

# Pacific Journal of Mathematics

**ADDITIVE FUNCTIONALS OF A MARKOV PROCESS**

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# ADDITIVE FUNCTIONALS OF A MARKOV PROCESS

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1. **Introduction.** We are concerned with functionals of the form  $L = \int_0^t V[x(\tau)]d\tau$  where  $x(t)$  is a temporally homogeneous Markov process in a locally compact Hausdorff space,  $X$ , and  $V$  is a non-negative measurable function on  $X$ . In studying the distribution of this functional various authors (e.g. [1], [3], and [7]) have considered the following function

$$(1.1) \quad r(t, x, A) = E\{e^{-uL} | x(0) = x; x(t) \in A\} p(t, x, A)$$

where  $p(t, x, A)$  is the transition probability function of  $x(t)$ . If one can determine  $r$  then one can in essence determine the distribution of  $L$  since ( $u > 0$ )

$$r(t, x, A) = \int_0^\infty e^{-u\lambda} d_\lambda P[L \leq \lambda | x(0) = x; x(t) \in A] \quad p(t, x, A).$$

Formally it is quite easy to see that if  $p$  satisfies an equation of diffusion type

$$(1.2) \quad \frac{\partial p}{\partial t} = \Omega p$$

that  $r$  should satisfy the equation

$$(1.3) \quad \frac{\partial r}{\partial t} = (\Omega - uV)r.$$

If  $x(t)$  is the Wiener process in  $E^N$  and  $V$  satisfies a Lipschitz condition of order  $\alpha > 0$  Rosenblatt [12] has given a rigorous derivation of (1.3). In this paper we use the theory of semi-groups to give a meaning to (1.3) for a wide class of processes without assuming any smoothness conditions on  $V$ . Rosenblatt's result does not follow from ours since our results only imply that  $r$  is a "weak" solution of (1.3). However, for many applications (e.g. [10]) this is all that is really required.

Because of certain difficulties connected with the definition of the conditional expectation in (1.1) we define  $r$  directly and prove that if  $p(t, x, A) > 0$  then  $\frac{r(t, x, A)}{p(t, x, A)}$  is the appropriate conditional expectation. Since we intend to apply analytic methods it is necessary to investigate the dependence of  $r$  on its various variables. This is done in § 2.

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Received May 8, 1957.

Beginning in § 3 we assume that  $p(t, x, A)$  has a density  $f(t, x, y)$  with respect to a Radon measure  $m$  and we show (§§ 4 and 5) that if  $U_t\varphi(x) = \int \varphi(y)p(t, x, dy)$  has infinitesimal generator  $\Omega$  on  $L_2(m)$  then  $T_t\varphi(x) = \int \varphi(y)r(t, x, dy)$  has infinitesimal generator  $\Omega - uV$  if  $V$  is bounded, subject to certain regularity conditions on  $f$ . If  $V$  is unbounded our results are less complete and are contained in Theorem 5.2. In the sequel we will suppress the parameter  $u$ .

We use throughout this paper the function space approach to stochastic processes. We also make use of certain elementary facts about integration in locally compact spaces. The reader is referred to [2], [4], and [5] for the basic facts required. In a future paper we plan to study the spectral properties of the operators defined here. In that paper  $X$  will be an open subset of an  $N$  dimensional Euclidean space.

I would like to thank Dr. R. M. Blumenthal for several enlightening discussions during the course of this research.

**2. A class of integrals over a function space.** Let  $X$  be a locally compact Hausdorff space and  $\mathfrak{B}(X)$  the Borel sets of  $X$ ; that is, the smallest  $\sigma$ -algebra of subsets of  $X$  containing the compact sets of  $X$ . Let  $\mathfrak{X}$  be the set of all functions from  $[0 \leq t < \infty]$  to  $X$  which are right continuous; that is,  $x(t) \rightarrow x(t_0)$  if  $t \downarrow t_0$ . Let  $p(t, x, A)$  be a transition probability function defined for  $t > 0$ ,  $x \in X$ , and  $A \in \mathfrak{B}(X)$ , such that given an arbitrary probability measure  $\mu$  on  $\mathfrak{B}(X)$  there exists a Markov process  $x_\mu(t)$  with paths which are right continuous and which has  $\mu$  as its initial distribution and  $p(t, x, A)$  as its transition probability. In other words, if  $\mathfrak{B}(\mathfrak{X})$  is the  $\sigma$ -algebra of subsets of  $\mathfrak{X}$  generated by sets of the form

$$A = \{x(\cdot) | x(t_j) \in A_j; j=0, 1, \dots, n; A_j \in \mathfrak{B}(X); 0 = t_0 < t_1 < \dots < t_n\}$$

then there exists a countably additive probability measure,  $P_\mu$ , on  $\mathfrak{B}(\mathfrak{X})$  such that

$$(2.1) \quad P_\mu(A) = \int_{A_0} \int_{A_1} \dots \int_{A_n} \mu(dx_0)p(t_1, x_0, dx_1)p(t_2 - t_1, x_1, dx_2) \cdot p(t_n - t_{n-1}, x_{n-1}, dx_n).$$

If  $\mu$  assigns mass one to a single point,  $x$ , we write  $P_x$  for  $P_\mu$ .

