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1. Introduction. In most of the self-adjoint differential eigenvalue problems occurring in mathematical physics we are concerned with finding the extremal values of the quotient of two integro-differential quadratic forms in a certain space of admissible functions. By setting up a suitable basis in this space the problem can be reduced to that of finding the extremal values of a quotient of the form $(\alpha X, X)/(\beta X, X)$, where α and β are infinite symmetric matrices and X is a vector. The ordinary Rayleigh-Ritz method of approximating the solutions of the latter problem is to replace the infinite matrices $\alpha=(a_{ij})_1^\infty$ and $\beta=(b_{ij})_1^\infty$ by their finite sections $\alpha^n=(a_{ij})_1^n$ and $\beta^n=(b_{ij})_1^n$. The extremal values of the quotient $(\alpha^n X^n, X^n)/(\beta^n X^n, X^n)$, where X^n is an n dimensional vector, are the roots λ of the equation

$$\det(\alpha^n - \lambda\beta^n)=0 , \tag{1}$$

and these are taken as approximations to the first n solutions of the original problem. If the roots of (1) are denoted by λ_k^n with $\lambda_1^n \geqq \lambda_2^n \geqq \cdots \geqq \lambda_n^n$, then for any fixed k, λ_k^n increases monotonically with n and its limit as $n\to\infty$ is the kth eigenvalue of the original problem. It should be stated here that the quotient of integro-differential quadratic forms in the original problem is taken as the reciprocal of the usual Rayleigh quotient so that the eigenvalues are all bounded.

If we let

$$\lambda_k=\lim_{n\to\infty}\lambda_k^n , \tag{2}$$

then the problem arises of estimating the difference $\lambda_k-\lambda_k^n$.

We shall consider this problem under certain assumptions with regard to the matrices α and β. These assumptions are that α and β are both positive definite, that the matrix $(b_{ij})_{n+1}^\infty$ has a positive lower bound independent of n, that the matrix $(a_{ij})_{n+1}^\infty$ has an upper bound which tends towards zero as $n\to\infty$, and that

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=n+1}^{\infty}a_{ij}^2=0 , \qquad \lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=n+1}^{\infty}b_{ij}^2=0 .$$

2. The simplest case, which we take up first, is that in which β

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is the unit matrix. Let $X_k^{(n)}$ be the orthonormal eigenvectors corresponding to the eigenvalues λ_k^n as defined above. Let numbers ε_n and ρ_n be defined by

$$(3) \qquad \varepsilon_n \geqq \left(\sum_{i=1}^{n} \sum_{j=n+1}^{\infty} a_{ij}^2 \right)^{1/2},$$

$$(4) \qquad \rho_n \geqq \sup_{x_i} \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} a_{ij} x_i x_j \Big/ \sum_{i=n+1}^{\infty} x_i^2 .$$

In general the exact values of the right-hand members of (3) and (4) will not be available, and for this reason we define ε_n and ρ_n as merely upper bounds for these quantities. The more closely these upper bounds can be estimated, the better will be the subsequent estimates of the eigenvalues. For the effectiveness of the method it is necessary that the values of ε_n and ρ_n can be made arbitrarily small for n sufficiently large. One method of defining ρ_n is to take it as an upper bound for $\left(\sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} a_{ij}^2 \right)^{1/2}$ in those cases where the latter series converges. A different method is given in the example of § 6.

We shall adopt the convention that, if X is a vector, $(x_i)_1^\infty$, then X^n stands for the n-dimensional vector $(x_i)_1^n$. Let $k \leqq n < N$. By the minimax principle,

$$(5) \qquad \lambda_k^N = \min_{U_i} \max_{X} \frac{(\alpha^N X^N, X^N)}{(X^N, X^N)}, \qquad (X^N, U_i^N) = 0,\ i = 1, 2, \cdots, k-1 .$$

Choose the vector U_i so that its first n components are equal respectively to those of $X_i^{(n)}$ and its remaining components are zero. Let

$$X = (x_i)_1^\infty, \quad y_1 = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}, \quad y_2 = (x_{n+1}^2 + x_{n+2}^2 + \cdots + x_N^2)^{1/2} .$$

Then

$$\lambda_k^N \leqq \max_{X} \frac{(\alpha^N X^N, X^N)}{(X^N, X^N)}, \qquad (X^n, X_i^{(n)}) = 0,\ i = 1, 2, \cdots, k-1$$

$$= \max_{X} \left[(\alpha^n X^n, X^n) + 2 \sum_{i=1}^{n} \sum_{j=n+1}^{N} a_{ij} x_i x_j + \sum_{i=n+1}^{N} \sum_{j=n+1}^{N} a_{ij} x_i x_j \right] \Big/ (y_1^2 + y_2^2)$$

$$(X^n, X_i^{(n)}) = 0,\ i = 1, 2, \cdots, k-1$$

$$\leqq \max_{y_i} \frac{\lambda_k^n y_1^2 + 2\varepsilon_n y_1 y_2 + \rho_n y_2^2}{y_1^2 + y_2^2} .$$

The last step is justified by use of the maximum principle for the first term of the numerator and the Schwarz inequality for the second term.

