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Introduction. One knows various operations on sets, e. g. join, intersection, limit, A-operation (Suslin), etc. In the present article we define, as an extension of operations we introduced in another paper (Kurepa [6], [7]) several operations of considerable generality and importance. It turns out that the well-known *distribution law* (cf. § 11) as well as the *De Morgan duality principle* (cf. § 5) are very special cases of our theorems. Moreover, a new *reciprocity phenomenon* occurs (cf. § 12). All depend on the interconnection between *maximal chains* and *maximal antichains* of ordered sets. By considering ordered sets one achieves considerable generality. By their use we get a *synthetic view* on (1) the *analytic operation*; (2) *c-analytic operation* (definition of complements of analytic sets); (3) the *distribution law*; (4) the *duality law*; and moreover, one arrives at (5) a *new reciprocity law*. In particular, in connection with the distributive law, the maximal chains and maximal antichains indicate respectively two distinct ways to reach the same result (cf. Theorems 4.2, 8.1). On the other hand, the parallel considerations of maximal chains and maximal antichains of S give rise to a new kind of interconnection of elements of $P^2 1$ (1 being any set; cf. the k-condition in § 8). This in turn opens a broad way to new investigations by consideration of the elements of $P^\alpha 1$ instead of those of $P^2 1$. Our results may be interpreted in mathematical logic too.

The results of this paper are connected to an idea we expressed in our Thesis [4], 135 n°40 (cf. A. Tarski [11]).

GLOSSARY AND NOTATIONS

Antichain; an ordered set having no couple of distinct comparable points.

Chain; an ordered set having no two distinct incomparable points.

1 or U means universal set.

γT (cf. 10.1)

Disjunctive family; a family composed of pairwise disjoint sets.

ε' denotes "not ε."

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j-connected (cf. 2.1)

k-condition (cf. 3.1., 8.,)

PS denotes the system of all subsets of S; in particular, the void set v is an element of PS; $P^2S=P(PS)$, $P^{\alpha+1}S=P(P^\alpha S)$, etc.

$\Omega \in \{O, O'\}$, $\overline{\Omega} \in \{\overline{O}, \overline{O'}\}$

$\bigcap'=\bigcup$, $\bigcup'=\bigcap$.

$\prod$ denotes the combinatorial multiplication.

Ramified set; an ordered set the predecessors of each of whose points form a chain.

Ramified table or tree; an ordered set S with the property that if $x \in S$ then the set $(.,x)_S$ is well-ordered.

ρ being a relation, ρ_1, ρ_2, ρ_3 designates its first part, second part, third part, e. g., in the equality (2) we use $(2)_1$ to designate the first (left) part of (2); $(2)_2$ designates the second part of (2). If (2) is a binary relation for sets, then $(2)_1$ is the set on the left side of (2).

$(x,.)_S$ denotes the set of all the points $y \in S$ such that $x < y$.

$(.,x)_S$ denotes the set of all the points $y \in S$ such that $y < x$.

$\perp$ denotes $\bigcap$ or $\bigcup$.

v=empty set.

1. The operator $(e, \perp, f)$. Let $e \in P^2 1$ and $\perp \in \{\bigcap, \bigcup\}$. Let f be any mapping of 1. This means that, for each $x \in 1$, $f(x)$ is a well-determined set; of course it may happen that $f(x)=v$ (void); by f' we denote the mapping $x \to f'(x)$ which to each $x \in 1$ associates the complement $f'(x)$ of the set $f(x)$; the complement is taken in respect to any set $\supseteq f(x)$ $(x \in 1)$. In the case that $f(x)$ consists of one point, say $f(x)=\{a\}$, we write $f(x)=a$ as well as $f(x)=\{a\}$. Let $\perp$ denote $\bigcup$ or $\bigcap$; let $\bigcup'=\bigcap$, $\bigcap'=\bigcup$.

We put

$$(1.1) \qquad (e, \perp, f)=\underset{e_1}{\perp'}\ \underset{e_0}{\perp}\ f(e_0) \qquad (e_0 \in e_1 \in e).$$

In particular, we put, by convention,

(1.2) $(v, \bigcap, f)=v$, $(v, \bigcup, f)=$universal set $\supseteq f(e)$ for each $e \in 1$.

More explicitly (1.1) reads

$$(1.3) \qquad (e, \bigcap, f)=\bigcup_{e_1} \bigcap_{e_0} f(e_0), \quad (e, \bigcup, f)=\bigcap_{e_1} \bigcup_{e_0} f(e_0)$$

where $e_0 \in e_1 \in e$. Thus, $e_0 \in 1$, $e_1 \in P1$.

The meaning of $(E, \perp, f')$, $(F, \perp, f')$ is obvious. Thus, $f'(x)$ denotes the complement of $f(x)$. In particular, one has the De Morgan Theorem.

THEOREM 1.1. $(e, \perp, f)' = (e, \perp', f')$.

In what follows, we shall denote by

(1.4) (e, e^*)

any ordered pair of elements of $P^2 1$. Given such a pair (e, e^*) we might consider various sets, as e. g.,

(1.5) $(e, \bigcap, f), (e, \bigcap, f'), (e, \bigcup, f), (e, \bigcup, f')$,

and similarly for e^*. In particular, we shall consider the sets

(1.6) $(e, \perp, f), (e^*, \perp, f')$.

Obviously, given e, $\perp$, f, the previous sets are well determined. The problem is to know their interconnections.

2. j-connection of (e, e^*).

THEOREM 2.1. *In order that for each* f

(2.1) $(e, \bigcap, f)' \supseteq (e^*, \bigcap, f')$ *or* $(e, \bigcap', f') \supseteq (e^*, \bigcap, f')$,

it is necessary and sufficient that

(2.2) $e_1 \bigcap e_1^* \neq v \qquad (e_1 \in e,\ e_1^* \in e^*)$.

Proof of necessity (2.1)$\Rightarrow$(2.2). Suppose, on the contrary, that (2.2) does not hold; i.e., that there exist

(2.3) $e_{1_0} \in e,\ e_{1_0}^* \in e^*$, so that $e_{1_0} \bigcap e_{1_0}^* = v$.

