

# Pacific Journal of Mathematics

**A NOTE ON ADDITIVE FUNCTIONS**

H. DELANGE AND HEINI HALBERSTAM

# A NOTE ON ADDITIVE FUNCTIONS

H. DELANGE AND H. HALBERSTAM

1. A real valued function  $f(n)$ , defined on the set of natural numbers, is called *additive* if  $f(mn) = f(m) + f(n)$  whenever  $(m, n) = 1$ , and *strongly additive* if also  $f(p^\alpha) = f(p)$  for  $p$  prime and  $\alpha = 2, 3, \dots$ . We define

$$(1) \quad A_n = \sum_{p < n} f(p)/p, \quad B_n = \sum_{p < n} f^2(p)/p,$$

and we assume throughout that

$$(2) \quad B_n \rightarrow \infty, \quad n \rightarrow \infty.$$

Additive functions for which  $B_n = O(1)$  have already been discussed thoroughly in Erdős and Wintner [4]. They proved the following theorem:

*Define*

$$f'(p) = \begin{cases} 1 & \text{for } |f(p)| > 1, \\ f(p) & \text{for } |f(p)| \leq 1. \end{cases}$$

*Then the additive function  $f(n)$  possesses a distribution function if, and only if, the series*

$$\sum_p f'(p)/p \quad \text{and} \quad \sum_p \{f'(p)\}^2/p$$

*converge.*

Moreover, it follows from a general result of P. Lévy [10] that this distribution function is continuous if, and only if, the series  $\sum_{f(p) \neq 0} f(p)/p$  diverges. Surveys of this subject are given in Kac [7] and Kubilyus [9]. A comprehensive account is being prepared by H. N. Shapiro.

Our knowledge of functions subject to (2) is not as complete. Outstanding is the result of Erdős and Kac [3] which states that if

$$(3) \quad f(p) = O(1),$$

the distribution of

$$\frac{f(m) - A_n}{B_n^{1/2}}, \quad m \leq n,$$

is asymptotically Gaussian. In a recent note H. N. Shapiro [11] has shown that the theorem of Erdős and Kac remains true even when (3) is replaced by

---

Received July 26, 1956 and in revised form April 11, 1957.

$$(4) \quad \lim_{n \rightarrow \infty} B_n^{-1} \sum_{\substack{p < n \\ |f(p)| > \varepsilon B_n^{1/2}}} f^2(p) / p = 0 \quad \text{for every } \varepsilon > 0 .$$

Since (4) is essentially the Lindeberg condition which is necessary and sufficient for the central limit theorem to hold, one is led to conjecture that (4) is not only the sufficient but also the necessary condition for the truth of the theorem of Erdős and Kac. However, it seems very difficult to establish the necessity (see Kubilyus [8] and Tanaka [12]).

Associated with such questions about the distributions of additive arithmetic functions is a number of ‘moment’ problems, which, if solved, lead to results of independent interest. Thus, for example, the following result is suggested by, and includes, the theorem of Erdős and Kac.

**THEOREM 1.** *Let  $f(m)$  be strongly additive and subject to (2) and*

$$(5) \quad f(p) = o(B_p^{1/2}) .$$

*Then we have for each fixed  $k=1, 2, 3, \dots$*

$$\lim_{n \rightarrow \infty} \frac{\sum_{m=1}^n (f(m) - A_n)^k}{n B_n^{k/2}} = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \omega^k e^{-\omega^2/2} d\omega .$$

(For proofs see Delange [1], [2], Halberstam [5], [6].)

The purpose of the present communication is to indicate briefly a proof that Theorem 1 remains true even when (5) is replaced by the weaker pair of conditions (4) and

$$(5a) \quad f(p) = O(B_p^{1/2}) .$$

That (5a) alone does not suffice can be seen readily from the case  $f(p) = \log p$ , which determines a very different kind of distribution. On the other hand, (4) alone would also be inadequate, as can be seen from the following example.