The quantity on the right side of this inequality is the larger root λ of the equation

$$\begin{vmatrix} \lambda_k^n - \lambda & \varepsilon_n \\ \varepsilon_n & \rho_n - \lambda \end{vmatrix} = 0 .$$

Hence,

$$\lambda_k^N \leq \frac{\lambda_k^n + \rho_n + \sqrt{(\lambda_k^n - \rho_n)^2 + 4\varepsilon_n^2}}{2} ,$$

and, since the right side is independent of N,

$$(6) \qquad \lambda_k^n \leq \lambda_k \leq \frac{\lambda_k^n + \rho_n + \sqrt{(\lambda_k^n - \rho_n)^2 + 4\varepsilon_n^2}}{2} .$$

If $\rho_n < \lambda_k^n$, this inequality gives the simpler, but less precise, one

$$(6a) \qquad \lambda_k^n \leq \lambda_k \leq \lambda_k^n + \frac{\varepsilon_n^2}{\lambda_k^n - \rho_n} .$$

The inequality (6) (or 6a) makes it possible to obtain arbitrarily close bounds for λ_k by taking n sufficiently large.

Better estimates for λ_k can be obtained if one makes full use of the available data, namely λ_k^n and $X_k^{(n)}$. With these it is possible to transform α into an equivalent matrix (one having the same eigenvalues) $\bar{\alpha} = (\bar{a}_{ij})$, where

$$\begin{aligned} &\bar{a}_{kk} = \lambda_k^n && (k = 1, 2, \cdots, n) , \\ &\bar{a}_{ij} = 0 && (i, j = 1, 2, \cdots, n;\ i \neq j) , \\ &\bar{a}_{ij} = a_{ij} && (i, j = n+1, n+2, \cdots) , \\ &\sum_{i=1}^{n} \bar{a}_{ij}^2 = \sum_{i=1}^{n} a_{ij}^2 && (j = n+1, n+2, \cdots) . \end{aligned}$$

The actual formula for $\bar{\alpha}$ is $\bar{\alpha} = \Gamma^{tr} \alpha \Gamma$ where $\Gamma = \begin{pmatrix} \Gamma^{(n)} & 0 \\ 0 & E \end{pmatrix}$, $\Gamma^{(n)} = (X_1^{(n)}, X_2^{(n)}, \cdots, X_n^{(n)})$ and the vectors $X_k^{(n)}$ are orthonormal.

Let

$$(7) \qquad \varepsilon_{nk} \geq \left(\sum_{j=n+1}^{\infty} \bar{a}_{kj}^2 \right)^{1/2} \qquad (k = 1, 2, \cdots, n) .$$

If any one of the numbers ε_{nk} is equal to zero, then the corresponding eigenvalue λ_k^n of $\bar{\alpha}^n$ is actually an eigenvalue of $\bar{\alpha}$ and the kth row and column of $\bar{\alpha}$ can be deleted before proceeding with any further calculations. We may therefore assume without loss of generality that all the numbers ε_{nk} appearing in subsequent formulas are different from zero.

Apply (5) with α^N replaced by $\bar{\alpha}^N$ and with U_i equal to the vector whose ith component is 1 and whose remaining components are zero. This gives, with $y=(x_{n+1}^2+\cdots+x_N^2)^{1/2}$

$$(8)\qquad \lambda_k^N \leqq \frac{\lambda_k^n x_k^2+\lambda_{k+1}^n x_{k+1}^2+\cdots+\lambda_n^n x_n^2+2\sum_{i=k}^{n}\sum_{j=n+1}^{N}\bar{a}_{ij}x_ix_j+\sum_{i=n+1}^{N}\sum_{j=n+1}^{N}a_{ij}x_ix_j}{x_k^2+x_{k+1}^2+\cdots+x_N^2}$$

$$\leqq \frac{\lambda_k^n x_k^2+\cdots+\lambda_n^n x_n^2+2\sum_{i=k}^{n}\varepsilon_{ni}|x_i|y+\rho_n y^2}{x_k^2+\cdots+x_n^2+y^2}\ .$$

The maximum value of the quotient

$$\frac{\lambda_k^n x_k^2+\cdots+\lambda_n^n x_n^2+2\sum_{i=k}^{n}\varepsilon_{ni}x_iy+\rho_n y^2}{x_k^2+\cdots+x_n^2+y^2}$$

can be attained when the variables $x_k, \cdots, x_n, y$ are restricted to non-negative values. Hence λ_k^N cannot exceed the largest root λ of the equation

$$(9)\qquad \begin{vmatrix} \lambda_k^n-\lambda & 0 & \cdots & 0 & \varepsilon_{nk} \\ 0 & \lambda_{k+1}^n-\lambda & \cdots & 0 & \varepsilon_{n,k+1} \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \lambda_n^n-\lambda & \varepsilon_{nn} \\ \varepsilon_{nk} & \varepsilon_{n,k+1} & \cdots & \varepsilon_{nn} & \rho_n-\lambda \end{vmatrix}$$

$$=(\rho_n-\lambda)\prod_{i=k}^{n}(\lambda_i^n-\lambda)-\sum_{j=k}^{n}\frac{\varepsilon_{nj}^2\prod_{i=k}^{n}(\lambda_i^n-\lambda)}{\lambda_j^n-\lambda}=0\ .$$

