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CONTINUOUS SPECTRA AND UNITARY EQUIVALENCE

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1. **Introduction.** In the differential equation

$$(1) \quad (px')' + (\lambda + f(t))x = 0,$$

let λ denote a real parameter and let $p(t) (> 0)$ and $f(t)$ be continuous real-valued functions on $0 \leq t < \infty$. Suppose that (1) is of the limit-point type, so that (1) and a linear homogeneous boundary condition

$$(2_\alpha) \quad x(0) \cos \alpha + x'(0) \sin \alpha = 0, \quad 0 \leq \alpha < \pi,$$

determine a boundary value problem with a spectrum $S = S_\alpha$ on the half-line $0 \leq t < \infty$; cf. [7]. The continuous spectrum C_α (if it exists) is determined by a continuous monotone nondecreasing basis function $\rho_\alpha(\lambda)$. It is known that the set of cluster points, S' , of S_α is independent of α , [7, p. 251]; the question as to whether the corresponding assertion for C_α is also true was raised by Weyl [7, 7. 252] but is still undecided.

Consider the self-adjoint operators $H_\alpha = \int \lambda dE_\alpha(\lambda)$ (all of which are extensions of the same symmetric operator) belonging to the various boundary value problems determined by (1) and (2_α) ; cf. for example, [2]. The object of this note is to show that any two H_α possessing purely continuous (hence, in view of the above remark concerning S' , necessarily identical) spectra are unitarily equivalent, at least if certain conditions concerning the nature of the sets C_α and the basis functions $\rho_\alpha(\lambda)$ are met. In fact there will be proved the following.

THEOREM (*). *Suppose that there exist two (distinct) values α_1 and α_2 ($0 \leq \alpha_k < \pi$) such that, for each of the two boundary value problems determined by (1) and (2_{α_k}) , the following three conditions are satisfied:*

- (i) $S_{\alpha_k} \neq (-\infty, \infty)$,
- (ii) *the point spectrum is empty, and*
- (iii) $\rho_{\alpha_k}(\lambda)$ *is absolutely continuous. Then H_{α_1} and H_{α_2} are unitarily equivalent.*

The condition (i) of (*) surely holds if, for instance, f is bounded or even bounded from below on $0 \leq t < \infty$. It should be noted however that every (real) λ belongs to an S_α for some α (depending on λ); [1].

For other results on the continuous spectra of boundary value pro-

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blems with absolutely continuous basis functions (on certain intervals), see [4].

The proof of (*) in § 2 will depend upon the following result of M. Rosenblum [5] concerning perturbations of self-adjoint operators: Let the self-adjoint operators $A_k = \int \lambda dE(\lambda)$ (for $k=1, 2, 3$) satisfy $A_1 - A_2 = A_3$. Suppose, in addition, that A_3 is completely continuous and such that $\int |\lambda| dE_3(\lambda)$ has a finite trace while (E_1x, y) and (E_2x, y) are absolutely continuous functions of λ for arbitrary x and y in Hilbert space. Then A_1 and A_2 are unitarily equivalent.

2. Proof of (*). In the sequel, the index α_k will be replaced by k . It is clear from the assumptions that there exists some real value $\lambda = \lambda^*$ not belonging to S_k for $k=1, 2$. Consequently, since $f(t)$ can be replaced in (1) by $f(t) + \lambda^*$, it can be supposed without loss of generality that $\lambda=0$ is not in either of the sets S_k . Then the operators H_k^{-1} , where

$$H_k^{-1} = \int_{\lambda}^{-1} dE_k(\lambda) = \int dF_k(\lambda) \quad (F_k(\lambda) = E_k(\lambda^{-1}))$$

are bounded, self-adjoint integral operators with kernels $G_k(s, t)$ on $0 \leq s, t < \infty$; cf. for example, [2], [7]. Furthermore,

$$G_1(s, t) - G_2(s, t) = cg(s)g(t),$$

where c denotes a constant and $g(t)$ is a function of class $L^2[0, \infty)$; cf. [7, p. 251]. Thus $(H_1^{-1} - H_2^{-1})x$ is a multiple of g for every element x of class $L^2[0, \infty)$. Hence $H_1^{-1} - H_2^{-1}$ is a multiple of a one-dimensional projection operator; in particular, $H_1^{-1} - H_2^{-1}$, corresponding to A_3 , satisfies the trace condition on that operator mentioned at the end of § 1.

In view of the assumptions (ii) and (iii) of (*), it follows from the formulas relating the basis functions $\rho_k(\lambda)$ to the projections $E_k(\lambda)$ (cf., for example, [2]) that $\|E(\lambda)x\|$ is an absolutely continuous function of λ for every x in the Hilbert space; therefore $(E_k(\lambda)x, y)$, hence also $(F_k(\lambda)x, y)$, is absolutely continuous for every pair x, y of this space. According to the above mentioned theorem of Rosenblum, it now follows that the operators H_k^{-1} (hence also the H_k) are unitarily equivalent, and the proof of (*) is now complete.

3. Consider the special case of (1) in which $f \equiv 0$. It is readily seen that there are no eigenvalues for either of the boundary value problems determined by $x'' + \lambda x = 0$ and the respective boundary conditions $x(0) = 0$ and $x'(0) = 0$. (These boundary conditions correspond to $\alpha = 0$,

$\pi/2$ in (2^α) ; in a somewhat more general connection, cf. [3, p. 792]). Thus, in each case, there is a purely continuous spectrum consisting of the half-line $0 \leq \lambda < \infty$. Moreover, the basis functions, which, in this instance, are even known explicitly [6, p. 59] are absolutely continuous. Consequently, Theorem (*) is applicable and shows that the self-adjoint operators belonging to the above mentioned boundary value problems are unitarily equivalent.

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