

Pacific Journal of Mathematics

**A PROPERTY OF DIFFERENTIAL FORMS IN THE CALCULUS
OF VARIATIONS**

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1. In the classical problems involving a simple integral

$$(1) \quad I_1 = \int L(t, q^i, \dot{q}^i) dt, \quad i=1, \dots, n,$$

one is led to the consideration of the Pfaffian form

$$(2) \quad \omega = L dt + \frac{\partial L}{\partial \dot{q}^i} \omega^i = \frac{\partial L}{\partial \dot{q}^i} dq^i - \left(\dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L \right) dt$$

where

$$\omega^i = dq^i - \dot{q}^i dt.$$

For example this form ω is the one which gives rise to the "relative integral invariant" of E. Cartan.

In a recent note [1] L. Auslander characterizes the form ω by a theorem equivalent to the following one.

THEOREM 1. *Among all semi-basic forms θ such that*

$$(3) \quad \theta \equiv L dt \pmod{\omega^i}$$

the form ω of (2) is the only one satisfying the condition

$$(4) \quad d\theta \equiv 0 \pmod{\omega^i}.$$

In this, a *semi-basic form* is a form for which the local expression contains only the differentials of t, q^i (not of \dot{q}^i). The integral I is defined over an arc \bar{c} of a space \mathscr{W} with local coordinates t, q^i, \dot{q}^i satisfying the equations $\omega^i = 0$: Therefore in (1) the form $L dt$ may be replaced by any θ satisfying (3).

Condition (4) is a special case of a congruence discovered by Lepage [5]. *The purpose of the present note is to give a natural reason for this congruence which goes beyond its nice algebraic expression.*

Let us observe that the space \mathscr{W} is the manifold of 1-dimensional contact elements of a manifold \mathscr{V} with local coordinates t, q^i . The map

$$(t, q^i, \dot{q}^i) \rightarrow (t, q^i)$$

is then the local expression of the natural projection $\pi: \mathscr{W} \rightarrow \mathscr{V}$. We

remark that we do not integrate (1) on any arc \bar{c} in \mathscr{W} satisfying $\omega^i=0$ but on such an arc the projection c of which in \mathscr{V} is regular.

2. Let U be the domain in \mathscr{V} of the coordinates t, q^i ; then the t, q^i, \dot{q}^i are defined in an open subset $W \subset \mathscr{W}$ of projection $\pi(W)=U$. If we denote by L_i n real undeterminates, we have coordinates t, q^i, \dot{q}^i, L_i in $W \times R^n$; we then define in this product the Pfaffian form

$$(5) \quad \Omega_W = L dt + L_i \omega^i .$$

Now, let us cover \mathscr{W} with open sets W, W', \dots ; this way we get a family of products $W \times R^n, W' \times R^n, \dots$ with forms $\Omega_W, \Omega_{W'}, \dots$. Using fibre bundle techniques, one proves that over a non-empty intersection $W \cap W'$ the products $W \times R^n$ and $W' \times R^n$ can be glued together in such a way that the forms induced on $W \cap W' \times R^n$ coincide. This yields a fibre bundle $E(\mathscr{W}, R^n)$ over \mathscr{W} as base, with fibre R^n . This bundle is covered by open subsets isomorphic with the products $W \times R^n$ and in which the t, q^i, \dot{q}^i, L_i are local coordinates; there is also on E a global Pfaffian form Ω of local expression (5). Combining the projections $E \rightarrow \mathscr{W}$ and $\mathscr{W} \rightarrow \mathscr{V}$ we obtain a map $E \rightarrow \mathscr{V}$ locally defined by

$$(t, q^i, \dot{q}^i, L_i) \rightarrow (t, q^i) .$$

We want to characterize in E the extremal arcs c^* of $\int \Omega$ which have a regular projection in \mathscr{V} .

An extremal arc c^* of $\int \Omega$ has to satisfy the local equations

$$\frac{\partial(d\Omega)}{\partial(dt)} = \frac{\partial(d\Omega)}{\partial(\omega^i)} = \frac{\partial(d\Omega)}{\partial(d\dot{q}^i)} = \frac{\partial(d\Omega)}{\partial(dL_i)} = 0 .$$

We have

$$d\Omega = \frac{\partial L}{\partial \dot{q}^i} \omega^i \wedge dt + \left(\frac{\partial L}{\partial \dot{q}^i} - L_i \right) d\dot{q}^i \wedge dt + dL_i \wedge \omega^i .$$

These equations are therefore

$$\omega^i = 0 , \quad \left(\frac{\partial L}{\partial \dot{q}^i} - L_i \right) dt = 0 , \quad \frac{\partial L}{\partial \dot{q}^i} dt - dL_i = 0 .$$

Since an arc c^* of regular projection in \mathscr{V} cannot satisfy simultaneously $\omega^i=0$ and $dt=0$ it has to lie in the submanifold F of E locally characterized by

$$\frac{\partial L}{\partial \dot{q}^i} = L_i$$

or equivalently by condition (4).

THEOREM 2. *Every arc c^* in E for which $\int \Omega$ is stationary and the projection of which in \mathcal{V} is regular necessarily lies in the submanifold F of E locally defined by the congruence (4). Furthermore the projection c of c^* in \mathcal{V} extremizes in the classical sense the integral (1). Finally if c is a regular extremal arc of (1) in \mathcal{V} let c^* be the arc of F the projection \bar{c} of which in \mathcal{W} is the arc of tangent directions to c ; then c^* extremizes $\int \Omega$.*

3. The submanifold F can be identified with \mathcal{W} in an obvious way so that \mathcal{W} can be considered as a submanifold of E . Then clearly Ω induces ω on \mathcal{W} .