If a number r appears $m+1$ times in the set $\lambda_k^n, \lambda_{k+1}^n, \cdots, \lambda_n^n$, then this number is an m-fold root of (9). If $\mu_1>\mu_2>\cdots>\mu_l$ are the distinct values in the set $\lambda_k^n, \lambda_{k+1}^n, \cdots, \lambda_n^n$, then (9) also has roots $r_1, r_2, \cdots, r_{l+1}$, where $r_1<\mu_1<r_2<\mu_2<\cdots<\mu_l<r_{l+1}$. The latter roots are all the roots of the equation

$$(9a)\qquad \lambda-\rho_n=\sum_{j=k}^{n}\frac{\varepsilon_{nj}^2}{\lambda-\lambda_j^n}\ .$$

3. As a simple example illustrating the estimates of the last section, let us take the problem of finding the eigenvalues Λ defined by

$$y''=-\Lambda(1+x)y\ ,\qquad (0<x<1)\ ,$$

$$y(0)=y(1)=0\ .$$

The reciprocals of these will be the extremal values $\lambda_1 \geqq \lambda_2 \geqq \lambda_3 \geqq \cdots$ of the quotient

$$Q(y) = \int_0^1 (1+x) y^2 \, dx \Big/ \int_0^1 y'^2 \, dx$$

in the space $\mathscr{F}$ consisting of all functions $y(x)$ with sectionally continuous first derivatives and with $y(0)=y(1)=0$. As a basis for this space we take

$$\varphi_n(x) = \frac{\sqrt{2}\,\sin n\pi x}{n\pi} \qquad (n=1, 2, \cdots)$$

and let

$$a_{ij} = \int_0^1 (1+x)\varphi_i\varphi_j \, dx = \begin{cases} \dfrac{3}{2i^2\pi^2} & \text{if} \quad i=j\,, \\ \dfrac{4[(-1)^{i-j}-1]}{\pi^4(i^2-j^2)^2} & \text{if} \quad i \neq j\,, \end{cases}$$

$$b_{ij} = \int_0^1 \varphi_i'\varphi_j' \, dx = \delta_{ij}\,.$$

If $y = \sum_{i=1}^{\infty} x_i \varphi_i$, then

$$Q(y) = \frac{(\alpha X, X)}{(\beta X, X)}\,,$$

where $\alpha = (a_{ij})_1^\infty$, $\beta = (b_{ij})_1^\infty$, $X = (x_i)_1^\infty$, so the problem is reduced to one of the type for which the estimates of the last section apply.

Let $n=3$. The equation for λ_1^3, λ_2^3, λ_3^3 is

$$\begin{vmatrix} \dfrac{3}{2\pi^2} - \lambda & -\dfrac{8}{9\pi^4} & 0 \\ -\dfrac{8}{9\pi^4} & \dfrac{3}{8\pi^2} - \lambda & -\dfrac{8}{25\pi^4} \\ 0 & -\dfrac{8}{25\pi^4} & \dfrac{3}{18\pi^2} - \lambda \end{vmatrix} = 0\,.$$

The eigenvalues and eigenvectors are:

$$\lambda_1^3 = .1527\ 0819\,, \qquad X_1^{(3)} = (.99684,\ -.07935,\ .00192)\,,$$

$$\lambda_2^3 = .0377\ 8273\,, \qquad X_2^{(3)} = (.07869,\ .98480,\ -.15482)\,,$$

$$\lambda_3^3 = .0163\ 7316\,, \qquad X_3^{(3)} = (.01040,\ .15449,\ .98794)\,.$$

We make the following estimates

$$\sum_{j=4}^{\infty} a_{1j}^2 = \frac{64}{\pi^8} \sum_{\sigma=2}^{\infty} \frac{1}{(4\sigma^2-1)^4}$$

$$< \frac{64}{\pi^8}\left[\frac{1}{15^4}+\frac{1}{35^4}+\frac{1}{63^4}+\sum_{\sigma=5}^{\infty}\frac{1}{(3\sigma^2)^4}\right]$$

$$< \frac{64}{\pi^8}\left[\frac{1}{15^4}+\frac{1}{35^4}+\frac{1}{63^4}+\frac{1}{81}\int_4^{\infty}\frac{dx}{x^8}\right] = 1.389\times 10^{-7} ,$$

