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**FAMILIES OF TRANSFORMATIONS IN THE FUNCTION
SPACES H^p**

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I. Introduction

Let the interior of the unit circle be denoted by \mathcal{A} ; and let the set of functions single-valued and analytic in \mathcal{A} be denoted by \mathfrak{A} .

It is well known that certain subsets of \mathfrak{A} can be made into Banach spaces by the introduction of suitable norms. In particular, if $f \in \mathfrak{A}$, and if, for $1 \leq p \leq \infty$,

$$(I.1) \quad \mathcal{M}_p(f, r) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p}, \quad p < \infty$$

$$\mathcal{M}_p(f; r) = \sup_{|z| < r} |f(z)|, \quad p = \infty$$

and if $\sup_{r < 1} \mathcal{M}_p(f; r) < \infty$, then f is said to be in the set H^p . Also, H^p is a Banach space with

$$(I.2) \quad \|f\|_{H^p} = \sup_{r < 1} \mathcal{M}_p(f; r)$$

A proof of these statements, together with a discussion of many properties of the spaces H^p , can be found in [8].

This paper is concerned with certain transformations in the spaces H^p ¹.

Let $\omega(z)$ be a function of z which is analytic in \mathcal{A} and such that $|\omega(z)| < 1$ for $z \in \mathcal{A}$. If $f \in \mathfrak{A}$, then so is the function defined by $f[\omega(z)]$. For $f \in \mathfrak{A}$, we define

$$(I.3) \quad T_\omega f = g \iff f[\omega(z)] = g(z) \text{ for } z \in \mathcal{A}.$$

T_ω is clearly an additive, homogeneous transformation.

It is well known [4] that if $f \in H^p$ and $\omega(0) = 0$, then $T_\omega f \in H^p$ and $\|T_\omega f\| \leq \|f\|$. In other words, if $\omega(0) = 0$, then $T_\omega \in [H^p]$ (the set of all linear bounded transformations on H^p to H^p), and $\|T_\omega\| \leq 1$. Our first task is to prove the following.

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¹ In the following, all statements about H^p refer to $1 \leq p \leq \infty$ unless further qualified.

THEOREM I.1. *If $\omega \in \mathfrak{A}$ and $|\omega(z)| < 1$ for $z \in \Delta$, and if $|\omega(0)| = \alpha < 1$, then $T_\omega \in [H^p]$ and $\|T_\omega\| \leq \left(\frac{1+\alpha}{1-\alpha}\right)^{1/p}$. There is at least one such ω for which the equality holds.*

Proof. For $p = \infty$, the theorem is trivial. For $1 \leq p < \infty$, a simple proof (for which the author is indebted to the referee) is as follows.

For $f \in H^p$, let u be the least harmonic majorant of $|f|^p$ in Δ (see [6]). Then $T_\omega u$ is a harmonic majorant of $|T_\omega f|^p$. Also,

$$\|f\| = \{u(0)\}^{1/p} \text{ and } \|T_\omega f\| \leq \{(T_\omega u)(0)\}^{1/p} = \{u(\beta)\}^{1/p}$$

where $\beta = \omega(0)$. The Poisson integral for u shows that

$$u(\beta) \leq u(0) \left(\frac{1+|\beta|}{1-|\beta|} \right)$$

Putting $\alpha = |\beta|$, it follows that

$$\|T_\omega f\| \leq \|f\| \left(\frac{1+\alpha}{1-\alpha} \right)^{1/p}.$$

To complete the proof, we note that the following statement holds. Define the transformation L_α ($0 \leq \alpha < 1$) by

$$L_\alpha f(z) = f\left(\frac{z+\alpha}{1+\alpha z}\right).$$

Then the function

$$f(z) = \left(\frac{z+1}{z-1}\right)^\eta$$

is an eigenfunction of $L_\alpha: L_\alpha f = \lambda f$, belonging to the eigenvalue

$$\lambda = \left(\frac{1+\alpha}{1-\alpha}\right)^\eta,$$

provided $|\Re \eta| < 1/p$. This follows trivially from the fact that $f \in H^p$ provided $|\Re \eta| < 1/p$.

The result stated in Theorem I.1 can be sharpened as follows.

