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LIE ALGEBRAS OF LOCALLY COMPACT GROUPS

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1. Introduction. We call an LP-group, a group which is the projective limit of Lie groups. Yamabe [8] has proved that every connected locally compact group is an LP-group. This permits the extension to locally compact groups of the notion of a Lie algebra. In §§ 2 and 3 we prove the existence and uniqueness of the Lie algebra of an LP-group and show the connection of the Lie algebra with the group by means of the exponential mapping.

In § 4, we extend the notion of a universal covering group for connected groups with the same Lie algebra. A covering group of a connected group g , in the extended sense used here, means a pair (\bar{g}, w) , where \bar{g} is a connected LP-group and w is a continuous representation of \bar{g} into g which induces an isomorphism of the Lie algebra of \bar{g} onto the Lie algebra of g (see Definition 4.5). The universal covering group of a connected locally compact group is not necessarily locally compact and may not map onto the group. It turns out that the arc component of the identity in \bar{g} is a covering space in the sense of Novosad [5] of the arc component of the identity of g (these components are dense subgroups, Lemma 3.7).

Finally, in § 5, we establish a one-to-one correspondence between "canonical LP-subgroups" of a group and subalgebras of its Lie algebra.

2. Projective limit of Lie algebras.

DEFINITION 2.1. By a *topological Lie algebra* (over the real numbers) we shall mean a (not necessarily finite dimensional) Lie algebra with an underlying topology such that the operations of addition, multiplication and scalar multiplication are continuous.

DEFINITION 2.2. Let J be an inductive set. Suppose given for each $a \in J$, a topological Lie algebra G_a such that if $a < b$ there exists a continuous representation $f_{ab}: G_b \rightarrow G_a$. Let $G = [\{X_a\} \in \prod_{a \in J} G_a$ such that $f_{ab}X_b = X_a$, all $a, b \in J$ with $a < b$]. Then G is a closed topological subalgebra of the direct product.

In analogy to A. Weil [7, p. 23], G will be called the *projective limit* of the G_a ($G = \lim G_a$) if the following hold:

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LP I: If $a < b < c$, $f_{ac}X_c = f_{ab}(f_{bc}X_c)$;

LP II: f_{ab} is a continuous open homomorphism of G_b onto G_a ;

LP III: f_a , the natural projection of G into G_a , is continuous and onto.

(Remark: f_a open is implied by f_{ab} open, (see [7, p. 24])

In particular, if the G_a are finite dimensional Lie algebras with the usual topology as a Euclidean space, we get the following.

THEOREM 2.3. *Let G_a , $a \in J$, be a system of finite dimensional Lie algebras satisfying LP I and*

LP II': f_{ab} is a representation onto.

Then G as defined in Definition 2.2 will necessarily satisfy LP II and LP III and hence $G = \lim G_a$.

Proof. LP II' implies LP II, since for finite dimensional vector spaces a representation onto is both continuous and open in the usual topology. That LP III is satisfied follows directly from the theory of linearly compact vector spaces [3 Ch. III, § 27]. In fact, this result holds for an inverse system of finite dimensional vector spaces.

DEFINITION 2.4. If $G = \lim G_a$, G_a finite dimensional Lie algebras, then G will be called an *LP algebra*.

LEMMA 2.5. *Let $G = \lim G_a$, where the G_a 's are complete topological Lie algebras, with homomorphisms f_a and f_{ab} satisfying LP I, II, and III. Let N_a be the kernel of f_a , then*

- A. *Every neighborhood of zero in G contains an N_a*
- B. *For each N_a, N_b ; there exists an $N_c \subset N_a \cap N_b$.*
- C. *G is complete.*

Proof. It is easy to show (see [7]) that a fundamental system of neighborhoods of zero in G is given by $f_a^{-1}(V_a)$, $a \in J$, and V_a running through a fundamental system of neighborhoods of zero in G_a . Condition A then follows directly.

If $c > a, b$ then N_c obviously satisfies B. Condition C is immediate from the definition of G .

LEMMA 2.6. *Given a topological Lie algebra G containing closed ideals N_a satisfying A, B, C; then $\lim G/N_a$ exists and is isomorphic to G (where we define $a < b$ if $N_a \supset N_b$ and let $f_{ab}: G/N_b \rightarrow G/N_a$ be the natural homomorphism).*

Proof. Since the Conditions A, B, C are identical to those for topological groups [7, p. 25] it follows that G is isomorphic to $\lim G/N_a$ as an additive topological group. The lemma now follows since the N_a are ideals.

THEOREM 2.7. *Suppose $G = \lim G_a$, G_a finite dimensional. If K is a closed subalgebra of G , then $K = \lim K_a$, where K_a is the image of K in G_a . If K is a closed ideal in G , then $G/K = \lim G_a/K_a$. In particular, G/K is complete.*

Proof. If $f_{ab}: G_b \rightarrow G_a$ ($a < b$), then $f_{ab}: K_b \rightarrow K_a$ satisfies LP I, II'. Hence $\lim K_a$ exists. Since K maps onto K_a , K is dense in $\lim K_a$, [7]. Since K is closed, $K = \lim K_a$.

Likewise G_a/K_a , $a \in J$, satisfy LP I, II where $\bar{f}_{ab}: G_b/K_b \rightarrow G_a/K_a$ is induced by f_{ab} . Hence $\lim G_a/K_a$ exists. The natural maps $p_a: G_a \rightarrow G_a/K_a$ evidently induce a map $p: G \rightarrow \lim G_a/K_a$ defined by: $p\{X_a\} = \{p_a X_a\}$, $\{X_a\} \in G$. This definition is legitimate since:

$$\bar{f}_{ab}(p_b X_b) = p_a(f_{ab} X_b) = p_a(X_a), \quad a < b.$$

This in turn induces a map $i: G/K \rightarrow \lim G_a/K_a$. We have to show that i is an isomorphism.