$$\sum_{j=4}^{\infty} a_{2j}^2 = \frac{64}{\pi^8} \sum_{\sigma=2}^{\infty} \frac{1}{[(2\sigma+1)^2-4]^4}$$

$$< \frac{64}{\pi^8}\left[\frac{1}{21^4}+\frac{1}{45^4}+\frac{1}{77^4}+\frac{1}{256}\int_4^{\infty}\frac{dx}{x^8}\right] = .368\times 10^{-7} ,$$

$$\sum_{j=4}^{\infty} a_{3j}^2 = \frac{64}{\pi^8} \sum_{\sigma=2}^{\infty} \frac{1}{(4\sigma^2-9)^4}$$

$$< \frac{64}{\pi^8}\left[\frac{1}{7^4}+\frac{1}{27^4}+\frac{1}{55^4}+\frac{1}{81}\int_4^{\infty}\frac{dx}{x^8}\right] = 28.234\times 10^{-7} ,$$

$$\sum_{j=4}^{\infty} a_{1j}a_{3j} = \frac{64}{\pi^8} \sum_{\sigma=2}^{\infty} \frac{1}{(4\sigma^2-1)^2(4\sigma^2-9)^2}$$

$$< \frac{64}{\pi^8}\left[\frac{1}{15^2\cdot 7^2}+\frac{1}{35^2\cdot 27^2}+\frac{1}{63^2\cdot 55^2}+\frac{1}{81}\int_4^{\infty}\frac{dx}{x^8}\right] = 6.206\times 10^{-7} ,$$

$$\sum_{j=4}^{\infty} a_{1j}a_{2j} = \sum_{j=4}^{\infty} a_{2j}a_{3j} = 0 ,$$

$$\sum_{j=4}^{\infty} (a_{1j}^2+a_{2j}^2+a_{3j}^2) < 29.991\times 10^{-7} = \varepsilon_3^2 ,$$

$$\sum_{i,j=1}^{\infty} a_{ij}^2 = \frac{9}{4\pi^4}\sum_{\sigma=4}^{\infty}\frac{1}{\sigma^4} + \frac{128}{\pi^8}\sum_{n=2}^{\infty}\sum_{\sigma=0}^{\infty}\frac{1}{[(2n+2\sigma+1)^2-4n^2]^4}$$

$$+ \frac{128}{\pi^8}\sum_{n=2}^{\infty}\sum_{\sigma=1}^{\infty}\frac{1}{[(2n+2\sigma)^2-(2n+1)^2]^4}$$

$$< \frac{9}{4\pi^4}\sum_{\sigma=4}^{\infty}\frac{1}{\sigma^4} + \frac{128}{\pi^8}\left\{\sum_{n=2}^{\infty}\sum_{\sigma=0}^{\infty}\frac{1}{[4n(1+2\sigma)]^4}\right.$$

$$\left. + \sum_{n=2}^{\infty}\sum_{\sigma=1}^{\infty}\frac{1}{[2(2n+1)(2\sigma-1)]^4}\right\}$$

$$= \frac{9}{4\pi^4}\sum_{\sigma=4}^{\infty}\frac{1}{\sigma^4} + \frac{8}{\pi^8}\sum_{\sigma=0}^{\infty}\frac{1}{(1+2\sigma)^4}\left\{\sum_{n=2}^{\infty}\frac{1}{(2n)^4}+\sum_{n=2}^{\infty}\frac{1}{(2n+1)^4}\right\}$$

$$=\sum_{\sigma=4}^{\infty}\frac{1}{\sigma^4}\left[\frac{9}{4\pi^4}+\frac{8}{\pi^8}\cdot\frac{\pi^4}{96}\right]=.00017\ 9117=\rho_3^2\ ,$$

$$\rho_3=.013\ 3835\ .$$

If the matrix α is transformed into the equivalent matrix $\bar{\alpha}$ in which the upper left hand 3×3 matrix is diagonalized, the formulas for the elements $\bar{a}_{ij}$ are (for $j\geq 4$):

$$\bar{a}_{1j}=.99684\ a_{1j}-.07935\ a_{2j}+.00192\ a_{3j}\ ,$$

$$\bar{a}_{2j}=.07869\ a_{1j}+.98480\ a_{2j}-.15482\ a_{3j}\ ,$$

$$\bar{a}_{3j}=.01040\ a_{1j}+.15449\ a_{2j}+.98794\ a_{3j}\ .$$

Hence,

$$\sum_{j=4}^{\infty}\bar{a}_{1j}^2<1.395\times 10^{-7}=\varepsilon_{31}^2\ ,$$

$$\sum_{j=4}^{\infty}\bar{a}_{2j}^2<1.042\times 10^{-7}=\varepsilon_{32}^2\ ,$$

$$\sum_{j=4}^{\infty}\bar{a}_{3j}^2<27.630\times 10^{-7}=\varepsilon_{33}^2\ .$$

The first three extremal values of the quotient $Q(y)$ can now be estimated by either (6), (6a), or (9a). From (6) we get