COROLLARY I.1. *For any ω ($\omega \in \mathfrak{A}$, mapping Δ into or onto itself),*

$$(I.4) \quad \|T_\omega\| \leq \inf_{\substack{\zeta \in \Delta \\ \eta \in \Delta}} \left\{ \left(\frac{1+|\zeta|}{1-|\zeta|} \right) \left(\frac{1+|\eta|}{1-|\eta|} \right) \left(\frac{1+|\Gamma_\omega(\eta, \zeta)|}{1-|\Gamma_\omega(\eta, \zeta)|} \right) \right\}^{1/p}$$

where

$$\Gamma_\omega(\eta, \zeta) = \frac{\omega(\eta) + \zeta}{1 + \zeta\omega(\eta)}$$

Proof. For $\zeta \in \Delta$, define L_ζ by

$$L_\zeta f(z) = f\left(\frac{z + \zeta}{1 + \zeta z}\right)$$

Then

$$T_\omega = L_{-\eta} L_\eta T_\omega L_\zeta L_{-\zeta}$$

where

$$\eta \in \Delta, \zeta \in \Delta$$

so that

$$\|T_\omega\| \leq \|L_{-\eta}\| \|L_{-\zeta}\| \|L_\eta T_\omega L_\zeta\|$$

Now, $\frac{z - \zeta}{1 - \zeta z}$ takes 0 into $-\zeta$; $\frac{z - \eta}{1 - \eta z}$ takes 0 into $-\eta$;

and $\omega\left(\frac{z + \eta}{1 + \eta z}\right) + \zeta / 1 + \bar{\zeta}\omega\left(\frac{z + \eta}{1 + \eta z}\right)$ takes 0 into $\frac{\omega(\eta) + \zeta}{1 + \zeta\omega(\eta)}$

Applying Theorem I.1, we obtain (I.4).

We are thus assured that a transformation T_ω defined by $T_\omega f(z) = f[\omega(z)]$ is a member of $[H^p]$, $1 \leq p \leq \infty$. § II is devoted to a study of semigroups and groups of these transformations. Section III contains a discussion of two examples which illustrate the theorems of § II.

II. Families of Transformations in H^p

A. Definitions and preliminary results. Consider a family of functions $\{\omega(z; t)\}$ —also denoted by $\{\omega_t(z)\}$ —where $z \in \Delta$ and t belongs to a set \mathcal{T} of complex numbers. The individual functions will be denoted by $\omega(z; t)$ or by $\omega_t(z)$, according to convenience.

Let the set \mathcal{T} satisfy the following conditions.

- (CII.1) (i) If $t_1, t_2 \in \mathcal{T}$, then $t_1 + t_2 \in \mathcal{T}$.
- (ii) \mathcal{T} contains the origin and some ray originating at the origin.
- (iii) Every two points in \mathcal{T} can be connected by a path² in \mathcal{T} .

² Here a path is defined to mean a finite number of rectifiable Jordan arcs joined together; see [3, pp 13, 14].

Let the family $\{\omega(z; t)\}$ satisfy the following conditions:

- (CII.2) (i) For each $t \in \mathcal{T}$, $\omega_t \in \mathfrak{A}$, and ω_t maps Δ into (or onto) itself.
 (ii) For $t_1, t_2 \in \mathcal{T}$, and $z \in \Delta$,

$$\omega_{t_2}[\omega_{t_1}(z)] = \omega_{t_1}[\omega_{t_2}(z)] = \omega_{t_1+t_2}(z)$$

- (iii) $\omega(z; 0) = z$ for $z \in \Delta$.
 (iv) For each $z \in \Delta$, $\omega(z; t)$ is differentiable³ with respect to t for $t \in \mathcal{T}$. Also, if

$$P(z) = \frac{\partial}{\partial t} \omega(z; t)|_{t=0},$$

then $P \in \mathfrak{A}$.

We can immediately state the following.

LEMMA II.1. For fixed $z \in \Delta$,

$$(II.1) \quad \frac{\partial}{\partial t} [\omega(z; t)] = P[\omega(z; t)]$$

Proof. $\omega[\omega(z; t); h] = \omega(z; t+h)$ for $t, h \in \mathcal{T}$

Therefore

$$\begin{aligned} \frac{\omega(z; t+h) - \omega(z; t)}{h} &= \frac{\omega[\omega(z; t); h] - \omega(z; t)}{h} \\ &= \frac{\omega[\omega(z; t); h] - \omega[\omega(z; t); 0]}{h} \end{aligned}$$

Letting $h \rightarrow 0$ (in \mathcal{T}), we obtain (II.1).

The family of transformations $\{T_{\omega_t}\}$ defined by (I.3) with $\omega = \omega_t$ will henceforth be denoted simply by $\{T_t\}$. This family forms a semi-group (possibly a group) of linear bounded transformations in the spaces H^p . (The boundedness is shown by Theorem I.1.)

We define the generator A of the family $\{T_t\}$ by

$$(II.2) \quad Af = \lim_{t \rightarrow 0} \frac{T_t f - f}{t}, \quad f \in H^p$$

the limit taken in the strong sense in H^p . The domain of A , denoted

³ Here and in the following, "differentiability with respect to t for $t \in \mathcal{T}$ " implies that the difference quotient approaches the *same* limit no matter how t is approached (as long as the approach is made entirely in \mathcal{T}).

by $\mathcal{D}(A)$, is defined to be the subset of H^p for which the limit in (II.2) exists as $t \rightarrow 0$, $t \in \mathcal{T}$ (the limit to be the same for all modes of approach within \mathcal{T} to 0).

