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BIORTHOGONAL SYSTEMS IN BANACH SPACES

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BIORTHOGONAL SYSTEMS IN BANACH SPACES

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1. Introduction. We shall be interested, in this paper, in the following question: Given a biorthogonal system (x_n, f_n) in a separable Banach space B , under what conditions can one assert that the sequence $\{x_n\}$ constitutes a basis? The system (x_n, f_n) is called a biorthogonal system if

$$x_n \in B, f_n \in B^* \quad \text{and} \quad f_n(x_m) = \delta_{nm}.$$

We shall assume throughout the paper that $\|x_n\|=1$ and the sequence $\{x_n\}$ is fundamental. When the sequence $\{x_n\}$ constitutes a basis it will be called *regular* otherwise *irregular*.

2. Irregular systems. Let $\{x_n\}$ be an irregular sequence. (For example the trigonometric functions for $C(-\pi, \pi)$). The following definitions will be used.

$$\varphi_n(x) = \sum_{i=1}^n f_i(x)x_i$$

$$\|x\| = \sup \{ \|\varphi_n(x)\|, n=1, 2, 3, \dots \}$$

Compare [4]

$$E_0 = \{x \mid \lim_{n \rightarrow \infty} \varphi_n(x) = x\}$$

$$E_1 = \{x \mid \|x\| < \infty\}$$

$$E_2 = \{x \mid \lim_{n \rightarrow \infty} \|\varphi_n(x)\| = \infty\}$$

$$E_3 = \{x \mid \|x\| = \infty\}.$$

We have $E_0 \subset E_1$ and $E_2 \subset E_3$. For regular systems $E_0 = E_1 = B$ and $E_2 = E_3 = \phi$ where ϕ is the null set. The system is regular if and only if the sequence $\{\|\varphi_n\|\}$ is bounded [2], and if the sequence $\{\|\varphi_n\|\}$ is not bounded the set

$$\bigcap_{n=1}^{\infty} \{x \mid \|\varphi_n(x)\| \leq K\}$$

is nowhere dense [2], hence for irregular systems the set

$$E_1 = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \{x \mid \|\varphi_n(x)\| \leq k\}$$

is of the first category. Also $E_3 = B - E_1$ is dense and of the second

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category. In the case of regular systems there exists a number $K \geq 1$ such that if $\|x\|=1$ then $1 \leq \|x\| \leq K$. The existence of such a bound, K , is equivalent to the equiboundedness of $\{\|\varphi_n(x)\|\mid \|x\|=1$ and therefore for irregular systems for any number a , there exists a point x such that $\|x\|=1$ and $\|x\| > a$, moreover such a point might be found in the linear manifold generated by $\{x_n\}$. (Equiboundedness of $\{\|\varphi_n(x)\|\}$ on a dense subset of the unit sphere would imply equiboundedness on the unit sphere.) It is interesting to note that for every number $a \geq 1$ there exists a point x such that $\|x\|=1$ and $\|x\|=a$. There exists a point y_n satisfying

$$y_n = \sum_{i=1}^n \alpha_i x_i, \quad \|y_n\|=1, \quad \|y_n\| > a.$$

On the other hand $\|x_1\|=1$ and $\|x_1\|=1$. Let $0 \leq t \leq 1$, then $(1-t)x_1 + ty_n \neq 0$. Define

$$g_\nu(t) = \left\| \varphi_\nu \left(\frac{(1-t)x_1 + ty_n}{\|(1-t)x_1 + ty_n\|} \right) \right\| \quad \nu = 12 \dots n$$

The functions $g_\nu(t)$ are continuous in t , and so is $g(t)$ where

$$g(t) = \sup \{g_\nu(t) \mid 1 \leq \nu \leq n\} .$$

$$g(0) = 1$$

and

$$g(1) = \sup \{\|\varphi_\nu(y_n)\| \mid 1 \leq \nu \leq n\} = \|y_n\| > a.$$

There exists a number t_0 such that

$$0 \leq t_0 < 1 \quad \text{and} \quad g(t_0) = a.$$

This following generalization of Baire’s theorem [1] will be used :
Let $\{u_n(x)\}$ be a sequence of real valued continuous functions defined on a metric space c , and $\lim u_n(x) = u(x)$, $|u_n(x)| \leq M$, then the set of points of discontinuity of u is of the first category.