Let f be the characteristic function of e_{1_0} such that $f(e_0)=1 \Longleftrightarrow e_0 \in e_{1_0}$. Since $e_{1_0} \in e$ and since $1 \in f(e_0)$ $(e_0 \in e_{1_0})$ one has obviously $1 \in (2.1)_1$. On the other hand, since $e_{1_0}^* \bigcap e_{1_0} = v$, $f(e_0^*) = v$ $(e_0^* \in e_{1_0}^*)$, thus $f'(e^*) = 1$ $(e_0^* \in e_{1_0}^*)$; in other words, $1 \in (2.1)_2$. Thus (2.3) implies $1 \in (2.1)_2 \setminus (2.1)_1$ which contradicts the hypothesis (2.1).

Proof of sufficiency. (2.2)$\Rightarrow$(2.1), that is, (2.2)$\Rightarrow(\xi \in (2.1)_2) \Rightarrow \xi \in (2.1)_1)$. Now the relation $\xi \in (e^*, \bigcap, f')$ means that there is a e_1^* such that $\xi \in f'(e_0^*)$ $(e_0^* \in e_1^*)$.

Again, let $e_1 \in e$; since $e_1 \bigcap e_1^* \neq v$ by hypothesis (2.2), let $z \in e_1 \bigcap e_1^*$; thus, $\xi \in {}'f(z)$; consequently, for each $e_1 \in e$ there is an $e_0 \in e_1$ such that $\xi \in {}'f(e_0)$. That means $\xi \in {}'(e, \bigcap, f)$, that is, $\xi \in (e, \bigcap, f)'$.

Since the condition (2.2) is symmetrical with respect to e, e^*, we get the following.

THEOREM 2.2. *The f-identity* $(e, \bigcap, f)' \supseteqq (e^*, \bigcap, f')$ *is equivalent to the f-identity* $(e^*, \bigcap, f)' \supseteqq (e, \bigcap, f')$.

The last two theorems give rise to the following.

DEFINITION 2.1. An ordered pair (e, e^*) of elements of P^21 is said to be *j-connected*, symbolically $(e, e^*) \in (j)$ if

$$e_1 \bigcap e_1^* \neq v\,, \qquad (e_1 \in e,\ e_1^* \in e^*).$$

THEOREM 2.3. *In order that* (2.1) *holds for each* f, *it is necessary and sufficient that the ordered pair* (e, e^*) *be j-connected.*

3. The k-condition. We will prove the following.

THEOREM 3.1. *In order that for each* f *one has*

$$(e, \bigcap, f)' \subseteqq (e^*, \bigcap, f')\,, \tag{3.1}$$

it is necessary and sufficient that for each $X \subseteqq 1$ *satisfying*

$$X \bigcap e_1 \neq v \qquad (e_1 \in e) \tag{3.2}$$

one has

$$PX \bigcap e^* \neq v\,, \tag{3.3}$$

that is, that there is an $e_1^* \in e^*$ *such that* $e_1^* \subseteqq X$.

Proof of necessity. Let X satisfy (3.2). Let f be the characteristic function of X. Then (3.2) implies $v \in {}'\bigcap f(e_0)$, $(e_0 \in e_1)$, for each $e_1 \in e$. Thus $v \in (3.1)_1$. As (3.1) holds, one has $v \in (3.1)_2$. Therefore there exists a $e_1^* \in e^*$ satisfying $v \in \bigcap\limits_{e_0^*} f'(e_0^*)$ $(e^* \in e_1^*)$. Consequently, $f(e_0^*) = 1$ for each $e_0^* \in e_1^*$, and that means exactly that $e_1^* \subseteqq X$.

Proof of sufficiency. If (3.2)$\Rightarrow$(3.3), then $\xi \in (3.1)_1$ implies $\xi \in (3.1)_2$. Let

$$X = \underset{x \in 1}{E}(\xi \in f'(x))\,, \tag{3.4}$$

that is, X denotes the set of all the $x \in 1$ for which $\xi \in {}'f(x)$. We say that (3.2) holds. In the opposite case, there would be an $e_{1_0} \in e$ such that $e_{1_0} \bigcap X = v$, thus $\xi \in f(e_0)$ $(e_0 \in e_{1_0})$ and therefore $\xi \in {}'(3, 1)_1$, contrary to the hypothesis that $\xi \in (3.1)_1$. The set (3.4) satisfying (3.2), there exists by supposition an element $e_{1_0}^* \in e^*$ such that $e_{1_0}^* \subseteqq X$. That means that $\xi \in f'(e_0^*)$ $(e_0^* \in e_{1_0}^*)$, that is, $\xi \in (3.1)_2$.

DEFINITION 3.1. The ordered pair (e, e^*) of elements of P^21 is said to satisfy the *k-condition*, symbolically

$$(e, e^*) \in (k), \tag{3.5}$$

provided the system

$$X \subseteqq 1,\ X \cap e_1 \neq v \qquad (e_1 \in e) \tag{3.6}$$

implies

$$PX \cap e^* \neq v. \tag{3.7}$$

Thus Theorem 3.1 may be expressed in the following form.

THEOREM 3.2. *The relation* $(e, e^*) \in (k)$ *is equivalent to the f-identity*

$$(e, \cap, f)' \subseteqq (e^*, \cap, f').$$

4. First fundamental theorem. Theorems 2.1 and 3.1 enable us to characterize the equality

$$(e, \cap, f)' = (e^*, \cap, f'). \tag{4.1}$$

THEOREM 4.1. *The equality* (4.1) *is equivalent to the relation*

$$(e, e^*) \in (j) \wedge (k). \tag{4.2}$$

(The last relation means that (e, e^*) satisfies both (j) and (k)).