We assume that

$(P_t)p(\cdot, \cdot, A)$  is jointly measurable<sup>1</sup> in  $(t, x)$  for each  $A \in \mathfrak{B}(X)$ . We also pick a fixed  $\mu$ , and  $x(t)$  will always denote the processes having  $\mu$  as

<sup>1</sup> Measurability conditions in  $t$  are understood to be with respect to the ordinary Borel sets of  $[0 \leq t < \infty]$ .

its initial distribution. Clearly (once we have established Theorem 2.1)

$$(2.2) \quad P[A|x(0)=x]=P_x(A) , \quad (A \in \mathfrak{B}(\mathfrak{X})) .$$

If  $A \in \mathfrak{B}(X)$  we define

$$(2.3) \quad A_t = \{x(\cdot)|x(\cdot) \in \mathfrak{X}; x(t) \in A\} \in \mathfrak{B}(\mathfrak{X}) .$$

If  $A \in \mathfrak{B}(\mathfrak{X})$  and  $A \in \mathfrak{B}(X)$  we define for  $t > 0$

$$(2.4) \quad P(A; x; t, A) = P_x[A \cap A_t] .$$

It is evident that  $P(\cdot; x; t, A)$  is a finite measure on  $\mathfrak{B}(\mathfrak{X})$  for fixed  $x, t, A$  and that  $P(A; x; t, \cdot)$  is a finite measure on  $\mathfrak{B}(X)$  for fixed  $A, x, t$ . It is easy to see that if  $t$  and  $A$  are such that  $p(t, x, A) > 0$  for all  $x$ , then (again assuming Theorem 2.1)

$$(2.5) \quad P[A|x(0)=x; x(t) \in A] = \frac{P[A; x; t, A]}{p(t, x, A)} .$$

**THEOREM 2.1.**  $P[A; \cdot; \cdot, A]$  is a measurable function of  $(t, x)$  for fixed  $A, A$ .

*Proof.* Let  $A$  be fixed and suppose

$$A = \{x(\cdot) | x(t_j) \in A_j; j=1, \dots, n\}$$

then  $P[A; x; t, A] = P_x[A \cap A_t]$  which is measurable in  $(t, x)$  in view of (2.1) and  $(P_t)$ . Hence  $P[A; x; t, A]$  is measurable in  $(t, x)$  for  $A$ 's which are finite disjoint unions of sets of the above form. But the measurability of  $P[A; x; t, A]$  is preserved under monotone limits of  $A$ 's and hence  $P[A; x; t, A]$  is measurable for all  $A \in \mathfrak{B}(\mathfrak{X})$ . See [8].

The following lemmas will be of use in the sequel.

**LEMMA 2.1.** Let  $(Y, \mathfrak{G})$  and  $(Z, \mathfrak{H})$  be measurable spaces and let  $m(A, B)$  be defined for  $A \in \mathfrak{G}$  and  $B \in \mathfrak{H}$ . Suppose that  $m(\cdot, B)$  is a measure on  $(Y, \mathfrak{G})$  for each fixed  $B \in \mathfrak{H}$  and that  $m(A, \cdot)$  is a measure on  $(Z, \mathfrak{H})$  for each fixed  $A \in \mathfrak{G}$ . Let  $f \geq 0$  be a measurable function on  $(Y, \mathfrak{G})$  then

$$(2.6) \quad q(B) = \int f(y)m(dy, B)$$

is a measure on  $(Z, \mathfrak{H})$ .

*Proof.* The only thing that requires proof is that  $q$  is countably additive. Let  $\{f_n\}$  be a sequence of simple functions such that  $f_n \geq 0$  and  $f_n \uparrow f$ . Clearly

$$q_n(B) = \int f_n(y)m(dy, B)$$

are measures and  $q_n(B) \uparrow q(B)$  for each  $B \in \mathfrak{G}$ . Let  $B = \bigcup_{j=1}^{\infty} B_j$  where the  $B_j$ 's are disjoint. Put  $B^{(k)} = \bigcup_{j=1}^k B_j$ , then  $\lim_n \lim_k q_n(B^{(k)}) = q(B)$ . Since  $q_n(B^{(k)})$  is increasing in both  $n$  and  $k$  we can interchange the limits obtaining

$$q(B) = \lim_k \lim_n q_n(B^{(k)}) = \lim_k q(B^{(k)}) = \sum_{j=1}^{\infty} q(B_j) .$$

LEMMA 2.2. Let  $(Y, \mathfrak{G})$  be a measurable space and let  $f(t, y)$  be an  $X$  valued function defined for  $t \geq 0$  and  $y \in Y$ . If  $f(\cdot, y)$  is right continuous for each  $y \in Y$  and  $f(t, \cdot)$  is  $\mathfrak{G}$ -measurable for each  $t$  then  $f(t, y)$  is jointly  $\mathfrak{B} \times \mathfrak{G}$  measurable. ( $\mathfrak{B}$  is the  $\sigma$ -algebra of ordinary Borel sets.)

*Proof.* Define  $g_n(t, y) = f((j+1)/n, y)$  if  $j/n < t \leq (j+1)/n$  for  $j = 0, 1, 2, \dots$  and  $n = 1, 2, \dots$ . Let  $B \in \mathfrak{B}(X)$  and define  $G_{jn} = f((j+1)/n, \cdot)^{-1}(B)$ , then since  $f(t, \cdot)$  is  $\mathfrak{G}$ -measurable  $G_{jn} \in \mathfrak{G}$ . Let  $A_{jn} = \{t | j/n < t \leq (j+1)/n\} \in \mathfrak{B}$ , then

$$g_n^{-1}(B) = \bigcup_{j=0}^{\infty} A_{jn} \times G_{jn}$$

which is in  $\mathfrak{B} \times \mathfrak{G}$ . Hence  $g_n$  is jointly  $\mathfrak{B} \times \mathfrak{G}$  measurable for each  $n$ , but  $g_n(t, x) \rightarrow f(t, x)$  as  $n \rightarrow \infty$  and thus  $f$  is  $\mathfrak{B} \times \mathfrak{G}$  measurable.

If  $\Phi[x(\cdot)]$  is a complex valued measurable<sup>2</sup> functional on  $\mathfrak{X}$  we write  $r[\Phi; t, x, A]$  for the integral of  $\Phi$  over  $\mathfrak{X}$  with respect to the measure  $P[\cdot; x; t, A]$ , provided the integral exists.

THEOREM 2.2. If  $\Phi \geq 0$  is a measurable functional on  $\mathfrak{X}$  then  $r[\Phi; t, x, A]$  is a measure on  $\mathfrak{B}(X)$  for fixed  $(t, x)$  and is measurable in  $(t, x)$  for fixed  $A$ .

*Proof.* This is an immediate consequence of Lemma 2.1 and Theorem 2.1.