$$.152\ 708\leq\lambda_1\leq .152\ 730\ ,$$

$$.037\ 782\leq\lambda_2\leq .037\ 905\ ,$$

$$.016\ 373\leq\lambda_3\leq .017\ 167\ ;$$

whereas (9a) yields the following more precise estimates:

$$.152\ 7081\leq\lambda_1\leq .152\ 7092\ ,$$

$$.037\ 7827\leq\lambda_2\leq .037\ 7871\ ,$$

$$.016\ 3731\leq\lambda_3\leq .017\ 1139\ .$$

4. **Returning to the general problem,** let us assume that, by a preliminary transformation, the matrices α and β are already diagonalized in the $n\times n$ upper left-hand corner; that is, that

$$a_{ii}=\lambda_i^n\ ,\quad b_{ii}=1 \qquad (i=1,2,\cdots,n)\ ,$$

$$a_{ij}=b_{ij}=0 \qquad (i,j=1,2,\cdots,n;\ i\neq j)\ .$$

Let the bounds ρ_n and ε_{nk} be defined by (4) and (7) (with $\bar{a}_{kj}$ replaced by a_{kj}): In addition let bounds δ_{nk} and r_n be defined by

(10) $$\delta_{nk} \geqq \left(\sum_{j=n+1}^{\infty} b_{kj}^2 \right)^{1/2} \qquad (k=1, 2, \cdots, n) ,$$

(11) $$r_n \leqq \inf_{x_i} \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} b_{ij} x_i x_j \Big/ \sum_{i=n+1}^{\infty} x_i^2 .$$

We assume that all these bounds exist, that

(12) $$r_n > \sum_{k=1}^{n} \delta_{nk}^2 ,$$

and that $\varepsilon_{nk} + \delta_{nk} \neq 0$ $(k=1, 2, \cdots, n)$ (see remark following (7)).

By the minimax principle with $k \leqq n < N$,

$$\lambda_k^N = \min_{U_i} \max_X \frac{(\alpha^N X^N, X^N)}{(\beta^N X^N, X^N)} , \qquad (\beta^N X^N, U_i) = 0, \ i=1, 2, \cdots, k-1 .$$

Proceeding as before, let U_i be the vector whose ith component is 1 and whose remaining components are zero. Then

$$\lambda_k^N \leqq \max_{x_i} \frac{\lambda_k^n x_k^2 + \cdots + \lambda_n^n x_n^2 + 2 \sum_{i=k}^{n} \sum_{j=n+1}^{N} a_{ij} x_i x_j + \sum_{i=n+1}^{N} \sum_{j=n+1}^{N} a_{ij} x_i x_j}{x_k^2 + \cdots + x_n^2 + 2 \sum_{i=k}^{n} \sum_{j=n+1}^{N} b_{ij} x_i x_j + \sum_{i=n+1}^{N} \sum_{j=n+1}^{N} b_{ij} x_i x_j}$$

$$\leqq \max_{x_i} \frac{\lambda_k^n x_k^2 + \cdots + \lambda_n^n x_n^2 + 2 \sum_{i=k}^{n} \varepsilon_{ni} |x_i| y + \rho_n y^2}{x_k^2 + \cdots + x_n^2 - 2 \sum_{i=k}^{n} \delta_{ni} |x_i| y + r_n y^2} ,$$

where $y = (x_{n+1}^2 + x_{n+2}^2 + \cdots + x_N^2)^{1/2}$. The condition (12) is equivalent to the positive definiteness of the denominator of the last expression. Hence, λ_k^N and therefore λ_k, cannot exceed the largest root λ of the equation

(13) $$\begin{vmatrix} \lambda_k^n - \lambda & \cdots & 0 & \varepsilon_{nk} + \lambda \delta_{nk} \\ \cdot & \cdots & \cdot & \cdot \\ 0 & \cdots & \lambda_n^n - \lambda & \varepsilon_{nn} + \lambda \delta_{nn} \\ \varepsilon_{nk} + \lambda \delta_{nk} & \cdots & \varepsilon_{nn} + \lambda \delta_{nn} & \rho_n - \lambda r_n \end{vmatrix}$$

$$= (\rho_n - \lambda r_n) \prod_{i=k}^{n} (\lambda_i^n - \lambda) - \sum_{j=k}^{n} (\varepsilon_{nj} + \lambda \delta_{nj})^2 \frac{\prod_{i=k}^{n} (\lambda_i^n - \lambda)}{\lambda_j^n - \lambda} = 0 ,$$

which is the same thing as the largest root of the equation

(13a) $$\lambda r_n - \rho_n = \sum_{j=k}^{n} \frac{(\varepsilon_{nj} + \lambda \delta_{nj})^2}{\lambda - \lambda_j^n} .$$

To analyze the location of the largest root of (13a), let

$$\varphi(\lambda)=\sum_{j=k}^{n}\frac{(\varepsilon_{nj}+\lambda\delta_{nj})^2}{\lambda-\lambda_j^n} .$$