We transform the previous conditions using De Morgan's theorem (c.f. Theorem 1.1). We have $(e, \cap, f)' = (e, \cup, f')$ so that (4.1) reads

$$(e, \cup, f') = (e^*, \cap, f');$$

and considering f' instead of f we obtain

$$(e, \cup, f) = (e^*, \cap, f).$$

Consequently we have the following,

THEOREM 4.2. (FIRST FUNDAMENTAL THEOREM). *Let* (e, e^*) *be a given ordered pair of elements of* P^21; *then the following properties are pairwise equivalent*:

I. $(e, e^*) \in (j) \wedge (k)$

II. For each mapping f of the set 1 the following duality law holds:

$$\left(\bigcup_{e_1} \bigcap_{e_0} f(e_0)\right)' = \bigcup_{e_1^*} \bigcap_{e_0^*} f'(e_0^*), \text{ that is, } (e, \cap, f)' = (e^*, \cap, f').$$

III. *For each mapping f of the set* 1, *one has the distributive law*

$$\bigcup_{e_1 \in e} \bigcap_{e_0 \in e_1} f(e_0) = \bigcap_{e_1^* \in e^*} \bigcup_{e_0^* \in e^*} f(e_0^*), \text{ that is, } (e, \cap, f) = (e^*, \cup, f).$$

IV. $(e^*, e) \in (j) \wedge (k)$.

Proof. In fact, I $\Leftrightarrow$ II (Theorem 4.1) and II$\Leftrightarrow$III as was shown by the application of the De Morgan theorem to $(\cup, \cap, f)'$. It remains to prove that IV is equivalent to I, II and III. First, the implication I $\Rightarrow$ III yields IV $\Rightarrow (e^*, \cap, f) = (e, \cup, f)$; from here, passing to complement III′ of III: $(e^*, \cap, f)' = (e, \cup, f)'$, that is, $(e^*, \cup, f') = (e, \cap, f')$. Writting f' instead of f, one gets III. Thus IV $\Rightarrow$ III. Conversely, III $\Rightarrow$ III′ (by implication III $\Rightarrow$ I) $\Rightarrow$ IV.

The equivalence I $\Leftrightarrow$ IV gives the following.

THEOREM 4.3. *Symmetry character of* $(j) \wedge (k)$: *If* $(e, e^*) \in (j) \wedge (k)$, *then also* $(e^*, e) \in (j) \wedge (k)$. *In other words, if* $(e, e^*) \in (j)$, *then* $(e, e^*) \in (k) \Leftrightarrow (e^*, e) \in (k)$.

THEOREM 4.4. *Symmetry of the k-property*[1]. *If* $(e, e^*) \in (k)$, *then* $(e^*, e) \in (k)$.

Proof. To begin with, if e is the null set, then for every e^*, $(e, e^*) \in {}'(k)$ and $(e^*, e) \in {}'(k)$. And if the null set is a member of e, then for every e^*, $(e, e^*) \in (k)$ and $(e^*, e) \in (k)$. It remains to consider cases where no sets involved are null. Suppose that $(e, e^*) \in {}'(k)$. Then there exists an x such that for every $e_1 \in e$, $e_1 \cap x \neq v$, and for every $e_1^* \in e^*$, $e_1^* \setminus x \neq v$. Let $y = \bigcup_{e_1^*} (e_1^* \setminus x)$ $(e_1^* \in e^*)$. Then for every $e_1^* \in e^*$, $y \cap e_1^* \neq v$; and if it can be proved that, for every $e_1 \in e$, $e_1 \setminus y \neq v$, it will follow that $(e^*, e) \in {}'(k)$. But for every $e_1 \in e$, $x_1 = e_1 \cap x \neq v$ and $x_1 \cap y = v$. Since $x_1 \neq v$, it follows that $x_1 \setminus y \neq v$ and therefore that $e_1 \setminus y \neq v$.

In what follows, the generality of Theorem 4.2 will be revealed. We will restrict ourselves to ordered sets. There we are naturally led to consider various operators which were the origin of the present investigations (cf. Kurepa [4], [6].)

5. Ordered sets, operators O, $\overline{O}$, O', $\overline{O}'$. Let S be any set ordered by $\leqq$. The operators O, $\overline{O}$, O', $\overline{O}'$ are defined in the following manner:

DEFINITION 5.1. OS designates the system of all *maximal chains* $\subseteqq S$.

[1] Theorem 4.4 and its proof are due to the referee.

DEFINITION $\overline{5.1}$. $\overline{O}S$ designates the system of all *maximal antichains* $\subseteqq S$.

DEFINITION 5.2. $O'S$ designates the system of all $X \in \overline{O}S$ such that

$$X \cap M \neq v \qquad (M \in OS).$$

DEFINITION $\overline{5.2}$. $\overline{O'}S$ designates the system of all $X \in OS$ such that

$$X \cap A \neq v \qquad (A \in \overline{O}S).$$

We shall be aware of a certain reciprocity between the notions chain and antichain, and in particular by passing from the system O, O' to the system $\overline{O}$, $\overline{O'}$.

To each ordered set S is associated the set consisting of

$$OS,\ \overline{O}S,\ O'S,\ \overline{O'}S \tag{5.1}$$

which are at most four elements of P^2S. The set (5.1) is of a great importance. Its elements form in a certain sense the spatial forms along which certain operations are to be taken. Each element e of (5.1) is as it were a system of paths for operations $(e, \perp, f)$, $(e, \perp', f)$, etc.

CONVENTION 5.1. The reciprocal of a statement s will be denoted $\bar{s}$. So the reciprocal of the Lemma 5.1 is denoted by Lemma $\overline{5.1}$. If X is a chain, then $\overline{X}$ is an antichain, etc, Here is an example.

LEMMA 5.1. *In order that $X \in O'S$, it is sufficient that X be an antichain of S such that $X \cap M \neq v$ $(M \in OS)$. In other words, if an antichain intersects each maximal chain of S it is necessarily a maximal antichain.*

The reciprocal result is as follows.

LEMMA $\overline{5.1}$. *In order that $X \in \overline{O'}S$, it is sufficient that X be a chain of S such that $X \cap A \neq v$ $(A \in \overline{O}S)$. In other words, if a chain X of S intersects each antichain of S, then X is necessarily a maximal one.*

Proof. Let X be an antichain satisfying $X \cap M \neq v$ $(M \in OS)$. To prove that $X \in O'S$, it is sufficient to prove that X is a *maximal* antichain, i.e., that each $b \in S$ is comparable to some point of X. Now, let $b \in B \in OS$. Then the point $B \cap X$ exists and is the required point of X which is comparable to b.

Reciprocally, let X be a chain such that $X \cap A \neq v$ $(A \in \overline{O}S)$. To

prove that X is a maximal chain, suppose, on the contrary, that there is a chain $C \supset X$. Let $d \in C \setminus X$ and let $d \in D \in \overline{O}S$. Then necessarily $D \cap X = v$, because if $x \in D \cap X$, one would have two distinct comparable points d, x in the antichain D.