Let  $\varphi$  be a complex valued measurable function on  $X$ , then for each  $t > 0$  we define a measurable functional,  $\varphi_t$ , on  $\mathfrak{X}$  as follows:  $\varphi_t[x(\cdot)] = \varphi[x(t)]$ . Also if  $\Phi$  is a measurable functional on  $\mathfrak{X}$  we denote its integral over  $\mathfrak{X}$  with respect to the measure  $P_x$  by  $E\{\Phi[x(\cdot)] | x(0) = x\}$ .

THEOREM 2.3. Let  $\Phi \geq 0$  be a measurable functional on  $\mathfrak{X}$  and  $\varphi$  a complex valued measurable function on  $X$ ; then

$$(2.7) \quad \int \varphi(y) r[\Phi; t, x, dy] = E\{\Phi \cdot \varphi_t | x(0) = x\} ,$$

provided either integral exists.

<sup>2</sup> Measurability of real or complex valued functions always means Borel measurability.

*Proof.* Suppose  $\varphi = I_A$  where  $I_A$  denotes the characteristic function  $A$ , then the left hand side of (2.7) is  $r[\Phi; t, x, A]$ . Now if  $\Phi = I_A$  then

$$r[I_A; t, x, A] = P[A; x; t, A].$$

But

$$(I_A)_t[x(\cdot)] = I_A[x(t)] = I_{A_t}[x(\cdot)],$$

where  $A_t = \{x(\cdot) | x(t) \in A\}$ . Thus

$$E\{I_A \cdot (I_A)_t | x(0) = x\} = P_x[A \cap A_t] = P[A; x; t, A].$$

Let  $\Phi_n$  be a sequence of simple functionals such that  $\Phi_n \uparrow \Phi$ , then  $\Phi_n \cdot (I_A)_t$  is a sequence of simple functionals increasing to  $\Phi \cdot (I_A)_t$ . Therefore

$$E\{\Phi_n \cdot (I_A)_t | x(0) = x\} \uparrow E\{\Phi \cdot (I_A)_t | x(0) = x\}.$$

On the other hand  $r(\Phi_n; t, x, A) \uparrow r(\Phi; t, x, A)$  by the monotone convergence theorem and since

$$E\{\Phi_n \cdot (I_A)_t | x(0) = x\} = r[\Phi_n; t, x, A]$$

it follows that if either of the integrals in (2.7) is finite the other is also and they are equal in the case  $\varphi = I_A$ .

If  $\varphi \geq 0$  let  $\varphi_n$  be a sequence of simple functions increasing to  $\varphi$  then if either of the integrals in (2.7) exists we have equality for each  $\varphi_n$  and by monotone convergence for  $\varphi$ . The result for a general  $\varphi$  now follows in the usual manner.

For each  $t \geq 0$  let  $x_t(\tau) = x(t + \tau)$  for all  $\tau \geq 0$ , then we define a map,  $S_t$ , from  $\mathfrak{X}$  into  $\mathfrak{X}$  by  $S_t x(\cdot) = x_t(\cdot)$ . Clearly  $S_t$  is a measurable transformation of  $\mathfrak{X}$  into  $\mathfrak{X}$ . If  $\Phi$  is a measurable functional we define  $S_t \Phi[x(\cdot)] = \Phi[S_t x(\cdot)]$ .

**THEOREM 2.4.** *Let  $\Phi$  be a functional measurable with respect to  $\mathfrak{B}_t$  and  $\Psi$  be measurable with respect to  $\mathfrak{B}_s$  such that  $0 \leq \Phi \leq M$  and  $0 \leq \Psi \leq M$ , then*

$$(2.8) \quad \int r[\Phi; t, x, dy] r[\Psi; s, y, A] = r[\Phi \cdot S_t \Psi; t + s, x, A].$$

*Proof.* Since  $\Phi$  and  $\Psi$  are non-negative and bounded it is clear that the integral in question exists. If  $\Phi = I_F$  and  $\Psi = I_G$  with  $F \in \mathfrak{B}_t$  and  $G \in \mathfrak{B}_s$  then

$$S_t I_G[x(\cdot)] = I_G[S_t x(\cdot)] = I_{S_t^{-1}G},$$

thus to prove (2.8) for  $I_F$  and  $I_G$  we must show that

<sup>3</sup>  $\mathfrak{B}[t_1, t_2]$  denotes the  $\sigma$ -algebra of subsets of  $\mathfrak{X}$  generated by sets of the form  $\{x(\cdot) | x(\tau_j) \in A_j; t_1 \leq \tau_j \leq t_2\}$ , and  $\mathfrak{B}_t = \mathfrak{B}[0, t]$ .

$$(2.9) \quad \int P[F; x; t, dy]P[G; y; s, A] = P[F \cap S_t^{-1}G; x; t+s, A] .$$

We first consider the case in which

$$F = \{x(\cdot) | x(t_j) \in A_j; j=1, \dots, n; t_j < t\}$$

$$G = \{x(\cdot) | x(t'_k) \in B_k; k=1, \dots, m; t'_k < s\} .$$

In this case

$$S_t^{-1}G = \{x(\cdot) | x(t+t'_k) \in B_k; k=1, 2, \dots, m\} ,$$

thus

$$\begin{aligned} & \int P[F; x; t, dy]P[G; y; s, A] \\ &= \int \int_{A_1} \dots \int_{A_n} p(t_1, x, dx_1) p(t_2-t_1, x_1, dx_2) \dots p(t-t_n, x_n, dy) \\ & \quad \cdot \int_{B_1} \dots \int_{B_m} p(t'_1, y, dy_1) \dots p(s-t'_m, y_m, A) \\ &= \int_{A_1} \dots \int_{A_n} \int_{B_1} \dots \int_{B_m} p(t_1, x, dx_1) \dots p(t+t'_1-t_n, x_n, dy_1) \\ & \quad \cdot p(t+t'_2-(t+t'_1), y_1, dy_2) \dots p(t+s-(t+t'_m), y_m, A) \\ &= P[F \cap S_t^{-1}G; x; t+s, A] . \end{aligned}$$

If  $t=t_n$ , or  $s=t'_m$ , or both, it is necessary to make only minor changes in the above argument.