By its definition i is evidently continuous and one-to-one into. We show that it is an isomorphism into. Since the natural map of G onto G/K is open, it is sufficient to show that if W is a neighborhood in G then $p(W) = p(W+K)$ is open. Now if W is a neighborhood in G , take $V+V \subset W$. Then V contains an N_a , kernel of $f_a: G \rightarrow G_a$ (Lemma 2.5). Then $p(W)$ contains $p(V) + p(N_a)$. Now $(p(V))_a = p_a f_a(V)$, is an open neighborhood in G_a/K_a ; and the preimage in $\lim G_a/K_a$ of a neighborhood in G_a/K_a is a neighborhood in $\lim G_a/K_a$. But if $(p(X))_a \in p_a f_a(V)$, $X \in G$ then $X_a \in f_a(V) + K_a$ and hence $X \in V + N_a + K \subset W + K$. But this implies that i is open in $i(G/K)$ and hence is an isomorphism into.

It remains to show that i is onto. But this follows since as an abstract vector space G is linearly compact [3]. Hence

$$\bigcap_{a \in J} f_a^{-1} p_a^{-1}(Y_a) \neq \phi, \quad \{Y_a\} \in \lim G_a/K_a$$

since the intersection is nonempty for any finite subset of J .

LEMMA 2.8. *Let θ be a continuous representation of an LP-algebra G into an LP-algebra H . Then the image of G in H is a closed subalgebra of H .*

Proof. Suppose $G = \lim G_a$, $a \in J$; $H = \lim H_b$, $b \in K$; G_a and H_b finite

dimensional. Consider the map $p_b: G \rightarrow H_b$, composed of θ and the projection of H onto H_b . This takes G onto a subalgebra H'_b of H_b . Obviously, the H'_b , $b \in K$, define a closed subalgebra $H' = \lim H'_b$ of H under the induced system of representations. Also $\theta(G) \subset H'$.

Let $Y = \{Y_b\}$ be in H' . The preimage $p_b^{-1}(Y_b)$ is a closed linear variety in G . Since G is linearly compact, $\bigcap p_b^{-1}(Y_b)$ is nonempty, as the intersection of a finite number is clearly nonempty. Hence there exists $X \in G$ such that $p_b(X) = Y_b$, all $b \in K$. Hence G maps onto H' . As H' is closed, this proves the Lemma.

LEMMA 2.9. *Let θ be a continuous representation of an LP-algebra G onto an LP-algebra H . Then θ is open.*

Proof. Let K be the kernel of θ . By Theorem 2.7, G/K is an LP-algebra. Hence it is clearly sufficient to prove the Lemma in the case that the map is also one-to-one.

Hence let θ be a continuous one-to-one representation of G onto H . Suppose $G = \lim G/K_a$; K_a , $a \in J$, closed ideals in G and G/K_a finite dimensional. By Lemma 2.8, $\theta(K_a)$ is a closed ideal in H . Further $H/\theta(K_a)$ is finite dimensional and hence is (topologically) isomorphic to G/K_a . Since θ is continuous every neighborhood of H contains some $\theta(K_a)$. It follows from Lemma 2.6 that

$$H = \lim H/\theta(K_a) = \lim G/K_a = G.$$

Clearly, the isomorphism so induced is the same map as θ .

THEOREM 2.10. *If G_1 and G_2 are two LP-algebras with the same underlying abstract algebra G , then G_1 and G_2 have the same topology.*

Proof. By Lemma 2.9 it is sufficient to construct an LP-algebra G_0 whose underlying abstract algebra is G and such that the identity maps of G_0 into G_1 and G_2 are both continuous. Let K_a , $a \in J$, be the set of all abstract ideals in G which have finite codimension. As is well known, if K_a and K_b have finite codimension, so does $K_a \cap K_b$. It follows that G/K_a , $a \in J$, satisfy the conditions of Theorem 2.3, where we define $b > a$ if $K_b \subset K_a$ and $f_{ab}: G/K_b \rightarrow G/K_a$ is the natural projection. Let $G_0 = \lim G/K_a$. We claim the underlying abstract algebra of G_0 is G .

Now G is linearly compact since it is the underlying algebra of G_1 and G_2 . Let $p_a: G \rightarrow G/K_a$ be the natural map. If $Y = \{Y_a\} \in G_0$, then $\bigcap p_a^{-1}(Y_a)$ is nonempty, since the intersection of a finite number is nonempty. Hence there exists $X \in G$ such that $p_a(X) = Y_a$, all $a \in J$. Hence the map $p: X \rightarrow p_a(X)$ take G onto G_0 . p is one-to-one, since for every $X \in G$ there is a K_a such that $p_a(X)$ is not zero, because the identity

map of say G onto G_1 is one-to-one.

Now $G_1 = \lim G_1/K_b$, $b \in J'$, J' a subset of J . $p^{-1}: G_0 \rightarrow G_1$ is an isomorphism of this underlying abstract algebras and is continuous since J' is a subset of J . Hence the theorem follows by Lemma 2.9.

COROLLARY 2.11. *Let G be an abelian LP-algebra. Then G is the direct product of 1-dimensional algebras. In particular, the underlying topological vector space of an LP-algebra is algebraically and topologically the direct product of 1-dimensional vector spaces.*

3. Lie algebra of an LP-group.

DEFINITION 3.1. Let g_a , $a \in J$ be Lie groups. Suppose $g = \lim g_a$, the limit satisfying LP I, II, III of A. Weil [7]. Then we call g an LP-group. (Note that g is complete since the g_a are complete.)

DEFINITION 3.2. Suppose $g = \lim g_a$, g_a connected Lie groups. Let G_a be the Lie algebra of g_a . Then the homomorphisms $f_{ab}: g_b \rightarrow g_a$ ($a < b$) induce homomorphisms $df_{ab}: G_b \rightarrow G_a$ satisfying LP I, II of Definition 2.1. Hence the G_a , $a \in J$, have a limit G . G is called the Lie algebra of g .

