_p)$ and $T(\Gamma_0(F_p))$ are isomorphic as permutation groups. Let $\Gamma_1\cdot\Gamma_2$ denote the direct product of the permutation groups Γ_1 and Γ_2 whose object-sets are disjoint.

The lines of the object-set of the configuration group $\Gamma_{H,p}$ are of three possible kinds:

I. neither vertex is in H
II. both vertices are in H
III. one vertex is in H and the other is not.

For each of these three cases, we find the permutation group on the corresponding subset of lines and then form their direct product to get $\Gamma_{H,p}$. In case I, every rearrangement of the lines with neither vertex in H which is induced by a permutation of the vertices of $G-H$ is compatible with the group of H, so that we have the group $\Gamma_1(F_{p-\alpha})$. For case II, we obtain the line group of the complement of H, that is, $\Gamma_1(H')$. The third "mixed" case yields the group $M(H, F_{p-\alpha})$ of degree $\alpha(p-\alpha)$ on those lines of F_p joining a vertex of H with one of $F_{p-\alpha}$, consisting of those permutations of these lines which are compatible with $\Gamma_0(H)$. Then $\Gamma_{H,p}$ is the direct product:

$$\Gamma_{H,p}=\Gamma_1(F_{p-\alpha})\cdot\Gamma_1(H')\cdot M(H, F_{p-\alpha}) \tag{4}$$

and by a remark of Pólya [7] to the effect that $Z(\Gamma_1\cdot\Gamma_2)=Z(\Gamma_1)\cdot Z(\Gamma_2)$, we have

$$Z(\Gamma_{H,p})=Z(\Gamma_1(F_{p-\alpha}))\cdot Z(\Gamma_1(H'))\cdot Z(M(H, F_{p-\alpha}))\ . \tag{5}$$

We note as a "dimensional check" that the degree of the groups of the right hand member of (4) are $(p-\alpha)(p-\alpha-1)/2$, $\alpha(\alpha-1)/2-\beta$, and $\alpha(p-\alpha)$ whose sum is $p(p-1)/2-\beta$, the degree of the configuration group.

Combining (5) and (3), we are now able to develop the counting polynomial for the dissimilar p vertex supergraphs of H. It is useful for this purpose to recall equation (10) of [2] which gives a formula for the first factor of (5). In this formula, which is equation (7) below, the letters g_i are employed for the indeterminates of the cycle index, S_p denotes the symmetric group of degree p, the sum is taken over all p-tuples (j) satisfying

$$1j_1+2j_2+\cdots+pj_p=p\ , \tag{6}$$

and $d(q, r)$, $m(q, r)$ denote the greatest common divisor and least common multiple respectively.

$$Z(\Gamma_1(F_p))=Z(T(S_p))=\frac{1}{p!}\sum_{(j)}\frac{p}{\prod_{i=1}^{n} i^{j_i} j_i!} \tag{7}$$

$$\times \prod_{n=0}^{[(p-1)/2]} g_{2n+1}^{nj_{2n+1}+(2n+1)\binom{j_{2n+1}}{2}}$$

$$\times \prod_{n=1}^{[p/2]} (g_n g_{2n}^{n-1})^{j_{2n}} \, g_{2n}^{2n\binom{j_{2n}}{2}}$$

$$\times \prod_{1\leq s<r\leq p} g_{m(s,r)}^{j_s j_r d(s,r)}$$

Equation (7) gives the first factor of the right hand member of (5). The second factor depends on the particular graph H whose supergraphs are being enumerated. The third factor also depends on H, but can be readily computed as soon as $Z(\Gamma_0(H))$, the cycle index of the automorphism group of H, is found, by the following procedure. It is well known that for S_p, the symmetric group of degree p, one has

$$Z(S_p)=\frac{1}{p!}\sum_{(j)}\frac{1}{1^{j_1}j_1!\,2^{j_2}j_2!\cdots p^{j_p}j_p!}b_1^{j_1}b_2^{j_2}\cdots b_p^{j_p} \tag{8}$$

where the sum is taken over all partitions (j) of p satisfying (6) and the letters b_k are indeterminates. We write $Z(\Gamma_0(H))$ using the letters a_i as indeterminates, and then form the product $Z(\Gamma_0(H))\cdot Z(S_{p-\alpha})$. This will be a polynomial whose general term, aside from its numerical coefficient is of the form

$$(a_1^{h_1}a_2^{h_2}\cdots a_\alpha^{h_\alpha})(b_1^{j_1}b_2^{j_2}\cdots b_{p-\alpha}^{j_{p-\alpha}})=\prod_{s=1}^{\alpha} a_s^{h_s}\prod_{r=1}^{p-\alpha} b_r^{j_r} \tag{9}$$