This equality clearly extends to finite disjoint unions of such  $F$ 's and  $G$ 's and since  $S_t^{-1}$  is a  $\sigma$ -homomorphism it extends to monotone limits of such  $G$ 's. Thus (2.9) holds for each  $F$  in the algebra of sets generated by sets of the given form and for each  $G \in \mathfrak{B}_s$ . For fixed  $G \in \mathfrak{B}_s$ , the left hand side of (2.9) is a measure in  $F$  by Lemma 2.1, hence (2.9) holds under monotone limits of such  $F$ 's and thus finally (2.9) holds for all  $F$  and  $G$  in the appropriate  $\sigma$ -algebras.

Let  $\Phi_n$  and  $\Psi_n$  be sequences of simple functionals increasing to  $\Phi$  and  $\Psi$ , then by monotone convergence

$$\int r[\Phi_n; t, x, dy]r[\Psi; s, y, A] = r[\Phi_n \cdot S_t \Psi; t+s, x, A] .$$

Applying an argument similar to that used in the proof of Lemma 2.1 the equality (2.8) results. (This also follows from Theorem 2.3.)

We conclude this section with the following theorem which is easily proved using standard approximation techniques.

**THEOREM 2.5.** *Let  $\Phi(t, x(\cdot)) \geq 0$  be jointly measurable in  $t$  and  $x(\cdot)$  then  $r[\Phi(t, x(\cdot)); t, x, A]$  is jointly measurable in  $(t, x)$ .*

**3. Additive functionals.** For each pair  $(t_1, t_2)$  with  $0 \leq t_1 < t_2$  let  $L[t_1, t_2; x(\cdot)]$  be a functional ( $L$  may be  $+\infty$ ) on  $\mathfrak{X}$  which is measurable with respect to  $\mathfrak{B}[t_1, t_2]$  and which is jointly measurable in  $t_1, t_2$ , and  $x(\cdot)$ . We further assume that for  $t_1 < t < t_2$  and each  $x(\cdot) \in \mathfrak{X}$  we have

$$(3.1) \quad L[t_1, t_2; x(\cdot)] = L[t_1, t; x(\cdot)] + L[t, t_2; x(\cdot)];$$

and that

$$(3.2) \quad S_t L[t_1, t_2; x(\cdot)] = L[t_1 + t, t_2 + t; x(\cdot)].$$

Such a functional will be called an additive functional on  $\mathfrak{X}$  (See [1]).

**THEOREM 3.1.** *Let  $V \geq 0$  be a measurable function on  $X$ , then*

$$L[t_1, t_2; x(\cdot)] = \int_{t_1}^{t_2} V[x(\tau)] d\tau$$

*is an additive functional on  $\mathfrak{X}$ .*

*Proof.* Define  $F(t, x(\cdot)) = x(t)$  then  $F$  is measurable in  $x(\cdot)$  for fixed  $t$  and right continuous in  $t$  for fixed  $x(\cdot)$ . Thus by Lemma 2.2  $F$  is jointly measurable in  $t$  and  $x(\cdot)$ . Since  $V[x(t)] = V[F(t, x(\cdot))]$  is the composition of measurable transformations  $V[x(\tau)]$  is jointly measurable in  $\tau$  and  $x(\cdot)$ , and therefore (a simple argument using Lemma 2.2 shows that)  $\int_{t_1}^{t_2} V[x(\tau)] d\tau$  is jointly measurable in  $t_1, t_2$ , and  $x(\cdot)$ . The other properties that  $L$  must satisfy are obvious.

We suppose that  $L[t_1, t_2; x(\cdot)] \geq -M$  where  $M > 0$  is independent of  $t_1, t_2$ , and  $x(\cdot)$ . We define

$$(3.3) \quad r(t, x, A) = r[e^{-L[0, t; x(\cdot)]}; t, x, A].$$

Theorems 2.2 and 2.5 imply that  $r(t, x, A)$  is a measure on  $\mathfrak{B}(X)$  for fixed  $(t, x)$  and is jointly measurable in  $(t, x)$  for fixed  $A \in \mathfrak{B}(X)$ . Moreover the fact that

$$(3.4) \quad 0 \leq r(t, x, A) \leq e^M p(t, x, A)$$

is a simple consequence of our definitions.

**THEOREM 3.2.**  $r(t+s, x, A) = \int r(t, x, dy) r(s, y, A)$ .

*Proof.* This is a corollary of Theorem 2.4 once we observe that

$$S_t e^{-L[0, s; x(\cdot)]} = e^{-L[0, s; S_t x(\cdot)]} = e^{-S_t L[0, s; x(\cdot)]} = e^{-L[t, t+s; x(\cdot)]},$$

and therefore



$$e^{-L[0,t;x(\cdot)]} \cdot S_t e^{-L[0,s;x(\cdot)]} = e^{-L[0,t+s;x(\cdot)]} .$$

At this point we assume that there exists a Radon measure,  $m$ , on  $\mathfrak{B}(X)$  such that  $p(t, x, A)$  has a density  $f(t, x, y) \geq 0$  with respect to  $m$  for  $t > 0$ ; that is

$$(3.5) \quad p(t, x, A) = \int_A f(t, x, y) m(dy) , \quad t > 0 .$$

We assume that  $f$  is jointly measurable in  $t, x$ , and  $y$ , but we do not assume that  $m$  is finite. We introduce the following conditions on  $f(t, x, y)$ :

(P<sub>2</sub>)  $\int f(t, x, y) m(dx) \leq k e^{\alpha t}$  where  $k$  and  $\alpha$  are positive constants independent of  $y$  and  $t$ .

(P<sub>3</sub>) Given  $\varepsilon > 0$  and a compact set  $A \subset X$  there exists a compact set  $B$  such that

$$\int_{x \notin B} f(t, x, y) m(dx) < \varepsilon \text{ for } y \in A \text{ and } t \leq 1 .$$

We define operators on appropriate function spaces as follows:

$$(3.6) \quad (T_t \varphi)(x) = \int \varphi(y) r(t, x, dy)$$

$$(3.7) \quad (U_t \varphi)(x) = \int \varphi(y) p(t, x, dy) = \int \varphi(y) f(t, x, y) m(dy) .$$

**THEOREM 3.3.** *If  $f(t, x, y)$  satisfies (P<sub>2</sub>) then  $\{T_t; t > 0\}$  and  $\{U_t; t > 0\}$  are semi-groups of bounded operators on  $L_2(m)$ .*

*Note.* All Borel sets are  $m$ -measurable [4; 5].