LEMMA 5.2. *$O'S \subseteqq \overline{O}S$, $\overline{O}'S \subseteqq OS$. (Each of the signs $\subseteqq$ here may be $=$ or $\subset$.) In particular there exists a non-void S such that*[2]

$$O'S = v, \quad \overline{O}'S = v \tag{5.2}$$

EXAMPLE 5.1. Let σ_0 denote the system of all non-void bounded well ordered sets of rational numbers ordered by means of the relation $\subseteqq$, where[3] (5.3) $x \subseteqq y$ or $y \supseteqq x$ means that x is an initial portion of y. In that case, $\overline{O}'\sigma_0 = v$, because, e,g., there is no chain in σ_0 intersecting each row of σ_0 (cf. [4, p. 95]). It is probable that $O'\sigma_0 = v$.

As an example of reciprocity considerations let us prove the following lemmas (5.3 and $\overline{5.3}$) which are mutually reciprocal and which will occur in distributive laws (cf. Theorem 9.1, Cases $2,\overline{2}$).

LEMMA 5.3. *If the maximal chains of S are pairwise disjoint, then the comparability relation in S is transitive, and conversely. Also*

$$O'S = \overline{O}S = \prod_{M} M, \tag{5.4}$$

where $\prod$ denotes the combinatorial product of sets M, M running over OS; and $OS = \overline{O}'S$.

Peciprocally we have the following.

LEMMA $\overline{5.3}$. *If the maximal antichains of S are pairwise disjoint, then the incomparability relation in S is transitive, and conversely. Also*

$$OS = \overline{O}'S = \prod_{A} A$$

where $\prod$ denotes the combinatorial product of all the sets A, A running over $\overline{O}S$; and $O'S = \overline{O}S$.

Proof of Lemma 5.3. If OS is disjoint, then as it is easy to show, the comparability relation in S is a congruence relation, and vice versa. Each $A \in \overline{O}S$ intersects each $M \in OS$ (thus $\overline{O}S = O'S$) in a single point,

[2] According to W. Gustin, there exists a denumerable ramified set S satisfying (5.2) [c.f. Gustin, *Math. Rev.* **14**, 255 (1953) in connection with the review of Kurepa [**8**]].

[3] The relation $\subseteqq$ is the very basis of the theory of ramified sets (cf. [**4**]).

since on the one hand OS is disjonint and on the other hand A is antichain; thus $A \in (5.4)_3 = \prod_M (M \in OS)$. Conversely, each $X \in (5.4)_3$ is an antichain because of the incomparability of each point of each $M \in OS$ to each point of each $M_0 \in OS$, $M_0 \neq M$. But X is also a maximal antichain. Analogously one proves the reciprocal of Lemma 5.3., that is Lemma $\overline{5.3}$.

REMARK 5.1. On Lemma 5.1 and Lemma $\overline{5.1}$ is based a very general distribution law (c.f. Theorem 9.1, Cases 2,$\overline{2}$).

6. Operations $(v, \bigcap, f)$, $(v, \bigcup, f)$ and $(\Omega, \perp, f)$ for each $\Omega \in (5.1)$ and each $\perp \in \{\bigcap, \bigcup\}$.

Let Ω be any element of the set

$$\{OS,\ O'S,\ \overline{O}S,\ \overline{O'}S\}; \tag{6.1}$$

then $\Omega \in P^2S$; so that for each Ω and each $\perp \in \{\bigcap, \bigcup\}$, the operator

$$(\Omega, \perp, f) \tag{6.2}$$

is well defined. In the particular case that $\Omega = v$, we put

$$(v, \bigcup, f) = \text{universal set},\ (v, \bigcap, f) = \text{void set}. \tag{6.3}$$

We shall consider ordered pairs (e, e^*) of elements of the set (6.1) and the corresponding sets (6.2) for $\Omega = e$ and $\Omega = e^*$, respectively.

EXAMPLE 6.1. Let

$$(T;\ \omega_0) \tag{6.4}$$

denote the system of all $< \omega_0$-complexes (finite complexes) of ordinals $< \omega_0$ ordered by means of the relation $\subseteq$ in (5.3). If f is a mapping of $(T;\ \omega_0)$ into the family of closed sets, then we can prove that $(O, \bigcap, f)$ and $(O', \bigcap, f')$, respectively, are the most general analytic set (A-set of Suslin) and the most general CA-set respectively (c.f. [10], [1], [2]; also [9]).[4]

Example 6.1 shows the importance of the operations (1.1) even in the particular cases (6.2) and $S = (T;\ \omega_0)$. (Cf. [6]).

7. Some simple lemmas.

LEMMA 7.1. *Either $O'S = v$ or each element of $O'S$ intersects each element of OS; and reciprocally, either $\overline{O}'S = v$ or $(\overline{O}S, \overline{O}'S)$ is a j-connected ordered*

[4] In our book [5] we defined A-sets just as sets $(OS, \bigcap, f)$ for the choice of S and f as in Example 6.1.

pair.

Lemma 7.1 and Theorem 2.1 yield the following.

THEOREM 7.1. $(OS, \bigcap, f) \subseteqq (O'S, \bigcap', f)$

and reciprocally,

$$(\overline{O}, \bigcap, f) \subseteqq (\overline{O}'S; \bigcap' f) .$$

In general, we have here the sign $\subset$ *instead of* $\subseteqq$. *The duals of that relation hold also.*

THEOREM 7.2. *The two sets,*

$$(OS, \bigcap, f'), (\overline{O}S, \bigcap', f)$$

may be non-comparable if S *is ramified.*

To see this, let D denote the set of all integers ordered as in this diagram:

$$\cdots \to -4 \to -2 \to 0 \to 2 \to 4 \to 6 \to \cdots$$

$$\cdots \searrow -3 \searrow -1 \searrow 1 \searrow 3 \searrow 5 \searrow 7 \searrow \cdots$$

Obviously, the set D is ramified; for the sets $2D-1$ and $2D$ of all odd and, respectively, even integers one has $2D-1 \in \overline{O}D$, $2D \in OD$, $2D \bigcap (2D-1)=v$.