It follows from (II.2) that, for $f \in \mathcal{D}(A)$, and each $z \in A$,

$$(II.3) \quad Af(z) = \lim_{t \rightarrow 0} \frac{T_t f(z) - f(z)}{t}$$

This is true since, for fixed $z \in A$, $f(z)$ is a bounded linear functional of f , [7].

Now

$$\begin{aligned} Af(z) &= \lim_{t \rightarrow 0} \frac{f[\omega(z; t)] - f(z)}{t} \\ &= \lim_{t \rightarrow 0} \frac{f[\omega(z; t)] - f[\omega(z; 0)]}{t} \\ &= \frac{\partial}{\partial t} f[\omega(z; t)]|_{t=0} = f'[\omega(z; t)] \frac{\partial}{\partial t} \omega(z; t)|_{t=0} \end{aligned}$$

or

$$(II.4) \quad Af(z) = P(z)f'(z) \quad z \in A, f \in \mathcal{D}(A)$$

It is thus clear that $\mathcal{D}(A)$ is contained in the subset of H^p consisting of those elements f for which $f'(z)P(z)$ defines an element of H^p .

B. Differentiability properties of the family $\{T_t\}$

THEOREM II.1. *Let f be in H^p , and t_0 be in \mathcal{T} ; let $g(z) = P(z)f'(z)$ and suppose that*

- (i) *There exists a neighborhood \mathcal{N}_{t_0} of t_0 and a positive constant M such that every point t of \mathcal{N}_{t_0} can be connected to t_0 by a polygonal line in $\mathcal{N}_{t_0} \cap \mathcal{T}$ of length $\leq M|t_0 - t|$;*
- (ii) *$T_t g \in H^p$ for $t \in \mathcal{N}_{t_0} \cap \mathcal{T}$;*
- (iii) *$\|T_t g - T_{t_0} g\| \rightarrow 0$ as $t \rightarrow t_0$ ($t \in \mathcal{T}$).*

Then, $T_t f$ is strongly differentiable with respect to t at t_0 and

$$(II.5) \quad \frac{d}{dt} T_t f|_{t=t_0} = T_{t_0} g.$$

Before giving the proof, the following formal derivation might be of interest

$$\begin{aligned} \lim_{t \rightarrow t_0} \frac{T_t f - T_{t_0} f}{t - t_0} &= \lim_{s \rightarrow 0} T_{t_0} \left\{ \frac{T_s f - f}{s} \right\} && (s = t - t_0) \\ &= T_{t_0} A f = T_{t_0} g \end{aligned}$$

This is however not a rigorous proof, even when $f \in \mathcal{D}(A)$, since $s = t - t_0$ may not be in \mathcal{S} for all $t \in \mathcal{N}_{t_0} \cap \mathcal{S}$.

A rigorous proof is as follows.

Let $f[\omega(z; t)] = h(z; t)$ and let

$$(II.6) \quad D(z; t; t_0) = \frac{h(z; t) - h(z; t_0)}{t - t_0} - T_{t_0} g(z)$$

If $z = r e^{i\theta}$, and if $\frac{\partial}{\partial t} h(z; t)$ is denoted by $h_t(z; t)$, then, from (II.1),

$$\begin{aligned} D(z; t; t_0) &= \frac{h(r e^{i\theta}; t) - h(r e^{i\theta}; t_0)}{t - t_0} - h_t(r e^{i\theta}; t_0) \\ &= \frac{1}{t - t_0} \int_{t_0}^t [h_t(r e^{i\theta}; \tau) - h_t(r e^{i\theta}; t_0)] d\tau \end{aligned}$$

where t is chosen in \mathcal{N}_{t_0} and the integral is taken along a polygonal line in $\mathcal{N}_{t_0} \cap \mathcal{S}$ connecting t and t_0 and of length $\leq M|t - t_0|$.