Let  $p_1, p_2, \dots, p_j, \dots$  be an increasing sequence of primes with the property that the number of primes which belong to this sequence and do not exceed  $x$  is  $o(\log \log x)$ . Now take

$$f(p) = \begin{cases} (p_j)^{1/2} & \text{if } p = p_j , \\ 1, & \text{if } p \text{ does not belong to the sequence.} \end{cases}$$

Then  $B_n \sim (\log \log n)$  and condition (4) is satisfied. However,

$$\sum_{m \leq p_j} (f(m) - A_{p_j})^4 \geq (f(p_j) - A_{p_j})^4 \sim p_j^2$$

whereas, if Theorem 1 were true in this case, we should have

$$\sum_{m \leq p_j} (f(m) - A_p)^4 \sim 3p_j (\log \log p_j)^2 .$$

The most general formulation of Theorem 1 remains an open question. The theorem shows, incidentally, that although the method of moments is in many ways more tractable for determining the distributions of given functions, it is not as wide in scope as the method evolved by Erdős and Kac.

2. We suppose throughout this section that (4) and (5a) hold. First of all, we rewrite (4) as

$$(6) \quad \lim_{n \rightarrow \infty} \phi(n, \varepsilon) = 0 \quad \text{for every } \varepsilon > 0 ,$$

where

$$(7) \quad \phi(n, \varepsilon) = B_n^{-1} \sum_{\substack{p < n \\ |f(p)| > \varepsilon B_n^{1/2}}} f^2(p) / p .$$

To simplify subsequent arithmetic we choose  $\varepsilon < 1/2$  and keep it fixed ; then we choose  $n$  so large that

$$(8) \quad \phi(n, \varepsilon) < \frac{1}{2} \varepsilon$$

as is possible by (6). We set

$$(9) \quad \alpha_n = n^{1/(3k)}$$

and observe that in view of (9) and the well-known relation

$$(10) \quad \sum_{p < y} p^{-1} = \log \log y + c + o(1)$$

where  $c$  is an absolute constant,<sup>1</sup>

$$(11) \quad \sum_{\alpha_n \leq p < n} p^{-1} = O(1) .$$

We define

$$(12) \quad A_y^* = \sum_{\substack{p < y \\ |f(p)| \leq \varepsilon B_n^{1/2}}} f(p) / p , \quad B_y^* = \sum_{\substack{p < y \\ |f(p)| \leq \varepsilon B_n^{1/2}}} f^2(p) / p$$

and

$$(13) \quad f^*(m) = \sum_{\substack{p < \alpha_n, p | m \\ |f(p)| \leq \varepsilon B_n^{1/2}}} f(p) .$$

By (7) and (12)

<sup>1</sup> The constants implied by the use of the  $O$ -notation depend throughout on at most  $k$ .

$$B_n^* = B_n(1 - \phi(n, \epsilon))$$

and this combines with (11) to give

$$(14) \quad B_{\alpha_n}^* = B_n(1 + O(\epsilon^2 + \phi(n, \epsilon))) .$$

LEMMA 1.  $A_n = A_{\alpha_n}^* + O(B_n^{1/2} \{ \epsilon + \epsilon^{-1} \phi(n, \epsilon) \}) .$

*Proof.* By (1)

$$A_n = \sum_{\substack{p < \alpha_n \\ |f(p)| \leq \epsilon B_n^{1/2}}} f(p)/p + \sum_{\substack{\alpha_n \leq p < n \\ |f(p)| \leq \epsilon B_n^{1/2}}} f(p)/p + \sum_{\substack{p < n \\ |f(p)| > \epsilon B_n^{1/2}}} f(p)/p .$$

The first sum on the right is  $A_{\alpha_n}^*$  by (12) with  $y = \alpha_n$ , the second sum is  $O(\epsilon B_n^{1/2})$  by (11), and the third is less than

$$\epsilon^{-1} B_n^{-1/2} \sum_{\substack{p < n \\ |f(p)| > \epsilon B_n^{1/2}}} f^2(p)/p = B_n^{1/2} \epsilon^{-1} \phi(n, \epsilon)$$

by (7). Hence the result.