Let f be the characteristic function of $2D-1$; one proves then easily that

$$(OD, \bigcap, f')=\{1\}, (\overline{O}D, \bigcap', f)=\{0\} ,$$

and that proves Theorem 7.2.

8. Ordered sets and k-condition. If we consider the pair $(OS, O'S)$ or its reciprocal $(\overline{O}S, \overline{O}'S)$, then the j-condition is satisfied; therefore one obtains Theorem 7.1. On the other hand, in general one has neither

$$(OS, O'S) \in (k) \text{ nor reciprocally } (\overline{O}S, \overline{O}'S) \in (\overline{k}) .$$

For the sake of simplicity, we present the following.

DEFINITION 8.1. The condition $(OS, O'S) \in (k)$ will be denoted $S \in (k)$ and reciprocally. Thus

(8.1) $$(OS,\ O'S) \in (k) \Leftrightarrow S \in (k)$$

($\overline{8.1}$) $$(\overline{O}S,\ \overline{O}'S) \in (\overline{k}) \Leftrightarrow S \in (\overline{k})$$

and we shall say that S *satisfies the* (k)*-condition and the* $(\overline{k})$*-condition* respectively.

In particular, $S \in (k)$ means the statement that each set $\leqq S$ which intersects each maximal chain of S contains a maximal antichain of S. Then Theorem 4.2. (implication I $\Rightarrow$ III) yields the following.

THEOREM 8.1. *For each ordered set* S *satisfying the* (k)*-condition, one has the following distribution law*[5]:

(8.2) $$\underset{e_1}{\perp'}\ \underset{e_0}{\perp} f(e_0) = \underset{A}{\perp}\ \underset{a}{\perp'} f(a), \quad (e_0 \in e_1 \in OS,\ a \in A \in O'S)$$

and reciprocally for ($\overline{8.2}$). ($\perp$ *designates* $\bigcap$ *or* $\bigcup$).

Usual distribution laws are special cases of (8.2). Thus if one takes the ordered set $S=\{1, 2, 3\}$ with diagram $\overset{2}{\underset{1}{\uparrow}}_3$ one has $OS=\{\{1, 2\}, \{3\}\}$, $O'S=\{\{1, 3\}, \{2, 3\}\}$ and the formula (8.2) yieds $f(3) \perp' (f(1) \perp f(2)) = (f(3) \perp' f(1)) \perp (f(3) \perp' f(2))$.

Analogously, considering the set $\overset{2}{\underset{1}{\uparrow}}\ \overset{4}{\underset{3}{\uparrow}}$ one has the binomial form of the distribution law. For other cases of distribution, cf. § 11.

9. Some classes of ordered sets satisfying (k) and $(\overline{k})$. We are going to prove that the conditions (k), $(\overline{k})$ are satisfied by ordered sets of some general classes—a fact which will give us a general distribution and duality law.

THEOREM 9.1. *The conditions* (k) *and* $(\overline{k})$ *are satisfied, provided* S *satisfies at least one of the following conditions*:

1) S *is a chain*;

$\overline{1}$) S *is an antichain*;

2) OS *is disjoint, i.e., the elements of* OS *are pairwise disjoint (this is equivalent to the statement that the comparability relation is transitive in* S);

$\overline{2}$) *The elements of* $\overline{O}S$ *are pairwise disjoint (this is equivalent to the transitivity of the incomparability relation in* S).

The cases 1), $\overline{1}$), 2), $\overline{2}$) are ranged according to relative importance. One sees that 1) and $\overline{1}$) as well as 2) and $\overline{2}$) are mutually reciprocal.

Let us prove, e.g., the case $\overline{2}$). At first, the elements of $\overline{O}S$ being

[5] The converse holds also.

pairwise disjoint, by Lemma 5.3, we have $OS=\prod_A A$ $(A\in\overline{O}S)$ and $\overline{O}S=O'S$. Now we prove $S\in(\bar{k})$. If no $A\in\overline{O}S$ were contained in an $X\subseteq S$, where X intersects each $M\in OS$, there would be a point $x(A)\in A\setminus X$ for each $A\in\overline{O}S$. The set $\bigcup_A x(A)$ $(A\in\overline{O}S)$ would be a maximal chain of S which does not intersect X, contrary to the hypothesis on X.

REMARK 9.1. Later we shall see that the fact that each chain (antichain) satisfies (k) and $(\bar{k})$ is reflected in the fact that our duality theorem has as a special case the De Morgan duality theorem (cf. Theorem 13.1).

10. The case of ramified tables. At many opportunities we considered ramified tables, i.e., ordered sets satisfying the condition that for each $x\in T$, the set $(.,x)_T$ of all its predecessors in T is well-ordered. Let us recall that for a table T,

(10.1) γT

denotes the first ordinal number α such that there is no point $x\in T$ such that the order type of $(.,x)_T$ is α; γT is called rank or degree (order) of T.

THEOREM 10.1. *Each ramified table T satisfies (k); or explicitly and more precisely, let T be a set such that for each $x\in T$, the set $(.,x)_T$ is well-ordered. Let $X\subseteq T$ and $M\cap X\neq v$ $(M\in OT)$. Then the set*

(10.2) R_0X

of all initial points of X is a maximal antichain of T; moreover, R_0X intersects each maximal chain of T. Thus, $R_0X\in O'T$.

THEOREM 10.2. *If $\gamma T<\omega_0$, then $T\in(\bar{k})$ and $OT=\overline{O}'T$, $\overline{O}T=O'T$. In particular, this holds for each finite table.*

Proof of Theorem 10.1. At first, $R_0X\in\overline{O}T$. As R_0X has no pair of distinct comparable points, it is sufficient to show that each $t\in T$ is comparable to a point $x_0(t)\in R_0X$. Now, by hypothesis, there exists at least one point $x(t)\in X$ comparable to t. Let $x_0(t)$ be the point in R_0X which is $\leq x(t)$. In fact, it $x_0(t)=x(t)$, or if $x(t)\leq t$, the comparability of t and $x_0(t)$ is obvious. On the other hand, if neither $x_0(t)=x(t)$ nor $x(t)\leq t$, then $x_0(t)<x(t)$, $t<x(t)$. Thus, $x_0(t)$, t belong to the set $(.,x(t))_T$ which by the supposition on T is a chain.