*Proof.* From (3.4) we obtain

$$\begin{aligned} |T_t \varphi(x)| &\leq \int |\varphi(y)| r(t, x, dy) \\ &\leq e^{\alpha t} \int |\varphi(y)| p(t, x, dy) = e^{\alpha t} U_t |\varphi|(x) \end{aligned}$$

and thus it will suffice to prove that  $U_t$  is a bounded operator on  $L_2(m)$  for each  $t > 0$ . But

$$\begin{aligned} |U_t \varphi(x)|^2 &= \left| \int f(t, x, y) \varphi(y) m(dy) \right|^2 \\ &\leq \int f(t, x, y) |\varphi(y)|^2 m(dy) , \end{aligned}$$

and therefore

$$\int |U_t \varphi(x)|^2 m(dx) \leq \int m(dx) \int f(t, x, y) |\varphi(y)|^2 m(dy) \leq ke^{at} \cdot \|\varphi\|^2 .$$

Thus  $\|U_t\|^2 \leq ke^{at}$  and  $\|T_t\|^2 \leq ke^{2M+at}$ . The fact that  $\{T_t; t > 0\}$  and  $\{U_t; t > 0\}$  are semi-groups now follows from Theorem 3.2 and the fact that  $p(t, x, A)$  satisfies the Chapman-Kolmogorov equation.

**THEOREM 3.4.** *If  $f(t, x, y)$  satisfies  $(P_2)$  and  $(P_3)$  and  $\lim_{t \rightarrow 0} L[0, t; x(\cdot)] = 0$  for all  $x(\cdot) \in \mathfrak{X}$  then the semi-groups  $\{U_t; t > 0\}$  and  $\{T_t; t > 0\}$  are strongly continuous<sup>4</sup> on  $L_2(m)$ .*

*Proof.* We prove the theorem for  $\{T_t; t > 0\}$  the results for  $\{U_t; t > 0\}$  being a special case (take  $L \equiv 0$ ). We must show that  $\|T_t \varphi - \varphi\| \rightarrow 0$  as  $t \rightarrow 0$  for all  $\varphi \in L_2(m)$ . Since  $\|T_t\|$  is uniformly bounded for  $t \leq 1$  it will be sufficient to show that  $\|T_t \varphi - \varphi\| \rightarrow 0$  as  $t \rightarrow 0$  for  $\varphi$  continuous with compact support, such functions being dense in  $L_2(m)$  since  $m$  is a Radon measure, [2]. We first show that  $T_t \varphi(x) \rightarrow \varphi(x)$  pointwise as  $t \rightarrow 0$  if  $\varphi$  is continuous with compact support. According to Theorem 2.3

$$T_t \varphi(x) = \int \varphi(y) r(t, x, dy) = E\{e^{-L[0, t; x(\cdot)]} \cdot \varphi(x(t)) | x(0) = x\} .$$

Using the right continuity of  $x(\cdot)$  and our assumption on  $L$  we see that

$$e^{-L[0, t; x(\cdot)]} \varphi[x(t)] \rightarrow \varphi[x(0)]$$

boundedly as  $t \downarrow 0$  and hence by the bounded convergence theorem

$$T_t \varphi(x) \rightarrow E\{\varphi[x(0)] | x(0) = x\} = \varphi(x) \quad \text{as } t \downarrow 0 .$$

Let  $A$  be the support of  $\varphi$ , then if  $B$  is compact and  $B \supset A$  we have

$$\begin{aligned} \|T_t \varphi - \varphi\|^2 &= \int_B |T_t \varphi(x) - \varphi(x)|^2 m(dx) + \int_{x \notin B} |T_t \varphi(x) - \varphi(x)|^2 m(dx) \\ &= I_1 + I_2 . \end{aligned}$$

But

$$|T_t \varphi(x)| \leq \int |\varphi(y)| r(t, x, dy) \leq \sup_{x \in A} |\varphi(x)| \cdot e^M ,$$

hence  $I_1 \rightarrow 0$  since  $B$  is compact. Now since  $B \supset A$  we have

$$I_2 \leq \int_{x \notin B} |T_t \varphi(x)|^2 m(dx) \leq e^{2M} \int_A |\varphi(y)|^2 \int_{x \notin B} f(t, x, y) m(dy) m(dx) ,$$

<sup>4</sup> By the strong continuity of a semi-group  $\{T_t; t > 0\}$  we will always mean strong continuity for  $t \geq 0$  where  $T_0$  is the identity.

and so, if  $B$  is chosen properly, using  $(P_3)$ , we see that  $I_2$  is small. This completes the proof of the fact that  $\{T_t; t > 0\}$  is strongly continuous on  $L_2(m)$ .

**4. The Darling-Siebert equations.** In [3] Darling and Siebert showed that  $r(t, x, A)$  has to satisfy two integral equations. We give a derivation of these equations based on the material of § 2. We assume that  $p(t, x, A)$  satisfies  $(P_1)$  and that

$$L[t_1, t_2; x(\cdot)] = \int_{t_1}^{t_2} V[x(\tau)]d\tau$$

where  $V$  is a bounded, non-negative, measurable function on  $X$ . The formal outline of the derivation given below is exactly that of Darling and Siebert.