THEOREM 1. *The set E_2 is of the first category.*

Proof. Define the functions $u_n(x)$ by

$$u_n(x) = \frac{\|\varphi_n(x)\|}{1 + \|\varphi_n(x)\|} .$$

We have $0 \leq u_n(x) \leq 1$ and if $x \in E_0 \cup E_2$ then

$$\lim u_n(x) = u(x)$$

where $u(x)=1$ for $x \in E_2$ and

$$u(x) = \frac{\|x\|}{1 + \|x\|} \text{ for } x \in E_0.$$

If E_2 is a set of the second category then there exists at least one point of continuity of u . Let us denote such a point by x_0 .

The set E_0 is dense in B . Let $\{y_n\}$ be a sequence of points in E_0 with $\lim y_n = x_0$, then

$$u(x_0) = \lim u(y_n) = \lim \frac{\|y_n\|}{1 + \|y_n\|} = \frac{\|x_0\|}{1 + \|x_0\|}$$

The set E_2 is dense in B . If $x \in E_2$ and $y \in E_0$ then $x+y \in E_2$. Let $\{z_n\}$ be a sequence of points in E_2 with $\lim z_n = x_0$, then $u(z_n) = 1$ and

$$\frac{\|x_0\|}{1 + \|x_0\|} = u(x_0) = \lim u(z_n) = 1$$

which is absurd.

THEOREM 2. *Let S be a subset of B such that each point $x \in S$ is the limit of some sequence $\{y_n\}$, $y_n \in B$, and the sequence $\{\|y_n\|\}$ is bounded, then S is of the first category.*

Proof. Define the functions $v_n(x)$ by

$$v_n(x) = \frac{\|\varphi_n(x)\|}{1 + \|\varphi_n(x)\|}$$

then $0 \leq v_1(x) \leq v_2(x) \leq \dots < 1$. If $x \in B$ let $\lim v_n(x) = v(x)$. $v(x) = 1$ for $x \in E_3$ and the set E_3 is dense, hence $v(x) = 1$ at every point of continuity of v . Let x be a point of continuity of v and $\{z_n\}$ a sequence with $\lim z_n = x$, then

$$\lim v(z_n) = v(x) = 1$$

therefore the sequence $\{\|z_n\|\}$ is unbounded. Thus the set S is contained in the set of points of discontinuity of v which is a set of the first category by Baire's theorem.

3. General criteria for regularity. From Theorems 1 and 2 we derive the following criteria.

THEOREM 3. *A necessary and sufficient condition for the regularity of the system (x_n, f_n) is.*

$$\sup \{ \|\varphi_n(x)\|, n=1, 2, \dots \} = \infty$$

implies

$$\lim \|\varphi_n(x)\| = \infty. \quad (\text{or } E_2 = E_3).$$

Proof. If the system is regular, then $E_2 = E_3 = \phi$. On the other hand, if the system is irregular E_2 is of the first category and E_3 of the second category.

Let
$$\phi_n(x) = \sum_{i=1}^n a_i^n x_i$$

denote the point nearest to x on the subspace spanned by

$$\{x_1, x_2, x_3, \dots, x_n\}.$$

THEOREM 4. *The system (x_n, f_n) is regular if and only if the sequence $\{\|\phi_n(x)\|\}$ is bounded for each x .*

Proof. If the system is regular, then there exists a positive number K , such that $\|x\| \leq K\|\phi_n(x)\|$. Then

$$\|\phi_n(x)\| \leq K\|\phi_n(x)\| \leq K(\|x\| + \|x - \phi_n(x)\|) \leq K2\|x\|,$$

hence the condition is necessary. Sufficiency is clear by Theorem 2.

4. Biorthogonal systems in Hilbert spaces. In this section we assume that B is a Hilbert space. In order to use Theorem 4 let us compute $\|\phi_n(x)\|$. $\phi_n(x) = \sum_{i=1}^n a_i^n x_i$ and the coefficient a_i^n may be computed from the equation