Then

$$\varphi'(\lambda)=\sum_{j=k}^{n}\left[\frac{2\delta_{nj}(\varepsilon_{nj}+\lambda\delta_{nj})}{\lambda-\lambda_j^n}-\frac{(\varepsilon_{nj}+\lambda\delta_{nj})^2}{(\lambda-\lambda_j^n)^2}\right],$$

$$\varphi''(\lambda)=2\sum_{j=k}^{n}\frac{(\varepsilon_{nj}+\lambda_j^n\delta_{nj})^2}{(\lambda-\lambda_j^n)^3},$$

$$\lim_{\lambda\to\infty}\varphi'(\lambda)=\sum_{j=k}^{n}\delta_{nj}^2 .$$

For $\lambda>\lambda_k^n$, $\varphi''(\lambda)>0$, and therefore in this range the graph of $\varphi(\lambda)$ can intersect that of the function $r_n\lambda-\rho_n$ in at most two points. Since $\lim_{\lambda\to\lambda_k^n+}\varphi(\lambda)=+\infty$ and since, by (12), $r_n\lambda-\rho_n>\varphi(\lambda)$ for all λ sufficiently large, there must be exactly one point of intersection, that is, one root of (13) or (13a), in the range $\lambda>\lambda_k^n$. This root is the upper bound which we obtain for λ_k.

Let us now assume that

$$r_n\lambda_k^n-\rho_n\geqq a>0 \tag{14}$$

for all n sufficiently large, and that

$$\lim_{n\to\infty}\sum_{j=1}^{n}(\varepsilon_{nj}^2+\delta_{nj}^2)=0 . \tag{15}$$

Then, for any $\varepsilon>0$, and for n sufficiently large, $\varphi(\lambda_k^n+\varepsilon)<r_n(\lambda_k^n+\varepsilon)-\rho_n$ and so the largest root of (13) or (13a) is less than $\lambda_k^n+\varepsilon$. Therefore, (14) and (15) are sufficient to ensure that the method gives arbitrarily close bounds on λ_k, for any k, by taking n sufficiently large.

5. To illustrate the method of the last section let us consider the problem:

$$\frac{d}{dx}\left((1+x)\frac{dy}{dx}\right)=-\Lambda y \qquad (0<x<1) ,$$

$$y(0)=y(1)=0 .$$

The reciprocals of the eigenvalues Λ of this problem are the extremal values of the quotient

$$Q(y)=\int_0^1 y^2\,dx \Big/ \int_0^1 (1+x)y'^2\,dx$$

on the space of functions $y(x)$ with sectionally continuous first derivatives and with $y(0)=y(1)=0$. If $\{\varphi_n(x)\}_1^\infty$ is a basis in this space and

$$a_{ij}=\int_0^1 \varphi_i\varphi_j\,dx\ , \qquad b_{ij}=\int_0^1 (1+x)\varphi_i'\varphi_j'\,dx\ ,$$

then the problem is reduced to that of finding the extremal values of the quotient $(\alpha X, X)/(\beta X, X)$, where $\alpha=(a_{ij})_1^\infty$, $\beta=(b_{ij})_1^\infty$.

Let the sequence $\{\varphi_n\}$ be defined as follows:

$$\varphi_i=\sum_{j=1}^{3} c_{ij}\sin j\pi x \qquad (i=1,\,2,\,3)\ ,$$

$$\varphi_i=\frac{\sqrt{2}\ \sin i\pi x}{i\pi} \qquad (i>3)\ ,$$

where the constants c_{ij} are chosen in such a way that

$$(b_{ij})_1^3=E\ ,$$

$$(a_{ij})_1^3=\begin{pmatrix} .0696\ 820 & 0 & 0 \\ 0 & .0173\ 553 & 0 \\ 0 & 0 & .0073\ 9145 \end{pmatrix}.$$

The values of the constants c_{ij} are given by the table:

$i\diagdown j$	1	2	3
1	.3713655	.0378935	.0039777
2	−.0189824	.1828646	.0301791
3	.0007276	−.0197241	.1199722

We now apply the method of the last section with $n=2$. Since the matrix α is of diagonal form, ε_{21} and ε_{22} may be taken as zero and ρ_2 may be taken as the maximum of the elements a_{ii} $(i\geq 3)$, namely $a_{33}=.0073\ 9145$.