LEMMA 2. *If  $r \leq k$ , then*

$$\sum_{m=1}^n (f(m) - f^*(m))^{2r} = O(n B_n^r \{ \epsilon + \epsilon^{-1} \phi(n, \epsilon) \}) .$$

*Proof.* By (13) and the definition of  $f(m)$

$$f(m) - f^*(m) = \sum_{\substack{p < n, p|m \\ |f(p)| > \epsilon B_n^{1/2}}} f(p) + \sum_{\substack{\alpha_n \leq p < n, p|m \\ |f(p)| \leq \epsilon B_n^{1/2}}} f(p) = \sum_{\substack{p|m \\ p \in \mathcal{E}_n}} f(p)$$

where  $\mathcal{E}_n$  is the set of those primes less than  $n$  which satisfy either

$$(i) \quad |f(p)| > \epsilon B_n^{1/2}$$

or

$$(ii) \quad |f(p)| \leq \epsilon B_n^{1/2}, \quad p \geq \alpha_n .$$

Then the sum of Lemma 2 is

$$\begin{aligned} & O\left( \sum_{\nu=1}^{2r} \sum_{\substack{r_1 + \dots + r_\nu = 2r \\ r_1 \geq \dots \geq r_\nu \geq 1}} \sum''_{p_1, \dots, p_\nu} |f^{r_1}(p_1) \cdots f^{r_\nu}(p_\nu)| \sum_{\substack{m=1 \\ (p_1 \cdots p_\nu) | m}}^n 1 \right) \\ & = O\left( \sum_{\nu=1}^{2r} \{ \max_{p \leq n} |f(p)|^{2r-\nu} \} \sum''_{p_1, \dots, p_\nu} \left[ \frac{n}{p_1 \cdots p_\nu} \right] |f(p_1) \cdots f(p_\nu)| \right) \end{aligned}$$

where  $\sum''$  indicates that the summation is carried out over all sets of distinct prime numbers  $p_1, p_2, \dots, p_\nu$  with  $p_i \in \mathcal{E}$  ( $i=1, 2, \dots, \nu$ ), and  $[y]$  stands for the integer part of  $y$ . Using (5a), (i) and (ii) this expression is

$$O\left(n \sum_{\nu=1}^{2r} B_n^{r-\frac{1}{2}\nu} \sum_{s=0}^{\nu} \left\{ \sum_{\substack{p < n \\ |f(p)| > \varepsilon B_n^{1/2}}} |f(p)|/p \right\}^s \left\{ \sum_{\substack{\alpha_n \leq p < n \\ |f(p)| \leq \varepsilon B_n^{1/2}}} |f(p)|/p \right\}^{\nu-s} \right),$$

which, as in the proof of Lemma 1, becomes

$$\begin{aligned} &O\left(n \sum_{\nu=1}^{2r} B_n^{r-\frac{1}{2}\nu} \sum_{s=0}^{\nu} \{B_n^{1/2}(\varepsilon^{-1}\phi)\}^s \{B_n^{1/2}\varepsilon\}^{\nu-s}\right) = O\left(n B_n^r \sum_{\nu=1}^{2r} \sum_{s=0}^{\nu} (\varepsilon^{-1}\phi)^s \varepsilon^{\nu-s}\right) \\ &= O(n B_n^r \{\varepsilon^{-1}\phi + \varepsilon\}) ; \end{aligned}$$

here we have used the restrictions on the magnitudes of  $\varepsilon$  and  $\phi$  imposed at the beginning of § 2 (see inequality (8)).

Next we set

$$M_k(n) = \sum_{m=1}^n (f(m) - A_n)^k, \quad M_r^*(n) = \sum_{m=1}^n (f^*(m) - A_{\alpha_n}^*)^r.$$

Then

$$M_k(n) = \sum_{m=1}^n \{(A_{\alpha_n}^* - A_n) + (f(m) - f^*(m)) + (f^*(m) - A_{\alpha_n}^*)\}^k,$$

so that by Lemmas 1 and 2 and Cauchy's inequality

$$\begin{aligned} &M_k(n) - M_k^*(n) \\ &= O\left(\sum_{\substack{r_1+r_2+r_3=k \\ r_3 \leq k-1}} |A_n - A_{\alpha_n}^*|^{r_1} \sum_{m=1}^n |f(m) - f^*(m)|^{r_2} |f^*(m) - A_{\alpha_n}^*|^{r_3}\right) \\ &= O\left(\sum_{\substack{r_1+r_2+r_3=k \\ r_3 \leq k-1}} B_n^{r_1/2} \{\varepsilon + \varepsilon^{-1}\phi\}^{r_1} \left\{ \sum_{m=1}^n (f(m) - f^*(m))^{2r_2} \right\}^{1/2} \{M_{2r_3}^*(n)\}^{1/2}\right) \\ &= O\left(n^{1/2} \sum_{r \leq k-1} B_n^{(k-r)/2} \{\varepsilon + \varepsilon^{-1}\phi\}^{1/2} \{M_{2r}^*(n)\}^{1/2}\right). \end{aligned}$$