It remains to prove that R_0X intersects each $M\in OT$. Again, by hypothesis, there exists a point $m\in X\cap M$; then the point $m'\in R_0X$

such that $m' \leqq m$ is a point of M. The set $(.,m]_T \cup M$ is a chain. By virtue of presupposed maximality of M, one has $(.,m]_T \subseteqq M$, thus $m' \in M$.

Proof of Theorem 10.2. At first we have the following.

LEMMA 10.1. *If* $\gamma T < \omega_0$, *then* $A \cap M \neq v$ $(A \in \overline{O}T,\ M \in OT)$, *thus* $OT = \overline{O}'T$, $O'T = \overline{O}T$ (cf. [8]).

Proof. Suppose, on the contrary, that T contains a maximal chain M and a maximal antichain A so that

$$A \cap M = v\,. \tag{10.3}$$

A being a maximal antichain of T, there exists for each $t \in T$ a point $\alpha(t) \in A$ such that $\{t,\ \alpha(t)\}$ is a chain; in particular, for each $m \in M$ the points m, $\alpha(m)$ are comparable. Now

$$m < \alpha(m)\,, \tag{10.4}$$

which is proved as follows. Since $M \in OT$, M is an initial portion of T. Consequently, if (10.4) did not hold, M would then contain also the point $\alpha(m)$ for at least a point $m_0 \in M$. Thus, $\alpha(m_0) \in A \cap M$ contrary to (10.3). Therefore (10.3)$\Rightarrow$(10.4). Now, since $\gamma T < \omega_0$, the chain M is finite.

Let l be the last point of M; l would be a last point of T also, contrary to the relation (10.4) for $m = l$. Thus the relation (10.3) is not possible, and Lemma 10.1 is proved.

To complete the proof of Theorem 10.2, we need to see that each $X \subseteqq T$ satisfying

$$X \cap A \neq v \qquad (A \in \overline{O}T) \tag{10.5}$$

contains a maximal chain of T. This holds for every T and we have the following statement which is reciprocal to Theorem 10.1.

THEOREM $\overline{10.1}$[6]. *Every ramified table* T *satisfies the* $\bar{k}$*-condition*: $T \in (\bar{k})$.

Proof. Suppose X satisfies (10.5). Since $R_0T \in \overline{O}T$, we have

$$X_0 \equiv X \cap R_0T \neq v\,. \tag{10.6}$$

The set (10.6) is an initial portion of X, that is,

$$x \in (10.6) \Rightarrow (.,x]_T \subseteqq (10.6)\,.$$

[6] Theorem $\overline{10.1}$ for $\gamma T \geqq \omega$ is due to the referee.

If X_0 contains a maximal chain of T, then Theorem 10.1 is proved. If $OX_0 \cap OT = v$, then

$$R_0(T \setminus X_0) \tag{10.7}$$

is a maximal antichain of T. As a matter of fact we have the following.

LEMMA 10.2. *If I is an initial portion of a ramified table T such that*

$$OI \cap OT = v\,, \tag{10.8}$$

then

$$R_0(T \setminus I) \in \overline{OT}\,. \tag{10.9}$$

To prove $(10.9)_1 \in \overline{OT}$, it suffices to show that each $i \in T$ is comparable to some point $i' \in (10.9)_1$. Obviously this holds for $i \in T \setminus I$. Suppose $i \in I$. Consider an M such that $i \in M \in OT$. By (10.8), $M \setminus I \neq v$. Let $P \in M \setminus I$, and let i' be the point such that $i' \in (10.9)_1$ and $i' \leqq P$. Since T is ramified and $i < P$, it follows that $i < i'$.

To prove Theorem 10.1, let us consider the sets

$$X_0,\ X_1,\ \cdots,\ X_\alpha,\ \cdots \tag{10.10}$$

defined as follows

$$\begin{aligned} &X_0 = X \cap R_0T,\ X_1 = X_0 \cup (X \cap R_0(T \setminus X_0)) \\ &X_\alpha = X_{\alpha-1} \cup (X \cap R_0(T \setminus X_{\alpha-1})) \\ &X_\alpha = \bigcup_{\alpha_0} X_{\alpha_0} \qquad (\alpha_0 < \alpha) \end{aligned} \tag{10.11}$$

depending upon whether α is isolated or a limit ordinal number.

Obviously, the sequence (10.10) is increasing and its terms $\subseteqq X$. Let δ be the first ordinal such that

$$X_\delta = X_{\delta+1}\,. \tag{10.12}$$

Of course, $\delta \leqq \gamma T$.

We say that

$$OX_\delta \cap OT \neq v \tag{10.13}$$

and hence

$$OX \cap OT \neq v \tag{10.14}$$

because $X_\delta \subseteqq X$.

First, each term of (10.10) is an initial portion of T-provable by an induction argument. Secondly, if the relation (10.13) were false, the

set

$$R_0(T \setminus X) \tag{10.15}$$

by virtue of Lemma 10.2 would be a maximal antichain of T. By hypothesis on X (see (10.5)) there would be a point $z \in X \cap$ (10.15). Therefore

$$z \in X_{\delta+1},\ x \in {}'X_\delta .$$

Hence, $z \in X_{\delta+1} \setminus X_\delta$ and $X_\delta \subset X_{\delta+1}$, contrary to (10.12). Hence (10.14) holds and Theorem 10.1 is proved.

11. General distribution laws. To see how the previous investigations are linked with distribution questions, let us prove the following distribution theorem which is the most general distribution law expressible in usual terms.

THEOREM 11.1. *Let $\mathfrak{F}$ be any non-void family of non-void sets $\subseteqq 1$, 1 being a standard set. Then for each mapping f of the set 1 we have*

$$\bigcap_{X \in \mathfrak{F}} \bigcup_{x \in X} f(x) = \bigcup_{A} \bigcap_{a \in A} f(a) \qquad \left(A \in \prod_{X \in \mathfrak{F}} X\right); \tag{11.1}$$

and dually.

Theorem 11.1 is a corollary to Theorem 4.2 (implication I$\Rightarrow$III). As a matter of fact, first the pair $(\mathfrak{F}, \prod_{X \in \mathfrak{F}} X)$ is j-connected; second it satisfies the k-conditions, as is easily probable.