$$(x - \sum_{i=1}^n a_i^n x_i, x_k) = 0 \quad k=1, 2, \dots, n$$

or

$$(x, x_k) = \sum_{i=1}^n a_i^n (x_i, x_k) \quad \text{see [5].}$$

We introduce the following notation

$$(x_i, x_k) = c_{ik}$$

$$C_n = (c_{ik}) \quad 1 \leq i \leq n \quad 1 \leq k \leq n$$

$$((x, x_1), (x, x_2), \dots, (x, x_n)) = (r)_n$$

$$(a_1^n, a_2^n, \dots, a_n^n) = (a)_n$$

Then

$$(\gamma)_n = (a)_n C_n \quad \text{or} \quad (a)_n = (\gamma)_n C_n^{-1}$$

since C_n^{-1} exists. Now

$$\left\| \sum_{i=1}^j a_i^n x_i \right\|^2 = \sum_{i,k=1}^j a_i^n \bar{a}_k^n c_{ik} = (a)_n E_j^n C_n E_j^n (a)_n^*$$

where E_j^n is the matrix $(e_{i,m})$ with

$$e_{i,m} = \begin{cases} 1 & l=m \leq j \\ 0 & \text{otherwise} \end{cases}$$

$C_n^* = C_n$ and $(a)_n = (\gamma)_n C_n^{-1}$ hence

$$\left\| \sum_{j=1}^j a_i^n x_i \right\|^2 = (\gamma)_n C_n^{-1} E_j^n C_n E_j^n C_n^{-1} (\gamma)_n^*$$

Orthogonalizing the sequence $\{x_n\}$ by Schmidt's process we get the sequence $\{y_n\}$ with

$$x_1 = y_1 \quad x_k = \sum_{\alpha=1}^k \bar{d}_{k,\alpha} y_\alpha$$

where

$$\bar{d}_{k,\alpha} = \begin{cases} (x_k, y_\alpha) & \alpha \leq k \\ 0 & \alpha > k \end{cases}$$

and $\bar{d}_{11} = 1$ see [3].

Let D_n denote the triangular matrix

$$(d_{k,\alpha}) \quad 1 \leq \alpha \leq n \quad 1 \leq k \leq n.$$

$$(x_i, x_j) = \sum_{\alpha} \bar{d}_{i,\alpha} \bar{d}_{j,\alpha} \quad \text{or} \quad C_n = D_n D_n^*.$$

The matrix D_n can be computed from this relation.

Let

$$(\delta)_n = ((x, y_1), (x, y_2), \dots, (x, y_n))$$

$$(x, x_k) = \sum_{\alpha=1}^k (x, y_\alpha) \bar{d}_{k,\alpha} \quad \text{or} \quad (\gamma)_n = (\delta)_n D_n^*$$

and hence

$$\begin{aligned} \left\| \sum_{i=1}^j a_i^n x_i \right\|^2 &= (\gamma)_n C_n^{-1} E_j^n C_n E_j^n C_n^{-1} (\gamma)_n^* = (\delta)_n D_n^* C_n^{-1} E_j^n C_n E_j^n C_n^{-1} D_n (\delta)_n^* \\ &= (\delta)_n (D_n^{-1} E_j^n D_n) (D_n^* E_j^n (D_n^*)^{-1}) (\delta)_n^* \end{aligned}$$

Let $A_j^n = D_n^{-1} E_j^n D_n$ then

$$||| \phi_n(x) ||| = \max \{ (\delta)_n A_j^n (A_j^n)^* (\delta)_n^* \mid 1 \leq j \leq n \}$$

The triangular matrix A_j^n is an operator defined on the Hilbert space. If

$$x = \sum_{i=1}^{\infty} \delta_i y_i$$

then

$$A_j^n(x) = (\delta_1, \dots, \delta_n) A_j^n.$$

By Theorem 4 and the above computation the system is regular if and only if for each x

$$\sup \{ || A_j^n(x) || \mid 1 \leq j \leq n \ n = 12 \dots \} < \infty$$

or by the uniform boundedness theorem.

THEOREM 5. *The system is regular if and only if the double sequence $\{ || A_j^n || \}$ is bounded, or, in other words, if and only if the set of characteristic roots of $A_j^n (A_n^n)^*$ is bounded.*

We shall use Theorem 3 to derive the following theorems.