If the letters c_k are the indeterminates of the third factor of (5), we then obtain $Z(M(H,\ F_{p-\alpha}))$ by substituting for (9) in $Z(\Gamma_0(H))\cdot Z(S_{p-\alpha})$ the expression:

$$\prod_{r,s} c_{m(r,s)}^{h_s j_r d(r,s)} \tag{10}$$

3. **Cliques.** We now specialize (5) to the case where H is a clique or complete graph, that is, to $H=F_\alpha$. For this to be meaningful, we define $Z(\Gamma_1(F_\alpha'))=1$, so that (5) becomes

$$Z(\Gamma_{F_\alpha,p})=Z(\Gamma_1(F_{p-\alpha}))\cdot Z(M(F_\alpha,\ F_{p-\alpha}))\,. \tag{11}$$

To illustrate (11), we take $p=4$, $\alpha=2$. Then the first factor is $Z(\Gamma_1(F_2))=c_1$ and the second factor is $Z(M(F_2,\ F_2))=\frac{1}{4}(c_1^4+3c_2^2)$. Therefore in thiscase, (3) yields the polynomial:

$$s_4^{F_2}(x) = x \cdot \frac{1}{4}(c_1^5 + 3c_1c_2^2,\ 1+x)$$

$$= x + 2x^2 + 4x^3 + 4x^4 + 2x^5 + x^6\,,$$

which can be readily verified pictorially by observing the number of dissimilar lines in all the graphs of 4 vertices: see Figure 1.

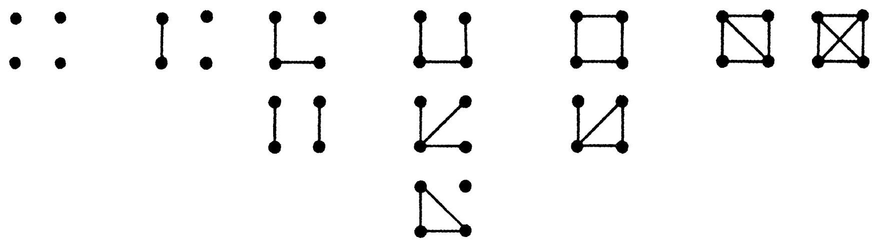

Figure 1

Equation (7) gives the first factor of (11) explicitly. One can also obtain an explicit formula for the second factor of (11) by applying (10) to two copies of (8) for the degrees α and $p-\alpha$. The result of this procedure is

$$Z(M(F_\alpha, F_{p-\alpha})) = \frac{1}{\alpha!(p-\alpha)!} \sum_{(j)} \sum_{(h)} \frac{\alpha!}{\prod_{i=1}^{\alpha} i^{h_i} h_i!} \cdot \frac{(p-\alpha)!}{\prod_{i=1}^{p-\alpha} i^{j_i} j_i!} \tag{12}$$

$$\times \prod_{r=1}^{p-\alpha} \prod_{s=1}^{\alpha} c_{m(r,s)}^{h_s j_r d(r,s)}\,.$$

When (12) is specialized to $\alpha=1$, and then substituted into (3), the formula in [2] for the number of rooted graphs results.

4. Cycles. A *cycle of length n*, or an *n-cycle*, of a graph is a collection of n lines of the form $A_1A_2, A_2A_3, \cdots, A_{n-1}A_n, A_nA_1$ in which the vertices A_i are distinct. Let C_n be a graph consisting of an n-cycle. We now specialize (5) to the case $H=C_\alpha$. Since a 3-cycle is also a 3-clique, the particular case $\alpha=3$ for cycles has already been treated. In general, however, $\Gamma_0(C_n) \equiv D_n$, the dihedral group of degree n and order $2n$. From *Pólya* [7], we have

$$Z(D_n) = \frac{1}{2n}\sum_{d \mid n} \phi(d) a_d^{n/d} + \begin{cases} \dfrac{1}{2} a_1 a_2^{m-1}, & \text{when } n = 2m-1 \\[2ex] \dfrac{1}{4}(a_1^2 a_2^{m-1} + a_2^m), & \text{when } n = 2m\,. \end{cases} \tag{13}$$