We show in Lemma 3.4 below that G is independent (in a natural sense) of the representation of g as a limit of Lie groups.

DEFINITION 3.3. Suppose $g = \lim g_a$, g_a connected Lie groups. Let G , $G = \lim G_a$, be the Lie algebra of g . Then we define a continuous map

$$\exp: G \rightarrow g, \quad \exp \{X_a\} = \{\exp X_a\}, \quad \{X_a\} \in G.$$

This mapping is legitimate, since if $f_{ab}: g_b \rightarrow g_a$ then

$$f_{ab}(\exp X_b) = \exp df_{ab} X_b = \exp X_a$$

and hence $\{\exp X_a\} \in g$.

LEMMA 3.4. *Suppose $g = \lim g_a$, g_a connected Lie groups, $a \in J$; $h = \lim h_b$, h_b connected Lie groups, $b \in K$. Let $G = \lim G_a$, $H = \lim H_b$ be the corresponding Lie algebras. If $\theta: g \rightarrow h$ is an isomorphism then we can define an isomorphism $d\theta: G \rightarrow H$ such that*

$$\theta(\exp X) = \exp d\theta(X), \quad X \in G$$

Proof. Let $f_a: g \rightarrow g_a$ and $\bar{f}_b: h \rightarrow h_b$ be the natural maps. Let n_a and \bar{n}_b be the kernels of f_a and \bar{f}_b respectively. Let $b \in K$. Since h_b is a Lie group there is a neighborhood V_b of h_b which contains no non-

trivial subgroups (that is, h_b doesn't have arbitrarily small subgroups). Since $\bar{f}_b\theta: g \rightarrow h_b$ is continuous, there is a neighborhood W in g which maps into V_b . But W contains some n_a and this n_a must go into the unit element of h_b . This defines a homomorphism $\theta_{ba}: g_a \rightarrow h_b$ such that

$$\theta_{ba}f_a = \bar{f}_b\theta,$$

this condition characterizing θ_{ba} .

If $a > a'$, a and a' in J , then $f_a = f_{aa'}f_{a'}$ and $\theta_{ba}f_{aa'}f_{a'} = \bar{f}_b\theta$. Hence $\theta_{ba}f_{aa'} = \theta_{ba'}$. Similarly, if $b' < b$, $\bar{f}_{b'b}\theta_{ba} = \theta_{b'a}$. The induced homomorphisms of the corresponding Lie algebras therefore satisfy

$$d\theta_{ba}df_{aa'} = d\theta_{ba'}, \quad d\bar{f}_{b'b}d\theta_{ba} = d\theta_{b'a}.$$

It follows that the maps $d\theta_{ba}$ define a continuous representation $d\theta$ of G into H , where $d\theta(X)$, $X = \{X_a\} \in G$, is defined by

$$(d\theta(X))_b = d\theta_{ba}(X_a).$$

This map is well defined because of the conditions satisfied above, and is continuous because $d\theta_{ba}$ is continuous.

Similarly for each $a \in J$, we can find a $b \in K$ and a homomorphism $\phi_{ab}: h_b \rightarrow g_a$ such that $f_a\theta^{-1} = \phi_{ab}\bar{f}_b$. This defines a continuous representation $d\phi$ of H into G . Because of the conditions satisfied by the maps one sees easily that: $d\theta d\phi: H \rightarrow H$ and $d\phi d\theta: G \rightarrow G$ are the identities, and hence that $d\theta$ is an isomorphism.

Since $d\theta_{ba}df_a = d\bar{f}_b d\theta$ by definition, we have

$$\bar{f}_b\theta(\exp X) = \theta_{ba}f_a(\exp X) = \exp d\theta_{ba}df_a(X) = \exp d\bar{f}_b d\theta(X).$$

By Definition 3.3, this implies that $\theta(\exp X) = \exp d\theta(X)$.

THEOREM 3.5. *Suppose $g = \lim g_a$, g_a connected Lie groups. Let $G = \lim G_a$ be the corresponding Lie algebra. Then g is the closed subgroup generated by the elements of the form $\exp X$, $X \in G$.*

Proof. Since G maps onto G_a , $\exp X_a$ for $X \in G$ generates g_a , and $\exp G$ generates a dense subgroup of g , proving the theorem.

LEMMA 3.6. *Suppose $G = \lim G_a$, G_a finite dimensional Lie algebras. Then the underlying space of G is arcwise connected.*

Proof. Since G is a topological vector space it is arcwise connected by straight lines.

LEMMA 3.7. *Suppose $g = \lim g_a$, g_a connected Lie groups. Then g is*

connected and the arcwise connected component of g is dense in g .

Proof. The map $\exp:G \rightarrow g$ is continuous. Hence if A is the image of G , A is arcwise connected. Hence A^n is arcwise connected. Hence $\bigcup_1^\infty A^n$ is arcwise connected. By Theorem 3.5 this is dense in g . Hence g is connected.

THEOREM 3.8. *Let (g_a) , $a \in J$ be a system of connected Lie groups satisfying LP I, II of A. Weil [1]; then $g = (\{x_a\} \in \prod_{a \in J} G_a; f_{ab}x_b = x_a$, all $a, b \in J$ with $a < b$) satisfies LP III, and hence:*

$$g = \lim g_a$$

Proof. Let G_a be the Lie algebra of g_a . Then the (G_a) , $a \in J$ satisfy LP I, II of Definition 1.1. Hence they have a limit G . Let $X \in G$, then if $X = \{X_a\}$ we have $\exp X_a \in g_a$ and $\{\exp X_a\} \in g$, since:

$$f_{ab} \exp X_b = \exp df_{ab} X_b = \exp X_a, \quad a < b$$

But elements of the form $\exp X_a$ generate g_a since G maps onto G_a . Hence g maps onto g_a . Hence $g = \lim g_a$.