We begin with the following identities which are easily verified ( $f$  measurable, non-negative, and bounded)

$$(4.1) \quad \exp \left[ - \int_0^t f(\tau)d\tau \right] = 1 - \int_0^t f(s) \exp \left[ - \int_s^t f(\tau)d\tau \right] ds$$

$$(4.2) \quad \exp \left[ - \int_0^t f(\tau)d\tau \right] = 1 - \int_0^t f(s) \exp \left[ - \int_0^s f(\tau)d\tau \right] ds .$$

Also using Theorem 2.4 we have

$$\begin{aligned} (4.3) \quad & r \left[ V[x(s)] \exp \left( - \int_s^t V[x(\tau)]d\tau \right); t, x, A \right] \\ & = r \left[ V[x(s)] \cdot S_s \exp \left( - \int_0^{t-s} V[x(\tau)]d\tau \right); (t-s) + s, x, A \right] \\ & = \int r[V[x(s)]; s, x, dy] r \left[ \exp \left( - \int_0^{t-s} V[x(\tau)]d\tau \right); t-s, y, A \right] \\ & = \int V(y)p(s, x, dy)r(t-s, y, A) \end{aligned}$$

provided we show that

$$(4.4) \quad \int f(y)r[V[x(s)]; s, x, dy] = \int f(y)V(y)p(s, x, dy)$$

for measurable, bounded  $f \geq 0$ . Suppose  $f = I_A$  and  $V = I_B$  then

$$\begin{aligned} \int f(y)r[V[x(s)]; s, x, dy] & = P[I_B[x(s)]; x; s, A] \\ & = P_x[B_s \cap A_s] = P(s, x, A \cap B) = \int f(y)V(y)p(s, x, dy) . \end{aligned}$$

The standard approximation technique now yields the desired result (4.4).

Putting  $f(\tau)=V[x(\tau)]$  in (4.1) and applying (4.3) we obtain (the interchange in the order of integration is valid since

$$V[x(s)] \cdot \exp\left(-\int_s^t V[x(\tau)] d\tau\right)$$

is bounded and jointly measurable in  $s$  and  $x(\cdot)$ )

$$(4.5) \quad r(t, x, A) = p(t, x, A) - \int_0^t ds \int V(y) r(t-s, y, A) p(s, x, dy) .$$

In a similar manner using (4.2) we find

$$(4.6) \quad r(t, x, A) = p(t, x, A) - \int_0^t ds \int V(y) p(t-s, y, A) r(s, x, dy);$$

and these are the Darling-Siebert equations. In deriving (4.6) one needs the relation

$$(4.7) \quad r[V[x(0)]; t, y, A] = V(y) p(t, y, A)$$

which is obtained in much the same manner as (4.4).

Taking Laplace transforms in (4.5) and (4.6) yields (the necessary interchange of order of integration is again justified since the integrand is bounded and jointly measurable in its variables)

$$(4.8) \quad \hat{r}(\lambda, x, A) = p(\lambda, x, A) - \int V(y) \hat{r}(\lambda, y, A) \hat{p}(\lambda, x, dy)$$

$$(4.9) \quad \hat{r}(\lambda, x, A) = \hat{p}(\lambda, x, A) - \int V(y) \hat{p}(\lambda, y, A) \hat{r}(\lambda, x, dy)$$

where  $\hat{r}$  and  $\hat{p}$  are the Laplace transforms of  $r$  and  $p$ .

**5. The infinitesimal generators.** Let  $\Omega$  and  $\Omega'$  be the infinitesimal generators of  $\{U_t; t > 0\}$  and  $\{T_t; t > 0\}$  respectively. We assume in this section that  $(P_1)$ ,  $(P_2)$ , and  $(P_3)$  are satisfied. It then follows, since the semi-groups involved are strongly continuous on  $L_2(m)$ , that  $\Omega$  and  $\Omega'$  are closed densely defined operators on  $L_2(m)$ . See [6] and [9].

We assume that

$$(5.1) \quad L[t_1, t_2; x(\cdot)] = \int_{t_1}^{t_2} V[x(\tau)] d\tau$$

where  $V$  is a non-negative measurable function on  $X$ . Note that in this case  $M=0$ .

**THEOREM 5.1.** *If  $V$  is bounded then  $\Omega' = \Omega - V$ .*

*Proof.* Let  $J_\lambda$  be the resolvent of  $\{T_t; t > 0\}$  then for  $\lambda > \alpha$  we have

$$J_\lambda \varphi(x) = \int_0^\infty e^{-\lambda t} T_t \varphi(x) dt,$$

and thus

$$|J_\lambda \varphi(x)|^2 \leq \frac{1}{\lambda} \int_0^\infty e^{-\lambda t} |T_t \varphi(x)|^2 dt.$$

Applying the Fubini theorem we see that  $J_\lambda \varphi(x)$  exists for almost all  $x(\lambda > \alpha)$  and is in  $L_2(m)$ , moreover for  $\lambda > \alpha$  we have

$$(5.2) \quad \|J_\lambda\|^2 \leq \frac{k}{\lambda(\lambda - \alpha)}.$$

In view of the above facts we can write

$$(5.3) \quad J_\lambda \varphi(x) = \int \varphi(y) \hat{r}(\lambda, x, dy).$$

From the general theory of semi-groups, [6] and [9], we know that for  $\lambda > \alpha$  the range of  $J_\lambda$  is independent of  $\lambda$  and is, in fact, the domain of  $\Omega'$ , which we denote by  $D_{\Omega'}$ . In addition it is known that

$$(5.4) \quad (\lambda - \Omega') J_\lambda \varphi = \varphi \quad \text{for all } \varphi \in L_2(m);$$

$$(5.5) \quad J_\lambda(\lambda - \Omega') \varphi = \varphi \quad \text{for all } \varphi \in D_{\Omega'}.$$

Let  $I_\lambda$  be the resolvent of  $\{U_t; t > 0\}$  and then in a similar manner we have

$$(5.7) \quad I_\lambda \varphi(x) = \int \varphi(y) \hat{p}(\lambda, x, dy) = \int \varphi(y) \hat{f}(\lambda, x, y) m(dy).$$

From (4.8) we see that

$$\begin{aligned} J_\lambda \varphi(x) &= I_\lambda \varphi(x) - \int \varphi(z) \int V(y) \hat{r}(\lambda, y, dz) \hat{p}(\lambda, x, dy) \\ &= I_\lambda \varphi(x) - I_\lambda [V \cdot J_\lambda \varphi](x) \\ &= I_\lambda [\varphi - V \cdot J_\lambda \varphi](x). \end{aligned}$$