First suppose that $1 \leq p < \infty$. Then

$$\begin{aligned} (II.7) \quad \mathcal{M}_p(D; r) &= \left\{ \frac{1}{2\pi} \int_0^{2\pi} |D(r e^{i\theta}; t; t_0)|^p d\theta \right\}^{1/p} \\ &= \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{1}{t - t_0} \int_{t_0}^t [h_t(r e^{i\theta}; \tau) - h_t(r e^{i\theta}; t_0)] d\tau \right|^p d\theta \right\}^{1/p} \end{aligned}$$

Let $\tau = \tau(s)$, $0 \leq s \leq 1$, $\tau(0) = t_0$, $\tau(1) = t$. Here s is a constant times the arc length. Then [4], [1]

$$\begin{aligned} \mathcal{M}_p(D; r) &= \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{1}{t - t_0} \int_0^1 [h_t(r e^{i\theta}; \tau) - h_t(r e^{i\theta}; t_0)] \tau'(s) ds \right|^p d\theta \right\}^{1/p} \\ &\leq \frac{1}{|t - t_0|} \int_0^1 |\tau'(s)| \left\{ \frac{1}{2\pi} \int_0^{2\pi} |h_t(r e^{i\theta}; \tau) - h_t(r e^{i\theta}; t_0)|^p d\theta \right\}^{1/p} ds \end{aligned}$$

Hence,

$$\|D\| = \left\| \frac{T_t f - T_{t_0} f}{t - t_0} - T_{t_0} g \right\| = \sup_{r < 1} \mathcal{M}_p(D; r)$$

$$\begin{aligned} &\leq \frac{1}{|t-t_0|} \int_0^1 |\tau'(s)| \left[\sup_{r<1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |h_i(re^{i\theta}; \tau) - h_i(re^{i\theta}; t_0)|^p d\theta \right\}^{1/p} \right] ds \\ &= \frac{1}{|t-t_0|} \int_0^1 |\tau'(s)| \|T_{\tau}g - T_{t_0}g\| ds \leq M \sup_{0 \leq s \leq 1} \|T_{\tau}g - T_{t_0}g\| \end{aligned}$$

Now, by (iii), as $t \rightarrow t_0$, the quantity $\sup_{0 \leq s \leq 1} \|T_{\tau}g - T_{t_0}g\|$ goes to zero. Thus $\|D\| \rightarrow 0$ as $t \rightarrow t_0$.

For $p = \infty$, the proof follows similar lines.

COROLLARY II.1-1. *Let f be in H^p , t_0 be in \mathcal{T} , and let $g(z) = P(z)f'(z)$. Suppose condition (i) of Theorem II.1 holds and in addition, suppose that*

- (a) $|\omega(z; t_0)| < r < 1$ for $z \in \Delta$
- (b) $\omega(z; t)$ is continuous with respect to t at t_0 , uniformly in z for $z \in \Delta$.

Then, $T_t f$ is differentiable with respect to t at t_0 and (II.5) holds.

Proof. By (b), there exists a neighborhood \mathcal{N}'_{t_0} of t_0 such that $|\omega(z; t)| < r' < 1$ for $z \in \Delta, t \in \mathcal{N}'_{t_0} \cap \mathcal{T}$.

Now, $g(z)$ is analytic in Δ . Therefore for $t \in \mathcal{N}'_{t_0} \cap \mathcal{T}$, $T_t g(z) = g[\omega(z; t)]$ is bounded in Δ and therefore $T_t g \in H^p$.

Also, $T_t g(z)$ is continuous with respect to t at t_0 , uniformly in z for $z \in \Delta$. Hence $\sup_{z \in \Delta} |T_t g(z) - T_{t_0} g(z)| \rightarrow 0$ as $t \rightarrow t_0$.

THEOREM II.2. *Suppose*

- (i) *Condition (i) of Theorem II.1 holds for $t_0 = 0$;*
- (ii) $\|T_t f - f\| \rightarrow 0$ as $t \rightarrow 0$ ($t \in \mathcal{T}$) for every $f \in H^p$.

Then, $\mathcal{D}(A)$, the domain of the generator A (defined by II.2), is the set of elements $f \in H^p$ for which $g(z) = f'(z)P(z)$ defines an element g of H^p .

Proof. Let \mathcal{G} denote the set of elements $f \in H^p$ such that $g(z) = f'(z)P(z)$ defines an element g of H^p . We already know (last paragraph of IIA) that $\mathcal{D}(A) \subset \mathcal{G}$. To show that $\mathcal{G} \subset \mathcal{D}(A)$, one must verify conditions (ii) and (iii) of Theorem II.1 for $f \in \mathcal{G}, t_0 = 0$.

Since $f \in \mathcal{G}$ implies $g \in H^p$, it follows from Theorem I.1 that $T_t g \in H^p$ for all $t \in \mathcal{T}$. Also, condition (iii) of Theorem II.1 is obtained for $t_0 = 0$ by applying condition (ii) of Theorem II.2 to the function g . Equation (II.5) becomes

$$(II.8) \quad Af=g \quad \text{where} \quad g(z)=P(z)f'(z).$$

THEOREM II.3. Under conditions (i) and (ii) of Theorem II.2, A is a closed transformation. Also $\mathcal{D}(A)$ is dense in H^p .