For $i=1$, 2 we have

$$\sum_{j=3}^{\infty} b_{ij}^2=\sum_{j=4}^{\infty} b_{ij}^2$$

$$=2\pi^2\sum_{j=4}^{\infty}\left(\int_0^1 (1+x)(c_{i1}\cos\pi x+2c_{i2}\cos 2\pi x+3c_{i3}\cos 3\pi x)\cos j\pi x\,dx\right)^2$$

$$=2\pi^2\sum_{j=4}^{\infty}\left[c_{i1}^2\left(\int_0^1 (1+x)\cos\pi x\cos j\pi x\,dx\right)^2+4c_{i2}^2\left(\int_0^1 (1+x)\cos 2\pi x\cos j\pi x\,dx\right)^2\right.$$

$$+ 9c_{i3}^2\left(\int_0^1 (1+x)\cos 3\pi x \cos j\pi x\,dx\right)^2 + 6c_{i1}c_{i3}\left(\int_0^1 (1+x)\cos \pi x \cos j\pi x\,dx\right)$$

$$\times\left(\int_0^1 (1+x)\cos 3\pi x \cos j\pi x\,dx\right)\Bigg]$$

$$= \frac{8}{\pi^2}\Bigg[c_{i1}^2 \sum_{\sigma=2}^{\infty} \frac{(1+4\sigma^2)^2}{(4\sigma^2-1)^4} + 4c_{i2}^2 \sum_{\sigma=2}^{\infty} \frac{(4+(2\sigma+1)^2)^2}{((2\sigma+1)^2-4)^4}$$

$$+ 9c_{i3}^2 \sum_{\sigma=2}^{\infty} \frac{(9+4\sigma^2)^2}{(4\sigma^2-9)^4} + 6c_{i1}c_{i3} \sum_{\sigma=2}^{\infty} \frac{(1+4\sigma^2)(9+4\sigma^2)}{(4\sigma^2-1)^2(4\sigma^2-9)^2}\Bigg].$$

We make the following estimates:

$$\sum_{\sigma=2}^{\infty} \frac{(1+4\sigma^2)^2}{(4\sigma^2-1)^4} < \frac{17^2}{15^4} + \frac{37^2}{35^4} + \frac{65^2}{63^4} + \frac{1}{15}\sum_{\sigma=5}^{\infty}\frac{1}{\sigma^4} = .00712722\,,$$

$$\sum_{\sigma=2}^{\infty} \frac{(4+(2\sigma+1)^2)^2}{((2\sigma+1)^2-4)^4} < \frac{29^2}{21^4} + \frac{53^2}{45^4} + \frac{85^2}{77^4} + \frac{5}{4}\sum_{\sigma=5}^{\infty}\frac{1}{(2\sigma+1)^4} = .00541918\,,$$

$$\sum_{\sigma=2}^{\infty} \frac{(9+4\sigma^2)^2}{(4\sigma^2-9)^4} < \frac{25^2}{7^4} + \frac{45^2}{27^4} + \frac{73^2}{55^4} + \frac{1}{8}\sum_{\sigma=5}^{\infty}\frac{1}{\sigma^4} = .26514737\,,$$

$$\sum_{\sigma=2}^{\infty} \frac{(1+4\sigma^2)(9+4\sigma^2)}{(4\sigma^2-1)^2(4\sigma^2-9)^2} < \frac{17\cdot 25}{15^2\cdot 7^2} + \frac{37\cdot 45}{35^2\cdot 27^2} + \frac{65\cdot 73}{63^2\cdot 55^2} + \frac{1}{8}\sum_{\sigma=5}^{\infty}\frac{1}{\sigma^4}$$

$$= .04125482\,.$$

This gives

$$\sum_{j=3}^{\infty} b_{1j}^2 < .0011490 = \delta_{21}^2\,,$$

$$\sum_{j=3}^{\infty} b_{2j}^2 < .0023514 = \delta_{22}^2\,.$$

To obtain a value for r_2 we let $F(x) = \sum_{i=3}^{N} x_i\varphi_i(x)$, where $(x_i)_3^N$ is any given vector. Then

$$\int_0^1 F'^2(x)\,dx = x_3^2\int_0^1 \varphi_3'^2 dx + \sum_{i=4}^{N} x_i^2$$

$$= .646936x_3^2 + \sum_{i=4}^{N} x_i^2 \geqq .646936 \sum_{i=3}^{N} x_i^2\,,$$

$$\int_0^1 (1+x)F'^2(x)\,dx = \sum_{i=3}^{N}\sum_{j=3}^{N} b_{ij}x_ix_j\,.$$

Hence,

$$\frac{\sum_{i=3}^{N}\sum_{j=3}^{N} b_{ij}x_i x_j}{\sum_{i=3}^{N} x_i^2} \geq \frac{\int_0^1 (1+x)F''^2(x)\,dx}{\int_0^1 F''^2(x)\,dx}(.646\ 936) \geq .646936 .$$

Since the bound on the right side is independent of N we may take

$$r_2 = .646936 .$$

The use of equation (13a) now gives the following results, where λ_1 and λ_2 are the reciprocals of the first two eigenvalues of the original problem :

$$.06968 \leq \lambda_1 \leq .06984 ,$$

$$.01735 \leq \lambda_2 \leq .01754 .$$

6. **In conclusion** we shall show how the method would work on the two dimensional problem of an oscillating square membrane of variable density ; namely,

$$u_{xx} + u_{yy} = -\Lambda g u \qquad \text{in} \quad R ,$$

$$u = 0 \qquad \text{on} \quad C ,$$

where R is the region $0 < x < 1$, $0 < y < 1$, C is the boundary of and g is a nonnegative function with the derivative g_{xy} sectionally continuous in $R+C$. The reciprocals of the eigenvalues Λ are the extremal values of the quotient

$$Q(u) = \int_0^1\int_0^1 gu^2\,dx\,dy \Big/ \int_0^1\int_0^1 (u_x^2 + u_y^2)\,dx\,dy$$

in the space of functions $u(x, y)$ with sectionally continuous first derivatives in $R+C$ and vanishing on C.