The above steps are justified since  $V \cdot J_\lambda \varphi \in L_2(m)$  under our assumption that  $V$  is bounded. Thus  $D_{\Omega'} \subset D_\Omega$  and conversely using (4.9)  $D_\Omega \subset D_{\Omega'}$ , that is,  $D_\Omega = D_{\Omega'}$ . Now

$$\begin{aligned} (\lambda - \Omega) J_\lambda \varphi &= (\lambda - \Omega) I_\lambda [\varphi - V J_\lambda \varphi] \\ &= \varphi - V J_\lambda \varphi, \end{aligned}$$

or equivalently,

$$[\lambda - (\Omega - V)] J_\lambda \varphi = \varphi \quad \text{for all } \varphi \in L_2.$$

Thus  $\Omega - V$  is an extension of  $\Omega'$ , but since  $V$  is bounded the domain of  $\Omega - V$  is  $D_\Omega = D_{\Omega'}$ . Hence  $\Omega' = \Omega - V$ .

**COROLLARY.** *If  $V$  is bounded and  $f(t, x, y) = f(t, y, x)$  then  $\Omega$  and  $\Omega'$  are self-adjoint operators.*

*Proof.* Since  $f(t, x, y) = f(t, y, x)$  each  $U_t$  is a bounded self-adjoint operator and hence  $\Omega$  is also self-adjoint, although not necessarily bounded. The boundedness of  $V$  implies that  $V$  considered as an operator on  $L_2(m)$  is bounded and self-adjoint, therefore  $\Omega - V$  is self-adjoint, [11]. Thus  $\Omega' = \Omega - V$  is a self-adjoint operator which in turn implies that each  $T_t$  is a bounded self-adjoint operator.

If  $V$  is not bounded our results are much less complete ( $V$  is no longer a bounded operator on  $L_2(m)$  and one runs into the usual "domain problems"). It is natural to try to approximate  $V$  by bounded functions and then use a limiting procedure. Accordingly we define

$$(5.8) \quad V_N(x) = \begin{cases} V(x) & \text{if } V(x) \leq N, \\ N & \text{if } V(x) \geq N \end{cases}$$

and it is evident that each  $V_N$  is measurable and bounded. Let

$$D_V = \{ \varphi | \varphi \in L_2(m); V \cdot \varphi \in L_2(m) \};$$

that is,  $D_V$  is the domain of  $V$  considered as an operator on  $L_2(m)$ . We are, of course, assuming that  $f(t, x, y)$  satisfies  $(P_1)$ ,  $(P_2)$ , and  $(P_3)$ .

**THEOREM 5.2.** *If  $V$  is non-negative and measurable then  $D_\Omega \cap D_V \subset D_{\Omega'}$  and if  $\varphi \in D_\Omega \cap D_V$  then  $\Omega' \varphi = (\Omega - V)\varphi$ .*

*Proof.* We define

$$r_N(t, x, A) = r \left[ e^{-\int_0^t V_N(x(\tau)) d\tau}; t, x, A \right]$$

and

$$T_t^{(N)} \varphi(x) = \int \varphi(y) r_N(t, x, dy).$$

For each  $N$  we know that  $\{T_t^{(N)}; t > 0\}$  is a strongly continuous semi-group of bounded operators on  $L_2(m)$  whose infinitesimal generator is  $\Omega - V_N$ . Since  $V_N \uparrow V$  we have by monotone convergence that

$$(5.9) \quad r_N(t, x, A) \downarrow r(t, x, A).$$

We first show that for each  $t > 0$  and all  $\varphi \in L_2(m)$

$$(5.10) \quad \|T_t^{(N)} \varphi - T_t \varphi\| \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Since  $\|T_t^{(N)}\| \leq ke^{\alpha t}$  it will suffice to prove (5.10) for  $\varphi$  continuous with compact support. Let  $\mu_N(A) = r_N(t, x, A) - r(t, x, A) \geq 0$ , then  $\mu_N(A) \downarrow 0$  for each fixed  $A$  and is a measure on  $\mathfrak{B}(X)$  for each fixed  $N$ . It is clear that

$$|T_t^{(N)}\varphi(x) - T_t\varphi(x)| \leq \int |\varphi(y)|\mu_N(dy).$$

Let  $\varphi_j$  be a sequence of simple functions decreasing to  $|\varphi|$ , then since  $\int \varphi_j(y)\mu_N(dy)$  is decreasing in both  $N$  and  $j$  we can interchange the limits obtaining  $|T_t^{(N)}\varphi(x) - T_t\varphi(x)| \rightarrow 0$  pointwise as  $N \rightarrow \infty$  at least if  $\varphi$  is continuous with compact support. If the support of  $\varphi$  is  $A$  then (5.10) follows exactly as in the proof of Theorem 3.4 since

$$\int_{x \notin B} |T_t^{(N)}\varphi(x)|^2 m(dx) \leq \int_A |\varphi(y)|^2 \int_{x \notin B} f(t, x, y) m(dx) m(dy)$$

for compact  $B$ . Thus (5.10) is established.

We prove next that  $D_\Omega \cap D_V \subset D_{\Omega'}$ . Let  $J_\lambda^{(N)}$  and  $J_\lambda$  be the resolvents of  $\{T_t^{(N)}; t > 0\}$  and  $\{T_t; t > 0\}$  respectively. Since  $\|T_t^{(N)}\| \leq ke^{\alpha t}$  and  $T_t^{(N)}\varphi \rightarrow T_t\varphi$  it follows that  $J_\lambda^{(N)}\varphi \rightarrow J_\lambda\varphi$  for each  $\varphi \in L_2(m)$  and  $\lambda > \alpha$ . Choose a  $\lambda > \alpha$  and let it be fixed for the remainder of the present proof. If  $\varphi \in D_\Omega \cap D_V$  then  $\varphi \in D_{\Omega - V_N}$  for each  $N$ , hence there exist  $\psi_N \in L_2(m)$  such that  $\varphi = J_\lambda^{(N)}\psi_N$ . Moreover  $[\lambda - (\Omega - V_N)]\varphi = \psi_N$  or  $\psi_N = \lambda\varphi - \Omega\varphi + V_N\varphi$ . Clearly  $V_N\varphi \rightarrow V\varphi$  pointwise and since  $|V_N\varphi| \leq |V\varphi|$  it follows that  $\|V_N\varphi - V\varphi\| \rightarrow 0$ . Thus  $\psi_N \rightarrow \lambda\varphi - \Omega\varphi + V\varphi = \psi$  as  $N \rightarrow \infty$  in  $L_2(m)$ . But