Proof. Let f_n be in $\mathcal{D}(A)$; $f_n \rightarrow f$ (in the norm of H^p) $Af_n \rightarrow g \in H^p$ (in the norm of H^p). Then [7]

$$\left. \begin{array}{l} f_n(z) \rightarrow f(z) \\ P(z)f_n'(z) \rightarrow g(z) \end{array} \right\} \text{uniformly on compact subsets of } \Delta,$$

that is, $g(z)=P(z)f'(z)$ for $z \in \Delta$.

Therefore, since $g \in H^p$, then, by Theorem II.2, $f \in \mathcal{D}(A)$ and $Af = g$. See [2, Chap. 11] for the fact that $\mathcal{D}(A)$ is dense in H^p .

C. The family of transformations generated by a given operator of the form $Af(z)=P(z)f'(z)$. Suppose P is a given function in \mathfrak{A} . The following question arises: Is there a set \mathcal{S} in the complex plane and a set of functions $\{\omega_t\}$ satisfying, respectively, conditions CII.1 and CII.2? If so, how, knowing just $P(z)$, can one determine the family $\{\omega_t\}$ and the maximum set \mathcal{S} ?

To investigate these questions, additional conditions will be imposed on the given function $P(z)$. First,

(CII.3) $1/P(z)$ is analytic in Δ except, possibly, for a single pole.

Let the function $Q(z)$ be defined by

$$(II.9) \quad Q(z) = \int_{z_0}^z \frac{d\zeta}{P(\zeta)} \quad z_0, z \in \Delta$$

The path of integration is chosen in Δ so as not to pass through any singularity of $1/P(z)$; also, z_0 is chosen so as not to be a singularity of $1/P(z)$. $Q(z)$ may be a many-valued function.

$Q(z)$ depends on the choice of z_0 ; however, as will become clear below, it is not worthwhile to express this dependence in the notation. Clearly, all definitions of Q (corresponding to different choices of z_0) differ from each other by additive constants.

The following property of Q is worth noting.

Let z_1 and z_2 be in Δ , and not singularities of $1/P(z)$; let $Q^{(1)}(z_1)$, $Q^{(2)}(z_1)$ be two values of Q at $z=z_1$; and let $Q^{(1)}(z_1) - Q^{(2)}(z_1) = h$. Let $Q^{(1)}(z_2)$ be a value of Q at $z=z_2$. There exists a value of Q at $z=z_2$, which may be denoted by $Q^{(2)}(z_2)$, such that $Q^{(1)}(z_2) - Q^{(2)}(z_2) = h$. This is clear from the definition of Q and from (CII.3).

We shall further assume:

(CII.4) *If z_1 and z_2 are in Δ , are not singularities of $1/P(z)$, and $z_1 \neq z_2$, then $Q(z_1) \neq Q(z_2)$.*

This may, of course, be regarded as a condition on $P(z)$.

Now suppose $P \in \mathfrak{A}$ is given satisfying (CII.3) and (CII.4), and that a set \mathcal{S} and a family $\{\omega_t\}$ exist satisfying (CII.1) and (CII.2). From (II.1) and (CII.2-iii), regarding z as fixed for the moment, one can write

$$(II.10) \quad \left. \begin{aligned} \frac{d}{dt} \omega(z; t) &= P[\omega(z; t)] \\ \omega(z; 0) &= z \end{aligned} \right\} \begin{array}{l} z \in \Delta \\ t \in \mathcal{S} \end{array}$$

Let z be fixed in Δ and not a singularity of $1/P(z)$. Then, from (II.10), $\omega(z; t)$ must satisfy

$$(II.11) \quad Q[\omega(z; t)] = Q(z) + t.$$

Now, for fixed $t \in \mathcal{S}$, $\omega(z; t)$ must be an analytic function of z in Δ , mapping Δ into itself.

Let I_Q be the image under Q of Δ (excluding the possible singularity of $1/P(z)$). The set I_Q includes *all* values of $Q(z)$ which can be obtained by integrating in (II.9) along paths which are entirely in Δ . If $\omega(z; t)$, for fixed $t \in \mathcal{S}$, is defined for all $z \in \Delta$, and such that $|\omega(z; t)| < 1$, then (II.11) implies that this t must translate I_Q into a subset of itself: $I_Q + t \subset I_Q$.

Let \mathcal{S}_Q be the set of translations of I_Q into or onto itself. (Clearly \mathcal{S}_Q does not depend on the choice of z_0 in defining Q .) Then $\mathcal{S} \subset \mathcal{S}_Q$.

On the other hand if P being given⁴, \mathcal{S}_Q contains a subset \mathcal{S}^* satisfying conditions (CII.1), then a family $\{\omega_t\}$ satisfying (CII.2) exists (with $t \in \mathcal{S}^*$).