LEMMA 3.9. *Let $g = \lim g_a$, g_a arbitrary Lie groups. Let g_a^0 be the connected component of the identity of g_a . Then the (g_a^0) , $a \in J$ form a system of groups satisfying LP I, II of A. Weil. Let $g^0 = \lim g_a^0$, then g^0 is the connected component of g .*

Proof. Since $f_{ab}:g_b \rightarrow g_a$ is continuous, open and g_b^0 is open in g_b , it takes g_b^0 onto g_a^0 . Hence the (g_a^0) , $a \in J$ satisfy LP I, II. By Theorem 3.8 they have a limit g^0 . g^0 may obviously be considered as a subgroup of g , closed since complete.

By Lemma 3.7, g^0 is connected and hence contained in the connected component of g . On the other hand, if g_1 is the connected component of g , $f_a(g_1)$ is connected and hence contained in g_a^0 . Hence g_1 is contained in the limit of the g_a^0 . Hence $g_1 = g^0$.

DEFINITION 3.10. Let g be a topological group. If the connected component of the identity of g is an LP-group, we define the Lie algebra of g as the Lie algebra of its connected component.

REMARK. According to the result of Yamabe [8], every locally compact group is a generalized Lie group. This implies in particular that its connected component is an LP-group. Hence every locally com-

compact group has a Lie algebra.

THEOREM 3.11. *Let g and h be topological groups for which Lie Algebras G and H are defined (Definition 3.10). Let f be a continuous representation of g into h , then f induces a unique continuous representation of G into H such that $f(\exp X) = \exp df(X)$, $X \in G$.*

Proof. Obviously f defines a continuous representation of the connected component of g into the connected component of h . Assume therefore that g and h are connected.

Suppose $g = \lim g_a$, $h = \lim h_b$ (g_a , h_b connected Lie groups). The map $g \rightarrow h \rightarrow h_b$ induces a map of $g_a \rightarrow h_b$ for some a , since h_b doesn't have arbitrarily small subgroups. This in turn induces a map of $G_a \rightarrow H_b$ and hence of G into H_b . As in the proof of Lemma 3.4, it is easy to see that the maps $G \rightarrow H_b$ induce a continuous representation of G into H , $df: G \rightarrow H$; such that $f(\exp X) = \exp df(X)$.

Suppose there are two such representations, say df and $\bar{d}f$. Then $(dfX)_b \neq (\bar{d}fX)_b$ some b and X . Since any neighborhood of zero generates G , and since df , $\bar{d}f$ are linear; we can choose X such that dfX and $\bar{d}fX$ are in any desired neighborhood of 0 in H_b . For a sufficiently small neighborhood \exp is one-to-one on H_b . Hence $(\exp dfX)_b \neq (\exp \bar{d}fX)_b$, a contradiction.

COROLLARY. *If g is connected and f, f_1 are two representations of g into h such that $df = df_1$, then $f = f_1$.*

Proof. Since $f(\exp X) = \exp dfX = \exp df_1X = f_1(\exp X)$ and since $\exp G$ generates a dense subgroup of g , we have $f = f_1$.

LEMMA 3.12. *Let g and h be locally compact topological group and g connected, G and H their Lie algebras. Let f be a continuous open homomorphism of g onto h , then df is a continuous open homomorphism of G onto H .*

Proof. According to A. Weil [7], if k is the kernel of f , we may take $g = \lim g_a$, $k = \lim k_a$, k_a the image of k in g_a , and $h = \lim g_a/k_a$, g_a and k_a Lie groups.

Then $G = \lim G_a$, G_a the Lie algebra of g_a . Let K be the Lie algebra of k ; then $K = \lim K_a$, K_a the Lie algebra of k_a . Then the Lie algebra of g_a/k_a is G_a/K_a [1]. Hence $H = \lim G_a/K_a$. By Theorem 2.5, $H = G/K$. It is easy to check that for $df: G \rightarrow G/K$ that $f(\exp X) = \exp dfX$. So that the natural map of G onto G/K is df . (Note: By a generaliza-

tion of Pontrjagin's theorem on groups satisfying the second axiom of countability; if f is continuous and onto it is automatically open.)

4. Universal covering group.

DEFINITION 4.1. Suppose $g = \lim g_a$, (g_a), $a \in J$ connected Lie groups. Let \bar{g}_a be the simply connected covering group of g_a . The map f_{ab} taking g_b onto g_a ($a < b$) induces an open homomorphism \bar{f}_{ab} of \bar{g}_b onto \bar{g}_a . Hence the (\bar{g}_a) , $a \in J$ satisfy LP I, II and therefore have a limit \bar{g} (Theorem 3.8). \bar{g} is a complete, connected group. \bar{g} is called the *universal covering group* of g .

PROPOSITION 4.2. Let G be the Lie algebra of g . Then \bar{g} has the Lie algebra G , and there exists a continuous representation w taking \bar{g} onto a dense subgroup of g such that $dw:G \rightarrow G$ is the identity.

Proof. The covering homomorphisms $w_a: \bar{g} \rightarrow g_a$ induce a continuous representation of \bar{g} into g . Since $dw_a: G_a \rightarrow G_a$ is the identity, it follows that dw is the identity. Since w_a maps \bar{g}_a onto g_a it follows that $w(\bar{g})$ is dense in g .

PROPOSITION 4.3. The kernel of w is totally disconnected and is in the center.

Proof. If k is the kernel of w , it follows from the definition of w that the image $f_a(k)$ of k in g_a belongs to the kernel of w_a . But this kernel is discrete and hence $f_a(k)$ is discrete, and therefore closed in g_a . It follows that the $f_a(k)$ satisfy LP I, II and since k is closed, $k = \lim f_a(k)$. Hence k is the projective limit of discrete groups. It follows from Lemma 3.9 that k is totally disconnected. Further, a totally disconnected normal subgroup of a connected group belongs to the center.

LEMMA 4.4. Let g be a connected LP-group with Lie algebra G . Let h be any other connected LP-group with Lie algebra H isomorphic to G . Then there exists an isomorphism of their universal covering groups $f: \bar{g} \rightarrow \bar{h}$ such that df is the given isomorphism of G onto H .