$$\|J_\lambda^{(N)}\psi_N - J_\lambda\psi\| \leq \|J_\lambda^{(N)}\psi_N - J_\lambda^{(N)}\psi\| + \|J_\lambda^{(N)}\psi - J_\lambda\psi\|$$

and therefore  $J_\lambda^{(N)}\psi_N \rightarrow J_\lambda\psi$  as  $N \rightarrow \infty$  since  $\|J_\lambda^{(N)}\|$  is uniformly bounded in  $N$ . However,  $\varphi = J_\lambda^{(N)}\psi_N$  for all  $N$  and hence  $\varphi = J_\lambda\psi$  which implies that  $\varphi \in D_{\Omega'}$ .

Since  $\varphi = J_\lambda\psi$  where  $\psi = \lambda\varphi - \Omega\varphi + V\varphi$  we see that  $(\lambda - \Omega')\varphi = \psi = \lambda\varphi - \Omega\varphi + V\varphi$  or equivalently that  $\Omega'\varphi = (\Omega - V)\varphi$  for  $\varphi \in D_\Omega \cap D_V$ . This completes the proof of Theorem 5.2.

**COROLLARY.** *If  $\Omega$  is self-adjoint (that is,  $f(t, x, y) = f(t, y, x)$ ) then  $\Omega'$  is self-adjoint. Let  $E_N(\lambda)$  denote the spectral resolution of  $\Omega - V_N$  and  $E(\lambda)$  the spectral resolution of  $\Omega'$ , then  $E_N(\lambda)\varphi \rightarrow E(\lambda)\varphi$  for all  $\varphi \in L_2(m)$  provided that  $\lambda$  is a continuity point of  $E(\lambda)$ .*

*Proof.* We use the same notation as in the proof of Theorem 5.2. From the corollary to Theorem 5.1 it follows that each  $T_t^{(N)}$  is self-adjoint and  $T_t$  being the strong limit of self-adjoint operators is self-adjoint

for each  $t > 0$ . Hence the infinitesimal generator,  $\Omega'$ , of  $\{T_t; t > 0\}$  is self-adjoint. The strong continuity of  $\{T_t; t > 0\}$  implies that  $T_t\varphi = 0$  if and only if  $\varphi = 0$ . A similar statement holds for  $T_t^{(N)}$ . Under these circumstances  $E_N^{(\lambda)} = F_N(e^\lambda)$  and  $E(\lambda) = F(e^\lambda)$  where  $F_N$  and  $F$  are the spectral resolutions of  $T_1^{(N)}$  and  $T_1$  respectively. See [11]. Thus if we show that  $F_N(\lambda)\varphi \rightarrow F(\lambda)\varphi$  at all continuity points of  $F$  we will have proved the corollary. Since  $T_1^{(N)}\varphi \rightarrow T_1\varphi$  this follows from a theorem of Rellich (See [11], p. 366).

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The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 10, 1-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

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# Pacific Journal of Mathematics

Vol. 7, No. 4

April, 1957

Robert Geroge Buschman, <i>A substitution theorem for the Laplace transformation and its generalization to transformations with symmetric kernel</i> .....	1529
S. D. Conte, <i>Numerical solution of vibration problems in two space variables</i> .....	1535
Paul Dedecker, <i>A property of differential forms in the calculus of variations</i> .....	1545
H. Delange and Heini Halberstam, <i>A note on additive functions</i> .....	1551
Jerald L. Ericksen, <i>Characteristic direction for equations of motion of non-Newtonian fluids</i> .....	1557
Avner Friedman, <i>On two theorems of Phragmén-Lindelöf for linear elliptic and parabolic differential equations of the second order</i> .....	1563
Ronald Kay Gettoor, <i>Additive functionals of a Markov process</i> .....	1577
U. C. Guha, <i><math>(\gamma, k)</math>-summability of series</i> .....	1593
Alvin Hausner, <i>The tauberian theorem for group algebras of vector-valued functions</i> .....	1603
Lester J. Heider, <i>T-sets and abstract (L)-spaces</i> .....	1611
Melvin Henriksen, <i>Some remarks on a paper of Aronszajn and Panitchpakdi</i> .....	1619
H. M. Lieberstein, <i>On the generalized radiation problem of A. Weinstein</i> ...	1623
Robert Osserman, <i>On the inequality <math>\Delta u \geq f(u)</math></i> .....	1641
Calvin R. Putnam, <i>On semi-normal operators</i> .....	1649
Binyamin Schwarz, <i>Bounds for the principal frequency of the non-homogeneous membrane and for the generalized Dirichlet integral</i> .....	1653
Edward Silverman, <i>Morrey's representation theorem for surfaces in metric spaces</i> .....	1677
V. N. Singh, <i>Certain generalized hypergeometric identities of the Rogers-Ramanujan type. II</i> .....	1691
R. J. Smith, <i>A determinant in continuous rings</i> .....	1701
Drury William Wall, <i>Sub-quasigroups of finite quasigroups</i> .....	1711
Sadayuki Yamamuro, <i>Monotone completeness of normed semi-ordered linear spaces</i> .....	1715
C. T. Rajagopal, <i>Simplified proofs of "Some Tauberian theorems" of Jakimovski: Addendum and corrigendum</i> .....	1727
N. Aronszajn and Prom Panitchpakdi, <i>Correction to: "Extension of uniformly continuous transformations in hyperconvex metric spaces"</i> .....	1729
Alfred Huber, <i>Correction to: "The reflection principle for polyharmonic functions"</i> .....	1731