

Pacific Journal of Mathematics

**SOME REMARKS ON A PAPER OF ARONSZAJN AND
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In the paper of the title [1], a number of problems are posed. Negative solutions of two of them (Problems 2 and 3) are derived in a straightforward way from a paper of L. Gillman and the present author [2].

Motivation will not be supplied since it is given amply in [1], but enough definitions are given to keep the presentation reasonably self-contained.

1. A Hausdorff space X is said to satisfy (Q_m) , where m is an infinite cardinal, if, whenever U and V are disjoint open subsets of X such that each is a union of the closures of less than m open subsets of X , then U and V have disjoint closures. In particular, a normal (Hausdorff) space X satisfies (Q_{\aleph_1}) if and only if disjoint open F_σ -subsets of X have disjoint closures. (For, an open set that is the union of less than \aleph_1 closed sets is a fortiori an F_σ . Conversely if U is the union of countably many closed subsets F_n , then since X is normal, for each n there is an open set U_n containing F_n whose closure is contained in U . Thus U is the union of the closures of the open sets U_n .) In Problem 3 of [1], it is asked if every compact (Hausdorff) space satisfying (Q_m) for some $m > \aleph_0$ is necessarily totally disconnected, and it is remarked that this is the case if the first axiom of countability is also assumed.

If X is a completely regular space, let $C(X)$ denote the ring of all continuous real-valued functions on X , and let $Z(f) = \{x \in X : f(x) = 0\}$, let $P(f) = \{x \in X : f(x) > 0\}$, and let $N(f) = P(-f)$. As usual, let βX denote the Stone-Ćech compactification of X . If every finitely generated ideal of $C(X)$ is a principal ideal, then X is called an F -space. The following are equivalent.

- (i) X is an F -space.
- (ii) If $f \in C(X)$, then $P(f)$ and $N(f)$ are completely separated [2, Theorem 2.3].
- (iii) If $f \in C(X)$, then every bounded $g \in C(X - Z(f))$ has an extension $\bar{g} \in C(X)$ [2, Theorem 2.6].

A good supply of compact F -spaces is provided by the fact that if X is locally compact and σ -compact, then $\beta X - X$ is an F -space [2, Theorem 2.7].

Received April 1, 1957. In revised form April 29, 1957. This paper was written while the author was an Alfred P. Sloan fellow.

We remark first that a normal (Hausdorff) space X satisfies (Q_{\aleph_1}) if and only if it is an F -space.

For, suppose first that X is an F -space, and let U, V be disjoint open F_σ -subsets of X . Since $X - (U \cup V)$ is a closed G_δ in a normal space, there is a bounded $f \in C(X)$ such that $Z(f) = X - (U \cup V)$. Hence by (iii), there is a $\bar{g} \in C(X)$ such that $\bar{g}[U] = 0$ and $\bar{g}[V] = 1$. In particular, U and V have disjoint closures, so X satisfies (Q_{\aleph_1}) . Conversely let X satisfy (Q_{\aleph_1}) , and let $f \in C(X)$. Then $P(f)$ and $N(f)$ are disjoint open F_σ -subsets of X , which by (Q_{\aleph_1}) have disjoint closures. So, by Urysohn's lemma, $P(f)$ and $N(f)$ are completely separated. Thus X is an F -space by (ii).

Compact connected F -spaces exist. In particular it is known that if R^+ denotes the space of nonnegative real numbers, then $\beta R^+ - R^+$ is such a space [2, Example 2.8]. Hence Problem 3 of [1] has a negative solution.

We remark finally that if the first axiom of countability holds at a point of an F -space, then the point is isolated [2, Corollary 2.4]. In particular, every compact F -space satisfying the first axiom of countability is finite.

2. In Problem 2 of [1], it is asked (in different but equivalent language) if for every totally disconnected compact space X satisfying (Q_m) for some $m > \aleph_0$, the Boolean algebra $B(X)$ of open and closed subsets of X has the property that every subset of less than m elements has a least upper bound. A lattice is said to be (conditionally) σ -complete if every bounded countable subset has a least upper bound and a greatest lower bound. In view of the above (and since every subset of $B(X)$ is bounded), in case $m = \aleph_1$, the problem asks if for every compact totally disconnected F -space X , the Boolean algebra $B(X)$ is σ -complete.

In [3, Theorem 18], it is shown that if X is compact and totally disconnected, then $B(X)$ is σ -complete if and only if $C(X)$ is σ -complete (as a lattice). It is noted in [2, Theorem 8.3, f.f.] that for a completely regular space Y , the lattice $C(Y)$ is σ -complete if and only if $f \in C(Y)$ implies $\bar{P}(f)$ and $\bar{N}(f)$ are disjoint open and closed subsets of Y ($\bar{P}(f)$ denotes the closure of $P(f)$). It is easily seen that Y has this latter property if and only if βY has [2, Lemma 1.6].

In [2, Example 8.10], an example is given of a completely regular space X such that βX is a totally disconnected F -space, and such that $C(X)$ is not σ -complete. By the above, it follows that $B(\beta X)$ yields a negative solution to Problem 2.

We remark also (as was pointed out by J. R. Isbell) that if N denotes the countable discrete space, then $\beta N - N$ is also a totally disconnected compact F -space such that $B(\beta N - N)$ is not σ -complete. The

former assertion follows easily from the remarks in § 1, and the latter follows from the fact that $B(\beta N - N)$ is isomorphic to the Boolean algebra of all subsets of N modulo the ideal of finite subsets of N (under the correspondence induced by sending a subset of N to the intersection of its closure in βN with $\beta N - N$). It is easily verified that this latter Boolean algebra is not σ -complete.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 10, 1-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

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Lester J. Heider, <i>T-sets and abstract (L)-spaces</i>	1611
Melvin Henriksen, <i>Some remarks on a paper of Aronszajn and Panitchpakdi</i>	1619
H. M. Lieberstein, <i>On the generalized radiation problem of A. Weinstein</i>	1623
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