Define, for $t \in \mathcal{S}^*$, $z \in \Delta$,

$$(II.12) \quad \omega(z; t) = \begin{cases} Q^{-1}[Q(z) + t], & z \text{ not a singularity of } \frac{1}{P(z)} \\ z, & z \text{ a singularity of } \frac{1}{P(z)} \end{cases}$$

where Q^{-1} denotes the function inverse to Q .

This definition defines ω uniquely. If $Q(z)$ refers to a particular branch of Q , then ω is uniquely determined (in Δ) because of (CII.4); moreover, by the property of Q mentioned on p. it is seen that the same point ω is defined no matter what branch of Q is used in (II:12).

⁴ $P \in \mathfrak{A}$ and satisfying (CII. 3) and (CII. 4).

It is also clear that $\omega(z; t)$ does not depend on the choice of z_0 .

The function $\omega(z; t)$ thus defined is analytic in z for each $t \in \mathcal{T}^*$. This is clear if z is not a singularity of $1/P(z)$. If z_1 is a singularity of $1/P(z)$ in Δ , it is necessary to show that $\omega(z; t)$ is (for fixed t) continuous at $z=z_1$; that is, (from II.12) $\omega_i(z) \rightarrow z_1$ as $z \rightarrow z_1$.

Since z_1 is a pole of $1/P(z)$, one can say, by the definition of Q , that there exist points $\omega_i(z)$ approaching z_1 as $z \rightarrow z_1$, such that (II.12) is satisfied. But, by (CII.4), these points are the only ones in Δ for which (II.12) is satisfied.

The other conditions of (CII.2) are readily verified for the functions $\omega(z; t)$ as defined by (II.12).

The preceding results may be summed up as follows.

THEOREM 11.4. *Let $P(z)$ be in \mathfrak{A} , satisfying (CII.3) and (CII.4). Let $Q(z)$ be defined by (II.9); let I_q be the image of Δ under Q , let \mathcal{T}_q be the set of translations of I_q into or onto itself.*

Then, there exists a set \mathcal{T} and a family $\{\omega_i\}$ satisfying (CII.1) and (CII.2), if and only if \mathcal{T}_q contains a subset \mathcal{T}^ satisfying (CII.1). The maximum set \mathcal{T} is the "direct sum" of all subsets of \mathcal{T}_q which satisfy (CII.1). Here "direct sum" is defined as follows: If $\{G_\alpha\}$ is a collection of subsets of the complex plane, each containing the origin, the direct sum of the sets $\{G_\alpha\}$ is defined to be the set consisting of all elements of the form $t=t_1+\dots+t_n$ where n is a finite (positive) integer and where $t_i \in \bigcup_{\alpha} G_\alpha$.*

The last statement follows from the fact that the direct sum of subsets of \mathcal{T}_q satisfying (C.II.1) is also a subset of \mathcal{T}_q which satisfies (C.II.1).

One result of the previous theorem is the following.

THEOREM II.5. *If $P(z) \in \mathfrak{A}$, satisfying (CII.3) and (CII.4), and if there exists a set \mathcal{T} and a family $\{\omega_i\}$ satisfying (CII.1) and (CII.2), then $1/P(z)$ can have only a pole of first order in Δ .*

Proof. If $1/P(z)$ had a pole of order higher than the first, then I_q would have a bounded (and non-null) complement; therefore \mathcal{T}_q would consist only of the point $t=0$.

Thus, if ζ_0 is the singularity of $1/P(z)$, then $Q(z)$ can be written

$$(II.13) \quad Q(z) = q_0 \ln(z - \zeta_0) + Q_1(z)$$

where $Q_1(z)$ is analytic in Δ .

Theorems II.6 and II.7 refer to families of transformations generated by $P(z)$ satisfying (CII.3) and (CII.4).

THEOREM II.6. *If $\omega(z_1; t)=z_1$, $z_1 \in \Delta$, for $t \neq 2\pi ikq_0$, $k=0, \pm 1, \pm 2, \dots$, then $z_1=\zeta_0$.*

Proof. $Q[\omega(z; t)]=Q(z)+t$ for $z \neq \zeta_0$.
 Therefore $Q[z_1]=Q[z_1]+t$ if $z \neq \zeta_0$.
 Therefore $t=2\pi ikq_0$, $k=0, \pm 1, \dots$.

THEOREM II.7. *If $\omega(z_1; t)=\omega(z_2; t)$, $t \in \mathcal{S}$, then $z_1=z_2$.*

Proof. Suppose first that $z_1, z_2 \neq \zeta_0$. Then $\omega(z_1; t)=\omega(z_2; t)$ would imply $Q(z_1)=Q(z_2)$ or, by (CII.4), $z_1=z_2$. On the other hand, if, say, $z_1=\zeta_0$, then $\omega(z_1, t)=z_1=\omega(z_2; t)$ and so $z_2=z_1$ by Theorem II.6.