But by the methods of Halberstam [5] or Delange [2] it is a straightforward matter to confirm that for  $n$  sufficiently large

$$M_l^*(n) = n(B_{\alpha_n}^*)^{l/2} (2\pi)^{-1/2} \int_{-\infty}^{\infty} \omega^l e^{-\omega^2/2} d\omega \{1 + O(\varepsilon)\}, \quad l \leq 2k,$$

so that by (14) and (8)

$$(15) \quad M_l^*(n) = n B_n^{l/2} (2\pi)^{-1/2} \int_{-\infty}^{\infty} \omega^l e^{-\omega^2/2} d\omega \{1 + O(\varepsilon)\}, \quad l \leq 2k,$$

and, in particular

$$M_{2r}^*(n) = O(n B_n^r), \quad r \leq k.$$

Hence

$$M_k(n) - M_k^*(n) = O(nB_n^{k/2} \{\varepsilon + \varepsilon^{-1}\phi\}^{1/2});$$

now, whilst still keeping  $\varepsilon$  fixed, we let  $n$  tend to infinity, and obtain

$$\overline{\lim}_{n \rightarrow \infty} \left| \frac{M_k(n)}{nB_n^{k/2}} - \frac{M_k^*(n)}{nB_n^{k/2}} \right| = O(\varepsilon^{1/2}).$$

Thus, by (15) with  $l=k$ ,

$$\overline{\lim}_{n \rightarrow \infty} \left| \frac{M_k(n)}{nB_n^{k/2}} - (2\pi)^{-1/2} \int_{-\infty}^{\infty} \omega^k e^{-\omega^2/2} d\omega \right| = O(\varepsilon^{1/2}).$$

Since the left side is entirely independent of  $\varepsilon$ , and yet the relation is true for every  $\varepsilon < 1/2$ , we have now proved that

$$\lim_{n \rightarrow \infty} \frac{M_k(n)}{nB_n^{k/2}} = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \omega^k e^{-\omega^2/2} d\omega$$

for every fixed  $k=1, 2, 3, \dots$ .

This concludes the proof of Theorem 1 with condition (5) replaced by the pair of conditions (5a) and (4).

#### REFERENCES

1. H. Delange, *Sur le nombre des diviseurs premiers de  $n$* , C. R. Acad. Sci. (Paris), **237** (1953), 542-544.
2. ———, *Sur un théorème d'Erdős et Kac*, Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 130-144.
3. P. Erdős and M. Kac, *The Gaussian law of errors in the theory of additive number-theoretic functions*, Amer. J. Math., **62** (1940), 738-742.
4. P. Erdős and A. Wintner, *Additive arithmetical functions and statistical independence*, Amer. J. Math., **61** (1939), 713-721.
5. H. Halberstam, *On the distribution of additive number-theoretic functions*, J. London Math. Soc., **30** (1955), 43-53.
6. ———, *On the distribution of additive number-theoretic functions, II*, **31** (1956), 1-14.
7. M. Kac, *Probability methods in some problems of analysis and number theory*, Bull. Amer. Math. Soc., **55** (1949), 641-665.
8. I. P. Kubilyus, *On the distribution of values of additive arithmetic functions*, Dokl. Akad. Nauk. U.S.S.R., **100** (1955), 623-626.
9. ———, *Probabilistic methods in the theory of numbers*, Uspechy Mat. Nauk. U.S.S.R., XI, 2 (**68**), (1956), 31-66.
10. P. Lévy, *Sur les séries dont les termes sont des variables éventuelles indépendantes*, Stud. Math., **3** (1931), 119-155.
11. H. N. Shapiro, *Distribution functions of additive arithmetic functions*, Proc. Nat. Acad. Sci. U.S.A., **42** (1956), 426-430.
12. M. Tanaka, *On the number of prime factors of integers*, Japan. J. Math., **25** (1955), 1-20.