THEOREM 6. *The system (x_i, f_i) is regular and $\sum_{i=1}^{\infty} |f_i(x)|^2 < \infty$ if and only if for every $x \in B$ there exists a real number $\alpha = \alpha(x)$ such that*

$$(1) \quad 2\Re \left\{ \sum_{i=1}^n \sum_{j=1}^{i-1} f_i(x) \overline{f_j(x)} c_{ij} \right\} > \alpha$$

Proof. If the system is regular and

$$\sum_{i=1}^{\infty} |f_i(x)|^2 < \infty \text{ then}$$

$$\begin{aligned} ||x||^2 - \sum |f_i(x)|^2 &= \sum_{i \neq j} f_i(x) \overline{f_j(x)} c_{ij} \\ &= 2\Re \left\{ \sum_{j < i} f_i(x) \overline{f_j(x)} c_{ij} \right\} \end{aligned}$$

Therefore the necessity of condition (1) is verified. Assume that condition (1) is satisfied then

$$\begin{aligned} ||\varphi_{n+p}(x)||^2 &= \sum_{i,j=1}^{n+p} f_i(x) \overline{f_j(x)} c_{ij} \\ &= ||\varphi_n(x)||^2 + \sum_{i=n+1}^{n+p} |f_i(x)|^2 + 2\Re \left\{ \sum_{i=n+1}^{n+p} \sum_{j=1}^{i-1} f_i(x) \overline{f_j(x)} c_{ij} \right\} \\ &\geq ||\varphi_n(x)||^2 + 2\alpha \end{aligned}$$

Therefore $\sup_n \|\varphi_n(x)\| = \infty$ implies $\lim \|\varphi_n(x)\| = \infty$. According to Theorem 3 the system is regular.

Moreover

$$\begin{aligned} \sum_{i=1}^n |f_i(x)|^2 &= \|\varphi_n(x)\|^2 - 2\Re \left\{ \sum_{j < i \leq n} c_{ij} f_i(x) f_j(x) \right\} \\ &\leq \|x\|^2 - \alpha < \infty. \end{aligned}$$

An immediate consequence is the following. The system is regular if $\sum_{i \neq j} |c_{ij}| < \infty$ and the sequence $\{\|f_i\|\}$ is bounded.

Professor R. C. James called my attention to the fact that this may be proved directly and without the assumption of boundedness of the sequence $\{\|f_i\|\}$ as follows. We may assume without loss of generality that $\sum_{i \neq j} |c_{ij}| = r < 1$

$$\begin{aligned} \left| \sum_{i \neq j} a_i \bar{a}_j c_{ij} \right| &\leq \max |a_i \bar{a}_j| \cdot r \leq \sum_{i=1}^n |a_i|^2 r \\ \left\| \sum_{i=1}^n a_i x_i \right\|^2 &= \sum_{i=1}^n |a_i|^2 + \sum_{i \neq j} a_i \bar{a}_j c_{ij} \leq 2 \sum_{i=1}^n |a_i|^2 \end{aligned}$$

Hence

$$\begin{aligned} \left\| \sum_{i=1}^{n+p} a_i x_i \right\|^2 &= \sum_{i=1}^{n+p} |a_i|^2 + \sum_{i \neq j} a_i \bar{a}_j c_{ij} \\ &\geq \sum_{i=1}^{n+p} |a_i|^2 (1-r) \geq \sum_{i=1}^n |a_i|^2 (1-r) \\ &\geq \frac{1-r}{2} \left\| \sum_{i=1}^n a_i x_i \right\|^2 \end{aligned}$$

and by [4] the system is regular.

Using the same method as in Theorem 6 we arrive at the following.

THEOREM 7. *The system is regular if and only if for each x*

$$(2) \quad \inf_{n,p} \Re \left\{ \sum_{i=1}^n \sum_{j=n+1}^p f_i(x) f_j(x) c_{ij} \right\} > -\infty$$

Proof.

$$\begin{aligned} \left\| \sum_{i=1}^{n+p} f_i(x) x_i \right\|^2 &= \left\| \sum_{i=1}^n f_i(x) x_i \right\|^2 + \left\| \sum_{i=n+1}^{n+p} f_i(x) x_i \right\|^2 \\ &\quad + 2\Re \left\{ \left(\sum_{i=1}^n f_i(x) x_i, \sum_{j=n+1}^{n+p} f_j(x) x_j \right) \right\}. \end{aligned}$$

If condition (2) is satisfied then according to Theorem 3 the system is regular. If the system is regular then

$$\left| \left(\sum_{i=1}^n f_i(x)x_i, \sum_{j=n+1}^{n+p} f_j(x)x_j \right) \right| \leq 2 \|x\|^2.$$

As a simple application we note the following.

If $c_{ij}=0$ when $|i-j|>N$ then the system is regular if and only if the sequence $\{\|f_i\|\}$ is bounded.

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