Thus, conditions (CII.3) and (CII.4) when imposed on the function $P(z)$ imply that the family $\{\omega_i\}$ is a family of schlicht functions.

It is clear that the functions ω_i as well as the set \mathcal{S} are unaltered if the definition of Q is altered by the addition of an arbitrary constant.

It is also easy to see that multiplying Q (that is, multiplying $1/P$) by a constant $c \neq 0$ yields essentially the same family of transformations:

Let $\mathcal{S}, \{\omega_i\}$ correspond to $P(z)$ and let $\mathcal{S}', \{\omega'_i\}$ correspond to $\frac{1}{c}P(z)$. (Here the primes do not, of course, imply differentiation.) Then clearly, $\mathcal{S}'=c\mathcal{S}$. Also, for $t' \in \mathcal{S}'$,

$$cQ[\omega'(z; t')]=cQ(z)+t',$$

or

$$Q[\omega'(z; t')]=Q(z)+\frac{t'}{c},$$

so that

$$(II.14) \quad \omega'(z; t')=\omega\left(z; \frac{t'}{c}\right); \quad t' \in \mathcal{S}', \quad \frac{t'}{c} \in \mathcal{S}.$$

In other words, there is a one-to-one correspondence between the transformations corresponding to $P(z)$ and those corresponding to $\frac{1}{c}P(z)$; the correspondence is given by (II.14).

Now consider, for $t \in \mathcal{S} \cap I_q$, the parameter defined by

$$(II.15) \quad \beta=Q^{-1}(t) \qquad t \in \mathcal{S} \cap I_q$$

Then $\beta \in \Delta$ and (II.12) becomes, writing $\omega[z; t(\beta)]$ simply as $\omega(z; \beta)$,

$$(II.16) \quad \omega(z; \beta)=Q^{-1}[Q(z)+Q(\beta)], \qquad z, \beta \in \Delta.$$

Here β is defined on $Q^{-1}[\mathcal{S} \cap I_Q]$.

It is always possible to define Q in such a way⁵ that $\mathcal{S} \subset I_Q$ and therefore $\mathcal{S} \cap I_Q = \mathcal{S}$. In such a case, (II.15) and (II.16) hold for all $t \in \mathcal{S}$. For example, in defining Q by (II.9), it is clear that $Q(z_0) = 0$ for $z_0 \in \Delta$. Thus, for Q defined as in (II.9) with $z_0 \in \Delta$, we have $\beta = Q^{-1}(t) = \omega(z_0; t)$.

It is, however, often possible and more convenient to define Q such that \mathcal{S} is the closure of I_Q . It is also often possible to extend the definition of Q to the boundary of Δ in such a way that the boundary of Δ goes (under Q) into the boundary of I_Q . (An example of this is given by the family of transformations studied in the next section.) In such cases, (II.15) holds for all $t \in \mathcal{S}$ and, in (II.16), β may be a point on the boundary of Δ .

The law of composition of the transformations $T_{\omega_\beta} = T_\beta$ in terms of the parameter β is

$$(II.17) \quad \begin{cases} T_{\beta_1} T_{\beta_2} = T_{\beta_3} \\ \beta_3 = \omega(\beta_1; \beta_2) \end{cases}$$

This can be shown as follows.

$$\omega[\omega(z; t_1); t_2] = \omega(z; t_1 + t_2),$$

so

$$\begin{aligned} \omega[\omega(z; \beta_1); \beta_2] &= \omega[z; t = Q(\beta_1) + Q(\beta_2)] \\ &= \omega[z; \beta = \omega(\beta_1; \beta_2)]. \end{aligned}$$

By simply looking at the set I_Q , one is usually able to determine many of the properties of the family $\{T_t\}$. For example, one may determine (a) whether or not such a family exists for the given $P(z)$; (b) what the maximum parameter domain \mathcal{S} is; (c) whether $\{T_t\}$ is a group or a semigroup; (d) which of the functions ω_t transform Δ onto itself and which transform Δ into but not onto itself;

D. Possible applications. The above results provide the basis for obtaining a variety of theorems by rephrasing known results in the theory of transformations in Banach space in terms of transformations in the function spaces H^p of the kind studied above. Three possible categories of results are:

(a) Representations of the transformations T_t in terms of the generator A or the resolvent of A ([2] contains many such formulas).

(b) Application of results in the theory of analytic Banach-space-

⁵ The addition of a constant to Q changes I_Q but leaves \mathcal{S} unaltered.

valued functions of a complex variable ([2], [7], [9])

(c) Other theorems concerning properties of semigroups or groups of transformations in Banach space.