Proof. Suppose $g = \lim g_a$, $h = \lim h_b$; then $\bar{h} = \lim \bar{h}_b$, $\bar{g} = \lim \bar{g}_a$ and $G = \lim G_a$, $H = \lim H_b$. The homomorphism $G \rightarrow H \rightarrow H_b$ induces a homomorphism $G_a \rightarrow H_b$ for some a , since H_b has no small subgroups when considered as an additive group. This in turn induces a homomorphism of \bar{g}_a onto \bar{h}_b . Similarly there exist homomorphisms of \bar{h}_b onto \bar{g}_a , $a \in J$,

some b . As in the proof of Lemma 3.4, it is easy to see that this implies that there exists an isomorphism $f:\bar{g} \rightarrow \bar{h}$. The induced homomorphism $df:G \rightarrow H$ such that $f(\exp X)=\exp df(X)$, is obviously the original isomorphism of G onto H .

DEFINITION 4.5. Let g and h be connected LP-groups, G and H their Lie algebras. A continuous representation w of g into h such that dw is an isomorphism of G onto H is called a *covering map* and g is called a *covering group* of h , (g, w) is called the *covering*.

PROPOSITION 3.6. $w(g)$ is dense in h .

Proof. In fact $w(\exp X)=\exp dw(X)$, $X \in G$. But $\exp dw(X)$, $X \in G$ generates a dense subgroup of h since dw is onto.

We now give a purely topological definition of covering space for arcwise connected spaces due to Novosad [5] and show that the arc component of the identity g^c of g in Definition 4.5 is actually a covering space in this sense of the arc component of the identity h^c of h . (Note that g^c is dense in g , Lemma 3.7) Similarly, we show the arc component of the identity of the universal covering group is a universal covering space.

DEFINITION 4.7. (Novosad) Let A be an arcwise connected space, $a \in A$. Let $f:(B, b) \rightarrow (A, a)$ be a continuous map of an arcwise connected space B into A taking b into a . Then (f, B, b) is called a *covering space* of (A, a) , if given any contractible space C , and point $c \in C$ which is a deformation retract of C , and a map $\alpha:(C, c) \rightarrow (A, a)$, then there exists a map $\bar{\alpha}:(C, c) \rightarrow (B, b)$ which is unique with respect to the property $f\bar{\alpha} = \alpha$.

Let $(P_A, a\#)$ be the pair consisting of the space of paths starting from $a \in A$, with the compact open topology, of an arbitrary topological space A , and the constant path $a\#$ at $a \in A$. A continuous map $f:(B, b) \rightarrow (A, a)$ induces a continuous map $f\#:(P_B, b\#) \rightarrow (P_A, a\#)$ defined by:

$$f\#(p)(t)=f(p(t)), p \in P_B, t \in I \text{ (the unit interval)}$$

It is then easy to see that Definition 4.7 is equivalent to the following.

DEFINITION 4.7'. Let $f:(B, b) \rightarrow (A, a)$ be a continuous map of an arcwise connected space B into an arcwise connected space A taking $b \in B$ into $a \in A$. Then (f, B, b) is called a *covering space* of (A, a) if $f\#:(P_B, b\#) \rightarrow (P_A, a\#)$ is a homeomorphism onto.

PROPOSITION 4.8. *Definitions 4.7 and 4.7' are equivalent.*

Proof.

(4.7) implies (4.7') by Lemma 2.3 of [5].

(4.7') implies (4.7); since, let $\lambda_A: P_A \rightarrow A$, $\lambda_A(p)=p(1)$, $p \in P_A$. Then λ_A defines a fiber space (in the sense of Serre) which obviously satisfies the covering homotopy theorem for arbitrary spaces. Hence if $\alpha: (C, c) \rightarrow (A, a)$ is homotopic to the constant map the homotopy may be lifted to $(P_A, a\#)$ and hence to $(P_B, b\#)$ by the homeomorphism of (4.7'). λ_B maps the image into (B, b) . The endpoint of the homotopy gives the desired covering of (4.7). The uniqueness follows since any point of C describes a path under the retraction and the image of this path in B is unique since covering paths are unique by (4.7').

DEFINITION 4.9. An arcwise connected space A is called *simply connected* if every covering space (4.7) of (A, a) is trivial. This property is independent of the base point $a \in A$ (see [5]).

Let Ω_A be the (closed) subspace of P_A consisting of closed paths (that is, the loop space). Let Ω_A^e be the arc component of $a\#$ in Ω_A .

THEOREM 4.10. *Let A be an arcwise and locally arcwise connected space, $a \in A$; then if Ω_A is connected (not necessarily arcwise connected) and Ω_A^e is dense in Ω_A , A is simply connected (Definition 4.9).*

Proof. Let (f, B, b) be a covering space of (A, a) . Then $f\#: P_B \rightarrow P_A$ is a homeomorphism, and hence $f\#$ maps Ω_B homeomorphically into Ω_A . But every loop in A contractible to a may be lifted to a unique loop in B (see proof of 4.8), hence $f\#(\Omega_B) \supset \Omega_A^e$. But $f\#(\Omega_B)$ is closed in P_A . Hence $f\#$ maps Ω_B onto Ω_A .

Now this means that $f: B \rightarrow A$ is one-to-one. For if $f\#(p)(1)=f\#(p')(1)$, $p, p' \in P_B$, then $p(1)=p'(1)$; since $f\#p$ and $f\#p'$ having the same endpoint form a loop in A , and hence must come from a loop in B by the above. Also $p(1)=p'(1)$ implies $(f\#p)(1)=(f\#p')(1)$. Hence since $\lambda_B: P_B \rightarrow B$, and $\lambda_A: P_A \rightarrow A$ (the endpoint maps) are onto, f is one-to-one.

Further λ_A is both continuous and open [4, Lemma 4] since A is locally arcwise connected, hence the continuous map $\lambda_B f^{-1}: P_A \rightarrow B$ induces a continuous map $f^{-1}: A \rightarrow B$. This proves the theorem.