UNIVERSITÉ DE CLERMONT, FRANCE

BROWN UNIVERSITY, PROVIDENCE, R. I. and UNIVERSITY OF CALIFORNIA, BERKELEY

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

H. L. ROYDEN  
Stanford University  
Stanford, California

R. A. BEAUMONT  
University of Washington  
Seattle 5, Washington

A. L. WHITEMAN  
University of Southern California  
Los Angeles 7, California

E. G. STRAUS  
University of California  
Los Angeles 24, California

## ASSOCIATE EDITORS

E. F. BECKENBACH  
C. E. BURGESS  
M. HALL  
E. HEWITT

A. HORN  
V. GANAPATHY IYER  
R. D. JAMES  
M. S. KNEBELMAN

L. NACHBIN  
I. NIVEN  
T. G. OSTROM  
M. M. SCHIFFER

G. SZEKERES  
F. WOLF  
K. YOSIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
OREGON STATE COLLEGE  
UNIVERSITY OF OREGON  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF UTAH  
WASHINGTON STATE COLLEGE  
UNIVERSITY OF WASHINGTON  
\* \* \*

AMERICAN MATHEMATICAL SOCIETY  
CALIFORNIA RESEARCH CORPORATION  
HUGHES AIRCRAFT COMPANY  
THE RAMO-WOOLDRIDGE CORPORATION

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors. All other communications to the editors should be addressed to the managing editor, E. G. Straus at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 10, 1-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.



# Pacific Journal of Mathematics

Vol. 7, No. 4

April, 1957

Robert Geroge Buschman, <i>A substitution theorem for the Laplace transformation and its generalization to transformations with symmetric kernel</i> .....	1529
S. D. Conte, <i>Numerical solution of vibration problems in two space variables</i> .....	1535
Paul Dedecker, <i>A property of differential forms in the calculus of variations</i> .....	1545
H. Delange and Heini Halberstam, <i>A note on additive functions</i> .....	1551
Jerald L. Ericksen, <i>Characteristic direction for equations of motion of non-Newtonian fluids</i> .....	1557
Avner Friedman, <i>On two theorems of Phragmén-Lindelöf for linear elliptic and parabolic differential equations of the second order</i> .....	1563
Ronald Kay Gettoor, <i>Additive functionals of a Markov process</i> .....	1577
U. C. Guha, <i><math>(\gamma, k)</math>-summability of series</i> .....	1593
Alvin Hausner, <i>The tauberian theorem for group algebras of vector-valued functions</i> .....	1603
Lester J. Heider, <i>T-sets and abstract (L)-spaces</i> .....	1611
Melvin Henriksen, <i>Some remarks on a paper of Aronszajn and Panitchpakdi</i> .....	1619
H. M. Lieberstein, <i>On the generalized radiation problem of A. Weinstein</i> .....	1623
Robert Osserman, <i>On the inequality <math>\Delta u \geq f(u)</math></i> .....	1641
Calvin R. Putnam, <i>On semi-normal operators</i> .....	1649
Binyamin Schwarz, <i>Bounds for the principal frequency of the non-homogeneous membrane and for the generalized Dirichlet integral</i> .....	1653
Edward Silverman, <i>Morrey's representation theorem for surfaces in metric spaces</i> .....	1677
V. N. Singh, <i>Certain generalized hypergeometric identities of the Rogers-Ramanujan type. II</i> .....	1691
R. J. Smith, <i>A determinant in continuous rings</i> .....	1701
Drury William Wall, <i>Sub-quasigroups of finite quasigroups</i> .....	1711
Sadayuki Yamamuro, <i>Monotone completeness of normed semi-ordered linear spaces</i> .....	1715
C. T. Rajagopal, <i>Simplified proofs of "Some Tauberian theorems" of Jakimovski: Addendum and corrigendum</i> .....	1727
N. Aronszajn and Prom Panitchpakdi, <i>Correction to: "Extension of uniformly continuous transformations in hyperconvex metric spaces"</i> .....	1729
Alfred Huber, <i>Correction to: "The reflection principle for polyharmonic functions"</i> .....	1731