III. Two Special Cases

A. The family $\{T_w\}$ defined by $T_w f(z) = f(wz)$, $|w| \leq 1$.

Let

$$(III.1) \quad P(z) = -z$$

and⁶

$$(III.2) \quad Q(z) = \int_1^z \frac{-d\zeta}{\zeta} = -\ln z.$$

Then I_Q is the open right half plane: $\Re(z) > 0$. \mathcal{J}_Q is the closed right half plane: $\Re(z) \geq 0$. Clearly, \mathcal{J}_Q itself satisfies conditions (CII.1) and is therefore the maximum domain \mathcal{S} of the parameter t . We have

$$(III.3) \quad \omega(z; t) = ze^{-t} \quad z \in \mathcal{A}, t \in \mathcal{J}_Q$$

or, if we let

$$(III.4) \quad w = e^{-t}$$

then, writing $\omega[z; t(w)]$ simply as $\omega(z; w)$,

$$(III.5) \quad \omega(z; w) = wz \quad z \in \mathcal{A}, |w| \leq 1$$

The corresponding family of transformations $\{T_w\}$ is then given by

$$(III.6) \quad T_w f = g$$

where

$$g(z) = f(wz)$$

The generator A is defined for those $f \in H^p$ for which the limit

$$Af = \lim_{w \rightarrow 1} \frac{T_w f - f}{1 - w} \quad |w| \leq 1$$

exists in the H^p norm. Thus,

$$(III.7) \quad Af(z) = -zf'(z) \quad \text{for } f \in \mathcal{D}(A).$$

For $1 \leq p < \infty$, $\mathcal{D}(A)$ is the set of functions $f \in H^p$ for which $f'(z)$ defines an element of H^p . This follows from Theorem II.2. The crucial point in applying Theorem II.2 is in verifying condition (ii) of

⁶ Here $z_0=1$ is not in \mathcal{A} , but in this case this is immaterial.

that theorem. This amounts to the following. Let h be in H^p ($1 \leq p < \infty$), and let $T_w h(z) = h(wz)$ for $|w| \leq 1$. Then $T_w h \rightarrow h$ in the norm of H^p as $w \rightarrow 1$ in the closure of Δ . It is not difficult to prove this.

Also, for $1 \leq p < \infty$, A is a closed operator with domain dense in H^p .

For $p = \infty$, (III.7) still holds, but one cannot verify condition (ii) of Theorem II.2 and it is easily seen that $\mathcal{D}(A)$ is not dense in H^∞ .

B. The family $\{L_\alpha\}$ defined by $L_\alpha f(z) = f\left(\frac{z+\alpha}{1+\alpha z}\right)$, $-1 < \alpha < 1$.

Let

$$(III.8) \quad P(z) = (1 - z^2)$$

and⁸

$$(III.9) \quad Q(z) = \int_0^z \frac{d\zeta}{1 - \zeta^2} = \tanh^{-1} z$$

Then I_Q is the strip $|\Im(z)| < \pi/4$. \mathcal{T}_Q is the real axis. Clearly \mathcal{T}_Q satisfies conditions (CII.1) and is therefore the maximum domain \mathcal{T} of the parameter t . We have

$$(III.10) \quad \omega(z; t) = \frac{z + \tanh t}{1 + z \tanh t} \quad t \in \mathcal{T}_Q, z \in \Delta.$$

If we let

$$(III.11) \quad \alpha = \tanh t, \quad t \in \mathcal{T}_Q$$

then, writing $\omega[z; t(\alpha)]$ simply as $\omega(z; \alpha)$,

$$(III.12) \quad \omega(z; \alpha) = \frac{z + \alpha}{1 + \alpha z}, \quad z \in \Delta, -1 < \alpha < 1.$$

The family of transformations $\{L_\alpha\}$ is given by

$$(III.13) \quad L_\alpha f = g$$

where

$$g(z) = f\left(\frac{z + \alpha}{1 + \alpha z}\right)$$

The norm of L_α is

$$(III.14) \quad \|L_\alpha\|_{H^p} = \left[\frac{1 + |\alpha|}{1 - |\alpha|} \right]^{1/p}$$

⁸ The path of integration lying entirely in Δ .

The generator A is defined for those $f \in H^p$ for which the limit

$$Af = \lim_{\alpha \rightarrow 0} \frac{L_\alpha f - f}{\alpha}$$

exists in the H^p norm. Hence

$$(III.15) \quad Af(z) = (1 - z^2)f'(z) \quad \text{for } f \in \mathcal{D}(A).$$

For $1 \leq p < \infty$, $\mathcal{D}(A)$ is the set of functions $f \in P^p$ for which $(1 - z^2)f'(z)$ defines an element of H^p ; also, A is a closed operator with domain dense in H^p . As with the previous example, these statements do not hold for H^∞ .

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