We now apply the above to LP -groups. We remark that if g is a topological group; then P_g , the space of paths of g beginning at the identity, may be made into a topological group by pointwise multiplication of paths. Then Ω_g is a closed normal subgroup.

LEMMA 4.11. *Let $g = \lim g_\alpha$, g_α Lie groups; then $P_g = \lim P_{g_\alpha}$.*

Proof. First $f_a: g \rightarrow g_a$ has a local cross-section. In fact, it is obvious that $df_a: G \rightarrow G_a$ has a cross-section since these spaces are linear, further g_a has a neighborhood which is homeomorphic to a neighborhood of G_a and $\exp: G \rightarrow g$ is continuous.

Now P_{g_a} is arcwise connected, hence $f_a^*: P_g \rightarrow P_{g_a}$ will be onto if the image covers a neighborhood of the identity. But this follows from the local cross-section of g_a in g . In fact, a fundamental system of neighborhoods of the identity in P_a is obtained from a fundamental system of neighborhoods of the identity in g by taking all those paths that are contained in a given neighborhood of the identity in g . Also it follows that f_a^* is open.

If V is a neighborhood of the identity in g that contains the kernel k_a of f_a , then the corresponding neighborhood in P_g contains P_{k_a} , and this last is clearly the kernel of f_a^* . Finally, P_g is complete since g is complete. Hence all the conditions for a projective limit are satisfied and the lemma follows.

LEMMA 4.12. *If $g = \lim g_a$, g_a simply connected Lie groups, then $\Omega_g = \lim \Omega_{g_a}$.*

Proof. The proof is the same as above, using the fact that since g_a is simply connected, Ω_{g_a} is arcwise connected.

LEMMA 4.13. *If $g = \lim g_a$, g_a simply connected, then Ω_g^c is dense in Ω_g and Ω_g is connected.*

Proof. Since $f_a: g \rightarrow g_a$ has local cross-sections (see 4.11), it defines a principal fiber bundle and hence satisfies the covering homotopy theorem of [6]. Hence since every loop in g_a is contractible each loop may be lifted to a contractible loop in g . Hence Ω_g^c maps onto Ω_{g_a} , all a . Hence Ω_g^c is dense in Ω_g . The last statement then follows.

LEMMA 4.14. *If $g = \lim g_a$, g_a simply connected, then $P_g/\Omega_g \approx g^c$.*

Proof. For Lie groups, $P_{g_a}/\Omega_{g_a} \approx g_a$, since g_a is locally arcwise connected and hence the map of P_{g_a} onto g_a is open (see [7]). Since P_g maps continuously onto P_{g_a} , P_g maps continuously onto $P_{g_a}/\Omega_{g_a} \approx g_a$. The induced map of P_g into $g = \lim g_a$ is obviously λ_g , and this induces a continuous one-to-one map of P_g/Ω_g onto $g^c \subset g$. Hence it is sufficient to show that P_g/Ω_g has the proper topology as a subgroup of g .

The neighborhoods of the identity in P_g/Ω_g are of the form $V\Omega_g$, where V is the preimage in P_g of a neighborhood V_a in P_{g_a} . But this is the preimage in P_g/Ω_g of the neighborhood $V_a\Omega_{g_a}$ in $P_{g_a}/\Omega_{g_a} \approx g_a$. Hence the Lemma follows.

THEOREM 4.15. *If $g = \lim g_\alpha$, g_α simply connected Lie groups, then g^c (the arc-component of the identity) is arcwise connected, locally arcwise connected, and simply connected (Definition 4.9).*

Proof. g^c is locally arcwise connected since P_g is obviously so. Hence by Lemmas 4.12 and 4.13 and Theorem 4.10, g^c is simply connected.

COROLLARY 4.16. *If g is the universal covering group (4.1) of a metrisable LP-group or a connected locally compact group, then g is arcwise connected, locally arcwise connected and simply connected.*

Proof. For metrisable groups, P_g is metrisable and complete. Hence $P_g/\Omega_g \approx g^c$ is complete and thus $g^c = g$. The result for locally compact groups will follow from Theorem 4.25.

We write again w for the map of Definition 4.5 cut down to g^c .

LEMMA 4.17. *(w, g^c, e) is a covering space (Definition 4.7) of (h^c, e) .*

Proof. First assume $g = \bar{h}$ the universal covering group of h . Then if $h = \lim h_\alpha$, $\bar{h} = \lim \bar{h}_\alpha$ and since $P_{\bar{h}_\alpha} \approx P_{h_\alpha}$ (this is obvious for Lie groups), $P_{\bar{h}} \approx \lim P_{\bar{h}_\alpha} \approx \lim P_{h_\alpha} \approx P_h$. Further, this isomorphism is clearly induced by the covering map w .

Now let (g, w) be any covering group of h . Then $\bar{h} \approx \bar{g}$ by (4.4) and hence $P_g \simeq P_{\bar{g}} \approx P_{\bar{h}} \approx P_h$. Since $\bar{g} \rightarrow g \rightarrow h$ is the same as $\bar{g} \rightarrow \bar{h} \rightarrow h$. The isomorphism $P_g \approx P_h$ is induced by w . Hence (w, g^c, e) satisfies (4.7)'.

THEOREM 4.18. *If g and h are LP-groups, G and H their Lie algebras, \bar{g} and \bar{h} their universal covering groups, P_g and P_h their group of paths, Ω_g^c and Ω_h^c the arc component of the identity in their group of loops, respectively; then the following are equivalent:*

- (a) G isomorphic to H
- (b) \bar{g} isomorphic to \bar{h}
- (c) P_g isomorphic to P_h such that the isomorphism takes Ω_g^c onto Ω_h^c .

Proof.