As a basis for this problem we take the functions

$$\frac{2 \sin m\pi x \sin n\pi y}{\pi(m^2+n^2)^{1/2}} , \qquad m, n = 1, 2, 3, \cdots ,$$

and arrange them in a sequence $\varphi_1, \varphi_2, \varphi_3, \cdots$ ordered according to the value of m^2+n^2; that is,

$$\varphi_i = \frac{2 \sin m_i\pi x \sin n_i\pi y}{\pi\sigma_i} , \qquad \sigma_i = (m_i^2 + n_i^2)^{1/2} ,$$

$$\sigma_1 \leq \sigma_2 \leq \sigma_3 \cdots .$$

As $N \to \infty$, $\sigma_N = O(\sqrt{N})$. Let

$$a_{ij}=\int_0^1\int_0^1 g\varphi_i\varphi_j\,dx\,dx\ ,$$

$$b_{ij}=\int_0^1\int_0^1\left(\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial x}+\frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial y}\right)dx\,dy=\delta_{ij}\ .$$

If $u=\sum_{i=1}^{\infty}x_i\varphi_i$, then

$$Q(u)=(\alpha X,\ X)/(\beta X,\ X)$$

where

$$\alpha=(a_{ij})_1^\infty\ ,\qquad \beta=(\delta_{ij})_1^\infty\ ,\qquad X=(x_i)_1^\infty\ .$$

In order to show that the method will give arbitrarily close estimates of the eigenvalues, we must show that the quantity defined in (4) can be determined and made arbitrarily small, and that $\sum_{i=1}^{n}\sum_{j=n+1}^{\infty}a_{ij}^2$ can be made arbitrarily small by taking n sufficiently large. The estimate ρ_n can be managed by noting that (4) is equivalent, in the present case, to

$$\rho_n\geq\sup_{v\in a_n}\int_0^1\int_0^1 gv^2\,dx\,dy\Big/\int_0^1\int_0^1(v_x^2+v_y^2)\,dx\,dy\ ,$$

where a_n is the set of admissible functions which are orthogonal to $\varphi_1,\varphi_2,\cdots,\varphi_n$. Let $g\leq M$ in R. Then we may define ρ_n by

(16) $$\rho_n=\sup_{v\in a_n}M\int_0^1\int_0^1 v^2\,dx\,dy\Big/\int_0^1\int_0^1(v_x^2+v_y^2)\,dx\,dy\ ,$$

and this gives

(17) $$\rho_n=\frac{M}{\pi^2\sigma_{n+1}^2}=O\left(\frac{1}{n}\right)$$

since the functions $\{\varphi_i\}$ are the extremal functions for the quotient in (16).

Next, the numbers a_{ij} satisfy

$$|a_{ij}|\leq\frac{C}{\sigma_i\sigma_j}\Delta_{ij}\bar{\Delta}_{ij}$$

where C is an absolute constant, and

$$\Delta_{ij}=\begin{cases}\dfrac{1}{|m_i-m_j|} & \text{if}\quad m_i\neq m_j\ ,\\[2ex] 1 & \text{if}\quad m_i=m_j\ ,\end{cases}$$

$$\bar{\Delta}_{ij}=\begin{cases}\dfrac{1}{|n_i-n_j|} & \text{if} \quad n_i\neq n_j\ ,\\ 1 & \text{if} \quad n_i=n_j\ .\end{cases}$$

Hence, for $1\leq i\leq n$,

$$\sum_{j=n+1}^{\infty} a_{ij}^2\leq\frac{C^2}{\sigma_i^2\sigma_{n+1}^2}\sum_{j=n+1}^{\infty}\Delta_{ij}^2\bar{\Delta}_{ij}^2\ ,$$

and

$$\sum_{j=n+1}^{\infty}\Delta_{ij}^2\bar{\Delta}_{ij}^2<\left(1+2\sum_{s=1}^{\infty}\frac{1}{s^2}\right)^2\ ,$$

so

$$\sum_{j=n+1}^{\infty} a_{ij}^2<\frac{C_1}{i(n+1)}\ .$$

Therefore,

$$\sum_{i=1}^{n}\sum_{j=n+1}^{\infty} a_{ij}^2<\frac{C_2\log n}{n} \qquad (n>1)\ ,$$

where C_1 and C_2 are absolute constants.

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