A direct proof of Theorem 11.1 is as follows.

First, $(11.1)_1 \subseteqq (11.2)_2$, that is, if $\xi \in (11.1)_1$ then $\xi \in (11.1)_2$. In fact $\xi \in (11.1)_1$ means $\xi \in \bigcup_{x \in X} f(x)$ $(X \in \mathfrak{F})$, that is, there exists an $X_e \in X$ such that $\xi \in f(X_e)$, $(X \in \mathfrak{F})$. Putting $A_e = \bigcup_X X_e (X \in \mathfrak{F})$, one has $\xi \in \bigcap_a f(a)$ $(a \in A_e)$ and $A_e \in \prod_X X$, thus $\xi \in (11.1)_2$.

Second, $(11.1)_2 \subseteqq (11.1)_1$: if $\xi \in (11.1)_2$, then $\xi \in (11.1)_1$. The relation $\xi \in (11.1)_2$ is equivalent to $\xi \in f(a)$ $(a \in A)$ for some $A \in \prod_x X$; since $A \cap X \neq v$, this implies $\xi \in \bigcup_{x \in X} f(x)$ for each $X \in \mathfrak{F}$, hence $\xi \in (11.1)_1$.

From the proof of Theorem 11.1 we obtain the following interesting result.

THEOREM 11.2. (Cf. Theorem 2.1) *Let $(\mathfrak{F}, \mathfrak{F}_0)$ be any ordered pair of systems of sets $\subseteqq 1$ such that*

(11.2) $$X \cap X_0 \neq v \qquad (X \in \mathscr{F},\ X_0 \in \mathscr{F}_0);$$

then for each mapping f:

(11.3) $$\bigcup_{X \in \mathscr{F}} \bigcap_{x \in X} f(x) \subseteqq \bigcap_{X_0 \in \mathscr{F}_0} \bigcup_{x_0 \in X_0} f(x_0);$$

and dually,

$$\bigcup' \bigcap' f'(x) \subseteqq' \bigcap'_{X^x \in \mathscr{F}^*} \bigcup'_{x^* \in X_0^*} f'(x_0)\,.$$

In general, one reads here $\subset$ instead of $\subseteqq$. The case $\mathscr{F}=\mathscr{F}_0$ is not excluded. Therefore the relation (11.3) holds even if one or both sets $\mathscr{F}$, $\mathscr{F}_0$ are vacuous. In particular, (11.3) holds if $\mathscr{F} \in \{OS,\ O'S,\ \overline{O}S,\ \overline{O}'S\}$ and $\mathscr{F}_0=\mathscr{F}'$ (obviously $(OS)'$ means $O'S$; $(O'S)'=OS$, $(\overline{O}'S)'=\overline{O}S$, even if $O'S=v=\overline{O}'S$. Consequently, we have the following.

THEOREM 11.3. *If* $\Omega \in \{O,\ \overline{O},\ O',\ \overline{O}'\}$, *then*

$$\bigcup_X \bigcap_x f(x) \subseteqq \bigcap_{X'} \bigcup_{x'} f(x') \qquad (x \in X \in \Omega S, x' \in X' \in \Omega' S)$$

and dually.

Passing to complements in the relation and using the De Morgan formula, we have the following.

THEOREM 11.4. *For any* $\Omega \in \{O,\ \overline{O},\ O',\ \overline{O}'\}$:

$$\bigcup_X \bigcap_x f(x)' \supseteqq \bigcup_{X_0} \bigcap_{x_0} f'(x_0) \quad (x \in X \in \Omega S,\ x_0 \in X_0 \in \Omega' S).$$

The question of whether sets forming $\mathscr{F}$ in Theorem 11.1 are pairwise disjoint or not disjoint is of no importance. However, without loss of generality, the system $\mathscr{F}$ may be supposed disjoint. In fact, let to each $X \in \mathscr{F}$ be associated the set X_a of all ordered pairs (X, x) $(x \in X)$; to each $x \in X$ we associate the pair (x, x). Instead of $\mathscr{F}$ we can consider the system $\mathscr{F}_a$ of all the X_a $(X \in \mathscr{F})$. Now, the family $\mathscr{F}_a$ is disjunctive and the system $\mathscr{F}_a$ can be interpreted either as OS or as $\overline{O}S$. If one orders totally each X_a and if one orders the set $S=\bigcup_X X_a$ $(X \in \mathscr{F})$ so that each element of X_a is incomparable to each element of each other element of $\mathscr{F}_a$ and if one leaves intact the ordering in each element of $\mathscr{F}_a$, then obviously

$$OS=\mathscr{F}_a,\ O'S=\prod_y y \qquad (y \in \mathscr{F}_a);$$

moreover

$$OS=\overline{O}'S,\ O'S=\overline{O}S,$$

the set satisfies the conditions (k) and $(\overline{k})$ and accordingly for the set S the distribution law (8.1) holds.

Combining Theorem 8.1 with Theorem 9.1, one has the following statements:

THEOREM 11.5. *If S is an ordered set of one of the cases* 1, $\overline{1}$, 2, $\overline{2}$, *in Theorem* 9.1, *then for each mapping f of S the following distribution law holds*:

$$\underset{M}{\perp}'\ \underset{m}{\perp}\ f(m)=\underset{A}{\perp}\ \underset{a}{\perp}'\ f(a) \qquad (m\in M\in OS,\ a\in A\in O'S); \tag{11.4}$$

and reciprocally. In (11.4) $\perp$ *denotes either* $\cap$ *or* $\cup$.

Theorems 9.1 and 10.1, 10.2 yield the following.

THEOREM 11.6. *For each ramified table T and each mapping f of T one has*

$$\underset{M}{\perp}'\ \underset{m}{\perp}(m)=\underset{A}{\perp}\ \underset{a}{\perp}'\ f(a) \qquad (m\in M\in OT,\ a\in A\in O'T); \tag{11.5}$$

and reciprocally.