- (a) implies (b) follows from (4.4).
- (b) implies (a) follows from (4.2).
- (b) implies (c): $P_g \simeq P_{\bar{g}}$ by (4.17). Also $\Omega_{\bar{g}}^c \simeq \Omega_g^c$ under the same

map since every contractible loop in g may be lifted to \bar{g} (see proof of 4.8).

(c) implies (b): writing $\bar{\Omega}_g^c$ for the closure of Ω_g^c in P_g , we have $\bar{g}^c \simeq P_g/\Omega_g \simeq P_g/\bar{\Omega}_g^c \simeq P_h/\bar{\Omega}_h^c \simeq P_h/\Omega_h \simeq \bar{h}^c$, since Ω_g^c is dense in Ω_g . Hence $\bar{g} \simeq \bar{h}$.

LEMMA 4.19. *Every LP-algebra is the Lie algebra of an LP-group.*

Proof. By assumption, if G is an LP-algebra then $G = \lim G_a$, G_a finite dimensional. Let g_a be the simply connected groups corresponding to G_a . The homomorphisms of G_a onto G_b ($a < b$), induce homomorphisms of g_a onto g_b which satisfy LP I, II. Hence they have a limit g (Theorem 3.8). But g obviously has Lie algebra G .

DEFINITION 4.20. The group g defined in the proof of (4.19) is called the *universal group corresponding to G* .

LEMMA 4.21. *Let g be a universal LP-group then every covering group (h, w) (Definition 4.5) is trivial, that is, w is an isomorphism of h onto g .*

Proof. h^c is a trivial covering of g^c . Since h and g are complete and h^c and g^c are dense, the lemma follows.

THEOREM 4.22. *Let h be an LP-group and let H be its Lie algebra. Let G be an LP-algebra and g the universal group corresponding to G . Let θ be a continuous representation of G into H . Then θ induces a continuous representation f of g into h such that $df = \theta$.*

Proof. If $h = \lim h_b$, h_b Lie groups; then $H = \lim H_b$, H_b finite dimensional Lie algebras. Suppose $G = \lim G_a$, G_a finite dimensional. Let g_a be the simply connected group corresponding to G_a , $g = \lim g_a$. The map $G \rightarrow H \rightarrow H_b$ induces a map $G_a \rightarrow H_b$, some a . But this in turn induces a map $g_a \rightarrow h_b$ and hence a map of $g \rightarrow h_b$ for every b . It is easy to see that this defines a map $f: g \rightarrow h$ such that $df = \theta$.

THEOREM 4.23. *The universal covering group \bar{g} of a connected locally compact group g is the direct product of simply connected Lie groups. More explicitly $\bar{g} \simeq h \times a \times s$, where h is a simply connected Lie group, a is the (possibly infinite) direct product of the reals and s is the (possibly infinite) direct product of simple simply connected compact Lie groups.*

Proof. According to Yamabe [8] and Iwasawa (Theorem 11 of [2]), g is locally the direct product of a local Lie group h' and a compact normal subgroup k . Now $k = \lim k_\alpha$, where $k_\alpha = k/n_\alpha$, n_α normal in k and hence in g (Theorem 4 of [2]). Hence $g = \lim g_\alpha$, where $g_\alpha = g/n_\alpha$. Evidently g_α is locally isomorphic to $h' \times k_\alpha$; and hence to $h' \times k_\alpha^0$, k_α^0 connected component of k_α (since k_α a Lie group).

Since $f_{ab}: g_b \rightarrow g_a$ takes k_b onto k_a , it takes k_b^0 onto k_a^0 . Hence f_{ab} induces a homomorphism of $h \times \bar{k}_b^0$ onto $h \times \bar{k}_a^0$, where h is the simply connected group associated with h' and \bar{k}_a^0 is the simply connected covering group of k_a^0 . If $\bar{k}^0 = \lim \bar{k}_\alpha^0$, then $h \times \bar{k}^0 = \lim h \times \bar{k}_\alpha^0$; $h \times \bar{k}^0$ is the simply connected covering group of g and hence $h \times \bar{k}^0$ is the universal covering group of g .

Since k^0 is the universal covering group of k , k^0 connected component of k ; the problem is reduced to considering the universal covering group of a compact connected group.

According to A. Weil (p. 91 of [7]), k^0 is isomorphic to $(a' \times s)/d$, where a' is a compact abelian connected group, s is the (possibly infinite) direct product of simple simply connected compact Lie groups, and d is totally disconnected. It is evident that $\bar{k}^0 = a \times s$, where a is the universal covering group of a' . Since a' in the projective limit of toroidal groups, a is the projective limit of vector groups, and hence is the direct product of the reals (2.11). This proves the theorem.

COROLLARY 4.24. *If g is a locally compact group, then its Lie algebra G has the form $G = H \times A \times S$, where H is a finite dimensional Lie algebra, A is the product (possibly infinite) of 1-dimensional Lie algebras, and S is the (possibly infinite) direct product of simple compact Lie algebras.*

EXAMPLE 1. Let $g = \prod T_\alpha$, $\alpha \in J$, T_α isomorphic to the torus group, all α . Then g is compact. But $\bar{g} = \prod R_\alpha$, R_α isomorphic to the additive group of reals, $\alpha \in J$. Hence for J infinite, \bar{g} is not locally compact.

EXAMPLE 2. Let P be the p -adic solenoid (See for example: Eilenberg and Steenrod, *Foundations of algebraic topology*, p. 230). P is a compact connected group and is the projective limit of torus groups. If T is the multiplicative group of all complex numbers z with $|z|=1$, the projections $\varphi: T \rightarrow T$ are given by $\varphi(z) = z^p$, p an integer. φ induces the map $\bar{\varphi}: R \rightarrow R$, $\bar{\varphi}(x) = px$, which is an isomorphism of the additive group of the reals onto itself. Hence the universal covering group of P , which is the projective limit of the reals under these isomorphisms, is itself the additive group of reals. Hence the Lie algebra of P is 1-

dimensional. As is well known (see above reference) R maps continuously, one-to-one onto a dense subgroup of P , not the whole group. The map is not open and not onto P . More generally we have the following.