THEOREM 3. *If the integral (1) is regular there exists a (one-to-one) correspondence between the regular extremal arcs c in \mathcal{V} of (1) and the extremal arcs \bar{c} of $\int \omega$ in \mathcal{W} which have a regular projection in \mathcal{V} . Starting from an extremal c , the corresponding \bar{c} is the arc the points of which are the tangent directions to c ; starting from \bar{c} the corresponding c is its projection in \mathcal{V} .*

In this statement, regularity of (1) means that the matrix $(\partial^2 L / \partial \dot{q}^i \partial \dot{q}^j)$ is everywhere non singular.

Theorem 2 and 3 give a complete justification of condition (4). Theorem 3 was actually proved by E. Cartan [2]. These theorems are special cases of similar theorems involving multiple integrals and even those in which the function L depends on higher order contact elements. Theorem 2 was first proved by the author [3], as well as the alluded generalizations.

Combining Theorems 2 and 3 yields the following.

THEOREM 4. *In the regular case, every arc \bar{c} in \mathcal{W} of regular projection in \mathcal{V} which extremizes $\int \omega$ with respect to variations confined to \mathcal{W} does also extremize $\int \Omega$ with respect to variations in the larger space E .*

4. There is a last question to be answered: why in Theorem 1 restrict oneself to semi-basic forms?

We can only add to $L \cdot dt$ a linear combination of Pfaffian forms vanishing with ω^i ; every such form is a linear combination of the ω^i

and is therefore semi-basic. Hence the restriction to semi-basic forms in Theorem 1 was actually redundant.

However, as mentioned above and as I have proved in various papers (e.g. [3, 4]), the above properties generalize to a multiple integral

$$(6) \quad I_p = \int L(t^\alpha, q^i, q_\alpha^i) dt ,$$

$$dt = dt^1 \wedge \dots \wedge dt^p, \quad \alpha = 1, 2, \dots, p; \quad i = 1, 2, \dots, n ,$$

to be integrated over a p -surface c defined by $q^i = q^i(t^\alpha)$ and where q_α^i stands for $\partial q^i / \partial t^\alpha$. Then \mathcal{V} is of dimension $n+p$ and \mathcal{W} (which is geometrically the manifold of p -dimensional contact elements of \mathcal{V}) is of dimension $n+p+np$. We can consider that we integrate (6) in \mathcal{W} over a p -surface \bar{c} of regular projection in \mathcal{V} and solution of the Pfaffian equations

$$\omega^i = dq^i - \sum q_\alpha^i dt^\alpha = 0 .$$

Such a p -surface \bar{c} is formed of the contact elements of dimension p to a regular p -surface in \mathcal{V} and will be called a p -multiplicity.

Now in (6) we can add to $L \cdot dt$ any p -form vanishing on all p -multiplicities and all such forms are no longer semi-basic if $p > 1$: for example $d\omega^i \wedge dt^3 \wedge \dots \wedge dt^p$ is such one. Nevertheless, the semi-basic forms satisfying the Lepage congruences [5]:

$$(7) \quad \theta \equiv L dt \quad \text{mod } \omega^i ,$$

$$(8) \quad d\theta \equiv 0 \quad \text{mod } \omega^i .$$

play an important role for a deeper reason which is actually a *transversality condition*. We briefly discuss this below referring the reader to my memoir [4] for further details.

5. Let \mathcal{H} be a p -dimensional manifold and K a domain of \mathcal{H} with regular boundary K . A map

$$c : K \rightarrow \mathcal{V}$$

is a domain of integration of (6); it gives rise canonically to a map

$$\bar{c} : K \rightarrow \mathcal{W}$$

such that for $k \in K$, $\bar{c}(k)$ is the contact element of dimension p to c at k . A *variation* (or *homotopy*) of c is a family of maps

$$c_t : K \rightarrow \mathcal{V}, \quad t \in R, \quad c_0 = c ;$$

this yields a variation of \bar{c} :

$$\bar{c}_t: K \rightarrow \mathscr{W}.$$

We also define $C: K \times R \rightarrow \mathscr{V}$, $\bar{C}: K \times R \rightarrow \mathscr{W}$ by

$$C(k, t) = c_t(k), \quad \bar{C}(k, t) = \bar{c}_t(k).$$

The corresponding variation of $\int \theta$ is then

$$\Delta = \int_{\bar{c}_t} \theta - \int_{\bar{c}_0} \theta$$

which may be expressed as a sum of two terms:

$$(9) \quad \Delta = \int_{\bar{c}_{0t}} d\theta + \int_{\lambda_{0t}\bar{c}} \theta.$$

The domains of integration \bar{c}_{0t} and $\lambda_{0t}\bar{c}$ are the restrictions of \bar{C} to $K \times I_{0t}$ and $\dot{K} \times I_{0t}$ respectively (where $I_{0t} = [0, t] \subset R$). We say that the variation \bar{C} is *transversal* to θ if this form vanishes on $\lambda\bar{C}$ (restriction of \bar{C} to $\dot{K} \times R$). This being the case, the last integral (or boundary term) in (9) is zero.

Now the variations usually considered are those for which the restriction of C to \dot{K} is constant (fixed boundary variations): for those, $\lambda\bar{C}$ has an everywhere non-regular projection in \mathscr{V} , so that every semi-basic form vanishes on $\lambda\bar{C}$. Therefore if we replace in (6) $L \cdot dt$ by a semi-basic p -form θ satisfying (7), all variations with fixed boundary are transversal to it. This would of course not be the case, should we add to $L \cdot dt$ a non-semi-basic p -form vanishing on all p -multiplicities.

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