12. A new duality law. We saw (Theorem 8.1) how the distribution law (11.4) is connected with the condition $(\overline{k})$. Now we will see the interconnection of the distribution law and of (k) or $(\overline{k})$ with some duality laws. Let us suppose that for each f one has

$$(OS,\ \perp,\ f)=(O'S,\ \perp'\ f)$$

(this happens if and only if $S\in(k)$ cf. Theorems 3.2, 4.1). In particular, since f is arbitrary, the same equality holds for the mapping f', f' being the complement of f; thus

$$(OS,\ \perp,\ f')=(O'S,\ \perp'\ f').$$

From here, passing to complements, one has

$$(OS,\ \perp,\ f')'=(O'S,\ \perp'\ f')'=(O'S,\ \perp'',\ f'')=(O'S,\ \perp,\ f),$$

(by De Morgan's formula). Thus we have the following.

THEOREM 12.1. GENERAL DUALITY LAW. *For each ordered set $S\in(k)$, one has*

$$(\Omega,\ \perp,\ f)'=(\Omega',\ \perp,\ f');\ \textit{where},\ \Omega=OS\ \textit{or}\ O'S, \tag{12.1}$$

$$\perp=\cap\ \textit{or}\ \cup.$$

Reciprocally, if $S \in (\overline{k})$, *then for each mapping* f *of* S:

$$(\Omega, \perp, f)' = (\Omega', \perp, f') . \tag{12.1}$$

Where, Ω *denotes* $\overline{O}S$ *or* $\overline{O}'S$, $\perp = \bigcap$ *or* $\bigcup$.

It is interesting to observe that the converse of Theorem 12.1 holds also.

THEOREM 12.2. $\quad \forall(f)\ (12.1) \Leftrightarrow S \in (k)$

$$\forall(f)\ (\overline{12.1}) \Leftrightarrow S \in (\overline{k}) ,$$

Let us express, e. g., the last equivalence directly.

THEOREM 12.3. *Given an ordered set* S: *in order that for each mapping* f *of* S, *one has*

$$(\bigcup_{A} \bigcap_{a \in A} f(a))' = \bigcup_{M} \bigcap_{m \in M} f'(m) \qquad (A \in \overline{O}S,\ M \in \overline{O}'S), \tag{12.2}$$

it is necessary and sufficient that S *satisfies the* $(\overline{k})$*-condition* (cf. Theorem 3.1).

13. Some special cases of the duality theorem.

THEOREM 13.1. *If* S *is a chain or an antichain, then the duality Theorem* 12.1 *yields the theorem of De Morgan.*

Let us consider an antichain S; thus $\overline{O}S = \{S\}$; $\overline{O}'S$ is the system of all one-point sets $x \in S$. Then for each $M \in O'S$, one has $M = \{x\}$ with $x \in S$; thus $\bigcap_{m \in M} f'(m) = f'(x)$ where $\{x\} = M$ and one has

$$\bigcup_{M} \bigcap_{m} f'(m) = \bigcup_{M} f'(x) = \bigcup_{x \in S} f'(x) \tag{13.1}$$

On the other hand, as $\overline{O}S = \{S\}$,

$$\bigcap_{a \in A} f(a) = \bigcap_{s \in S} f(a)$$

and

$$\bigcup_{A \in \overline{O}S} \bigcap_{a \in A} f(a) = \bigcup_{A \in \{s\}} \bigcap_{s \in S} f(s) = \bigcap_{s \in S} f(s) . \tag{13.2}$$

By virtue of (13.1) and (13.2) the equality (12.2) yields

$$(\bigcap_{s \in S} f(s))' = \bigcup_{s \in S} f'(s)$$

and this is just the equality of De Morgan. Since each family, or set, may be considered as an antichain, we see that Theorem 12.3 (its sufficient condition) for S an antichain gives the equality of De Morgan in its most general form.

THEOREM 13.2. *For each ramified table T and each mapping f of T one has*

$$(\underset{M}{\perp'}\ \underset{m}{\perp}\ f(m))' = \underset{A}{\perp'}\ \underset{a}{\perp}\ f'(a) \tag{13.3}$$

$$(m \in M \in OT,\ a \in A \in O'T,\ \perp \in \{\cap,\ \cup\})$$

In particular ($\perp = \cap$):

$$(\bigcup_{M}\ \bigcap_{m} f(m))' = \bigcup_{A}\ \bigcap_{a} f'(a) \quad (m \in M \in OT,\ a \in A \in O'T), \tag{13.4}$$

and reciprocally.

If one bears in mind the generality and importance of ramified tables (a tool for complete subdivisions or atomizations of sets), one is conscious of the importance and generality of Theorem 13.2.

REMARK 13.1. From a logical point of view it is very important that (13.4) as well as its reciprocal hold, especially for each table whose chains are finite.

Actually, we observe that such tables occur even in psychological processes, in subdivisions, evolution, etc. Thus is seems that the evolution processes follow a ramification scheme, as will be shown elsewhere.

REFERENCES

1. P. Alexandroff, *Sur l'invariances des ensembles complémentaires aux ensembles A*, Mat. Sb. **31** (1923), 310–318.
2. ———, *Sur les ensembles complémentaries aux ensembles A*, Fund. Math. **6** (1924), 160–165.
3. A. Fraenkel, *Abstract set theory*, Amsterdam, (1953).
4. G. Kurepa, *Ensembles ordonnés et ramifiés*, Thèse. Paris, 1935. Publ. Math., Belgrade, **4** (1935), 1–138.
5. ———, Teorija skupova, Zagreb (1951).
6. ———, *Ensembles partiellement ordonnés*, Proc. Int. Congr. Math. 1950 **1** (1952), 460–461.
7. ———, *Généralisation de l'opération de Suslin, de celle d'Alexandroff et de la formule de De Morgan*, Bull. Int. Acad. Sci., Zagreb, **5** (1955), 97–107.
8. ———, *On a characteristic property of finite sets*, Pacific J. Math. **2** (1952), 323–326.
9. W. Sierpinski, *Sur une généralisation des ensembles mesurables B*, Bull. Acad. Sci., Cracovie, (1918), 161–167.
10. M. Suslin, *Sur une définition des ensembles mesurables B sans nombres transfinis*, C.R. Acad. Sci., Paris **164** (1917), 88.
11. A. Tarski, *Zur Grundlegung der Boole'schen Algebra I*, Fund. Math. **24** (1935) 177–198.

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