EXAMPLE 3. Let g be a connected, but not locally connected, locally compact group. Let (\bar{g}, w) be its universal covering group. Then $w:\bar{g} \rightarrow g$ is not both open onto. Consequently, if \bar{g} is locally compact, w is neither open (on the image) nor onto.

In fact \bar{g} is locally connected. Hence if w is open and onto, g would be locally connected. If \bar{g} is locally compact then if w is open on $w(\bar{g})$, $w(\bar{g})$ would be locally compact, hence closed, hence $w(\bar{g})=g$. On the other hand, if w is onto and \bar{g} connected locally compact, w is open.

Hence, in particular we have the following. Let g be a connected, but not locally connected finite dimensional locally compact group. Then \bar{g} is locally compact (a Lie group) and hence w is neither open nor onto.

EXAMPLE 4. Not every complete topological Lie algebra is an LP-algebra. In fact an infinite dimensional Banach space cannot contain arbitrarily small subspaces and cannot be an abelian LP-algebra.

5. Subgroups and subalgebras.

DEFINITION 5.1. Let g be an LP-group. An LP-group h is called an LP-subgroup of g if h is an abstract subgroup and the inclusion map $f:h \rightarrow g$ is a continuous representation such that $df:H \rightarrow G$ is an isomorphism into.

THEOREM 5.2. Let $g=\lim g_a$ be a LP-group; $G=\lim G_a$ its Lie algebra. Let H be a closed subalgebra of G , then $H=\lim H_a$ where H_a is the image of H in G_a . Let h_a be the analytic subgroup of g_a corresponding to H_a . Then $h=\lim h_a$ exists and is a connected LP-subgroup of g with Lie algebra H .

Proof. $H=\lim H_a$ follows from Theorem 2.5. Now $f_{ab}:g_b \rightarrow g_a$ induces $\bar{f}_{ab}:h_b \rightarrow h_a$, onto since the image of h_b in g_a is the analytic subgroup of g_a whose Lie algebra is H_a [1]. Therefore the h_a satisfy LP I, II and hence have a limit h (Theorem 3.8).

Obviously h is an abstract subgroup of g and the maps $h_a \rightarrow g_a$ induce a continuous one-to-one representation of h into g , namely the inclusion map. Obviously df is the inclusion map of H into G and hence an isomorphism into.

LEMMA 5.3. *Let h be the LP-subgroup of g defined in Theorem 5.2. Let h' be any other connected LP-subgroup with Lie algebra H' and such that the inclusion map $f':h' \rightarrow g$ induces an isomorphism df' of H' onto H . Then h' is a covering group of h and the covering map is abstractly an inclusion.*

Proof. Suppose $h' = \lim h'_b$, h'_b Lie groups. The map $h' \rightarrow g \rightarrow g_a$ induces a map $h'_b \rightarrow g_a$, for some b . But the image of h'_b in g_a is the analytic subgroup whose Lie algebra is H_a , the image of H'_b in G_a . Hence, $h'_b \rightarrow h_a$ is continuous and open, and induces $h' \rightarrow h_a$ continuous, open. This induces a continuous representation $\theta:h' \rightarrow h$ such that $d\theta$ is the isomorphism dg' . Since h' is contained abstractly in g , its elements are determined by their coordinates in g_a . Hence θ is an abstract inclusion of h' in h .

COROLLARY. *The subgroup h defined in Theorem 5.2 is uniquely characterized by Lemma 5.3.*

Proof. Suppose h and h' are two connected LP-subgroups such that h is a covering group of h' and h' is a covering group of h , and such that the covering maps are abstract inclusions. Then the maps $h \rightarrow h' \rightarrow h$ and $h' \rightarrow h \rightarrow h'$ are both the identity. Hence $h = h'$.

DEFINITION 5.4. The LP-subgroup h of g defined in Theorem 5.2 is called the *canonical LP-subgroup* corresponding to the subalgebra H of G .

We have proved the following.

THEOREM 5.5. *Let g be an LP-group, G its Lie algebra. There exists a one to one correspondence between canonical PL-subgroups of g and closed subalgebras of G .*

THEOREM 5.6. *Let g be the universal group corresponding to an LP-algebra G . Let k be a closed normal connected subgroup of g . Then k is the canonical LP-subgroup corresponding to an ideal in G . Conversely, the canonical LP-subgroup corresponding to an ideal in G is a closed normal connected topological subgroup of g .*

Proof. Let k be a closed, normal, connected subgroup of g . Suppose $g = \lim g_a$, g_a simply connected Lie groups. Let the image of k in g_a be k_a . The closure \bar{k}_a of k_a in g_a is a closed connected normal subgroup of g_a . Let K_a be the ideal of G_a corresponding to \bar{k}_a .

The image by f_{ab} of \bar{k}_b in g_a ($a < b$), is the analytic subgroup of g_a corresponding to the image of K_b in G_a . But the image is an ideal in

G_a , hence the corresponding analytic subgroup is a closed normal connected subgroup of g_a . Hence $\bar{k}_a \subset f_{ab}(\bar{k}_b)$. On the other hand since f_{ab} is continuous, $f_{ab}(\bar{k}_b) \subset \bar{k}_a$. Hence $f_{ab}(\bar{k}_b) = \bar{k}_a$.

Hence $k' = \lim \bar{k}_a$ exists and is a closed normal subgroup of g , and is the canonical LP-subgroup corresponding to $K = \lim K_a$. On the other hand, $k < K'$; but k maps onto k_a , a dense subgroup of \bar{k}_a . Hence k is dense in k' and $k = k'$ since k is closed.

The converse is obvious since the analytic subgroups of g_a corresponding to ideals are closed topological subgroups.

EXAMPLE. Consider the p -adic solenoid of Example 2, § 4. The additive group of reals R may be considered as an LP-subgroup of the p -adic solenoid P . Then P itself is the canonical LP-subgroup corresponding to the Lie algebra of R .

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