When the cycle index of $Z(D_\alpha)$ is multiplied by $Z(S_{p-\alpha})$ from (8), and

(10) is applied, one obtains a formula for $Z(M(C_\alpha, F_{p-\alpha}))$ analogous to (12). Substituting $H=C_\alpha$ into (5), we see that

$$(14) \qquad Z(\Gamma_{C_\alpha, p}) = Z(\Gamma_1(F_{p-\alpha}))\cdot Z(\Gamma_1(C'_\alpha))\cdot Z(M(C_\alpha, F_{p-\alpha}))\,.$$

The only factor of the right-hand member of (14) for which we have not yet developed a formula is $Z(\Gamma_1(C'_\alpha))$.

To describe $Z(\Gamma_1(C'_\alpha))$, it is convenient to use a special case of the "Kranzgruppe" of Pólya [7]. Let Γ be any permutation group of degree d, and let E_n be the group of degree n and order 1. Then by $\Gamma[E_n]$, the *crown-group* of Γ around E_n, is meant the permutation group of degree nd obtained from Γ by replacing the d elements of the object-set acted on by the permutations belonging to Γ, by d disjoint sets of n elements each. Thus $Z(\Gamma[E_n])$ is obtained from $Z(\Gamma)$ when one replaces each factor $f_k^{j_k}$ occurring in each term of $Z(\Gamma)$ by $f_k^{nj_k}$.

For α odd, $\alpha=2n+1$, one sees that

$$(15') \qquad \Gamma_1(C'_{2n+1}) = D_{2n+1}[E_{n-1}]\,,$$

from which $Z(\Gamma_1(C'_{2n+1}))$ is readily computed.

For α even, $\alpha=2n$ the group can be described using A. Cayley's[1] term "dimediation." For example the permutation group $\Gamma_1(C'_6)$ is generated by (123456)(789) and (12)(36)(45)(7)(89). Thus $\Gamma_1(C'_6)$ is isomorphic to D_6 as an abstract group, but as a permutation group it can be constructed from one copy of D_6 and two different copies of D_3. Abbreviating dimediation by "dim" following Cayley, we have in general

$$(15'') \qquad \Gamma_1(C'_{2n}) = D_{2n}[E_{n-2}] \dim D_n\,.$$

One can compute $Z(\Gamma_1(C'_{2n}))$ by multiplying each term of $Z(D_{2n}[E_{n-2}])$ by the appropriate term of $Z(D_n)$.

A *Hamilton cycle* of a graph is a cycle passing through all its vertices. Thus the number of dissimilar Hamilton cycles occurring in all (p, q) graphs is the number of dissimilar (p, q) supergraphs of C_p. In this situation, (14) becomes simplified to:

$$(16) \qquad Z(\Gamma_{C_p, p}) = Z(\Gamma_1(C'_p))\,.$$

We illustrate (16) for $p=5$. Here (15′) becomes $\Gamma_1(C'_5)=D_5[E_1]=D_5$, and by (13):

$$Z(D_5) = \frac{1}{10}(a_1^5 + 4a_5 + 5a_1a_2^2)\,,$$

[1] See for example: A. Cayley, On the substitution groups for two, three, four, five, six, seven, and eight letters, Quart. J. Math. 25 (1890) especially p. 74.

so that applying (3), we get the counting polynomial for the number of dissimilar Hamilton cycles of length 5:

$$s_5^{C_5}(x) = x^5 + x^6 + 2x^7 + 2x^8 + x^9 + x^{10} .$$

This polynomial is verified by the graphs of Figure 2, in each of which the Hamilton cycle is drawn as the exterior cycle.

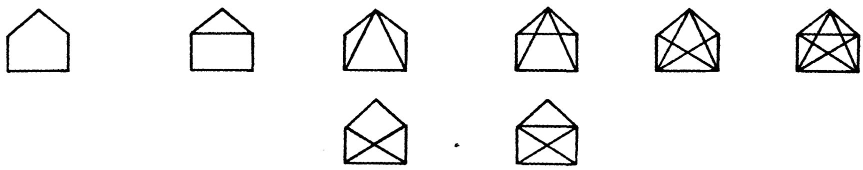

Figure 2.

For $p=5$, it turns out that each similarity type of Hamilton cycle occurs in a different graph; but this is not always so for larger p.

5. Problems. We discuss two unsolved problems implicit in [4] and [8] respectively.

I. It was shown in [4] that for any linear graph G; the dissimilarity characteristic equation:

$$v-(k-k_e)+(c-c_e)=1 \tag{17}$$

holds, where v, k, k_e denote the number of *dissimilar* vertices, lines, exceptional lines[2] respectively, and c, c_e denote the number of cycles, exceptional cycles respectively which appear in any dissimilarity cycle basis[3] of G. In the past, dissimilarity characteristic equations for trees and for Husimi trees [6] have proven useful in enumerating these kinds of graphs. The unsolved problem is to sum (17) over all (p, q) graphs, then multiply the resulting equation through by x^q and sum over $q=0$ to $\binom{p}{2}$. When this is done, the term 1 which is the right-hand member of (17) becomes $g_p(x)$, the counting polynomial for all p vertex graphs [2] and the term v clearly is manipulated into $G_p(x)$, the polynomial for p vertex rooted graphs [2]. By a result of Pólya [7], the enumeration of configurations in which all figures are distinct may be accomplished by using $Z(A_n)$—$Z(S_n)$, where A_n is the alternating group of degree n. But this is precisely the nature of the term k—k_e of (17), which is the number of dissimilar lines of G whose vertices are not similar to each other. One sees by inspection from Figure 1 that for

[2] An *exceptional line* of a graph is one whose vertices are similar to each other.

[3] A *dissimilarity cycle basis* of a graph G is a minimal collection of cycles independent mod similarity on which all cycles of G depend mod similarity. Consult [**4**] for more details.

$p=4$, the counting polynomial induced by the term $k-k_e$ is

$$x^2+2x^3+2x^4+x^5 .$$

To derive the general formula of which the preceding polynomial is the special case $p=4$, let us regard $\overline{F}_2$ as a line whose vertices are not similar. Then replacing F_α by $\overline{F}_2$ in (11), we get

$$Z(\Gamma_{F_2,p})=Z(\Gamma_1(F_{p-2}))\cdot Z(M(\overline{F}_2, F_{p-2})) \tag{18}$$

An explicit formula for the second factor is computed by noting that we may take $Z(\Gamma_0(\overline{F}_2))=Z(A_2)-Z(S_2)=\frac{1}{2}(a_1^2-a_2)$ by the above-mentioned result of Pólya, then multiplying this cycle index by (8) in which p is replaced by $p-2$, and applying (10).

The only term of (17) which we have been unable to sum is $c-c_e$. This appears to offer a nontrivial combinatorial problem, which if solved would provide a functional equation for $g_p(x)$ of the form

$$g_p(x)=G_p(x)-Z(\Gamma_{F_2,p},\ 1+x)+\text{the missing term.}$$

Using (14) for $\alpha=3, 4, \cdots, p$ one can enumerate *all* the dissimilar cycles among all (p, q) graphs, but this does not count just those in a dissimilarity cycle basis.

II. An *n-cube* can be described briefly as a graph whose vertices are the 2^n n-digit binary numbers in which 2 vertices are adjacent whenever they differ in exactly one place. An interesting unsolved problem with some potential applicability to switching theory is to determine the number h_n of dissimilar Hamilton cycles in an n-cube. It is well known that $h_2=h_3=1$ and it has been shown by E. N. Gilbert (unpublished) that $h_4=9$. From the formula of [3] one can find the number of dissimilar (p, p) subgraphs of any (p, q) graph, and of course, all the Hamilton cycles of the graph are included among these subgraphs. On the other hand, (16) gives a formula for the number of dissimilar Hamilton cycles occurring in all (p, q) graphs. However, each of these observations merely provides an upper bound for h_n and leaves the problem open. The more general problem of determining the number of dissimilar occurrences of a fixed graph H as a subgraph of a fixed graph G is also interesting.

One can give the results of this paper an interpretation in binary relations, following [1], and can also generalize them to directed graphs by employing the ordered-pair group of [2] instead of the pair group, but we shall not spell this out. We note finally that (3) implies that each such counting polynomial has end-symmetry with respect to its coefficients. This is explained geometrically by the one-to-one correspondence between the collection of all supergraphs G of H and the collection of their relative complements G'_H with respect to H defined by $G'_H = G' \cup H$.

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