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A SUBSTITUTION THEOREM FOR THE LAPLACE TRANSFORMATION AND ITS GENERALIZATION TO TRANSFORMATIONS WITH SYMMETRIC KERNEL

R. G. BUSCHMAN

In the problem of the derivation of images of functions under the Laplace transformation, the question arises as to the type of image produced if t is replaced by $g(t)$ in the original. Specific examples have been given by Erdélyi [3, vol. I §§ 4.1, 5.1, 6.1], Doetsch [1, 75–80], McLachlan, Humbert, and Poli [6, pp. 11–13] and [7, pp. 15–20], and Labin [5, p. 41] and a general formula is also listed by Doetsch [1, 75–80].

The Laplace transformation will be taken as

$$f(s)=\int_0^\infty e^{-st}F(t)\,dt$$

in which the integral is taken in the Lebesgue sense and which, as suggested by Doetsch [2, vol. I, p. 44], will be denoted by

$$F(t)\underset{t\quad s}{\overset{\mathscr{L}}{\circ\!-\!\bullet}}f(s)\,.$$

(The symbol will be read " $F(t)$ has a Laplace transform $f(s)$ ".)

THEOREM 1. *If*

(i) *k, g, and the inverse function $h=g^{-1}$ are single-valued analytic functins, real on $(0, \infty)$, and such that $g(0)=0$ and $g(\infty)=\infty$ (or $g(0)=\infty$ and $g(\infty)=0$)*;

(ii) *$F(t)\underset{t\quad s}{\overset{\mathscr{L}}{\circ\!-\!\bullet}}f(s)$ and this Laplace integral converges for $0<\Re s$*;

(iii) *there exists a function $\Phi(s, u)$, $\Phi(s, u)\underset{u\quad p}{\overset{\mathscr{L}}{\circ\!-\!\bullet}}\phi(s, p)$ and this Laplace integral converges for $0<\Re p$, and $\phi(s, p)=e^{-sh(p)}k[h(p)]\,|h'(p)|$; and*

(iv) *$\int_0^\infty\left[\int_0^\infty|e^{-up}\Phi(s, u)F(p)|\,du\right]dp$ converges for $a<\Re s$*;

then

$$k(t)\,F[g(t)]\underset{t\quad s}{\overset{\mathscr{L}}{\circ\!-\!\bullet}}\int_0^\infty\Phi(s, u)f(u)\,du$$

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and this Laplace integral converges for $a < \Re s$.

Proof. From (iii) and (iv) it follows that

$$\int_0^\infty e^{-sh(p)} k[h(p)]\,|h'(p)|\,F(p)dp$$

is absolutely convergent for $a < \Re s$. There are two cases to be considered. Since from (i) both g and h are single valued, h is monotonic.

Case 1. If $g(0) = 0$ and $g(\infty) = \infty$, then $0 \leqq h'(p)$.

Case 2. If $g(0) = \infty$ and $g(\infty) = 0$, then $h'(p) \leqq 0$.
In either case, therefore, if the substitution $t = h(p)$ is made in the integral

$$\int_0^\infty e^{-st} k(t) F[g(t)]dt\,,$$

then

$$k(t)\,F[g(t)] \overset{\mathscr{L}}{\underset{t\quad s}{\circ\!-\!\bullet}} \int_0^\infty e^{-sh(p)} k[h(p)]\,|h'(p)|\,F(p)\,dp\,.$$

From (iii) $\Phi(s, u)$ can be introduced and by (iv) the order of integration changed so that

$$k(t)\,F[g(t)] \overset{\mathscr{L}}{\underset{t\quad s}{\circ\!-\!\bullet}} \int_0^\infty \left[\int_0^\infty e^{-up} F(p)\,dp\right] \Phi(s, u)\,du\,.$$

Finally, from (ii)

$$k(t)\,F[g(t)] \overset{\mathscr{L}}{\underset{t\quad s}{\circ\!-\!\bullet}} \int_0^\infty \Phi(s, u) f(u)\,du\,.$$

To show that there are functions $\phi(s, p)$ as assumed in (iii), let, for example, $g(t) = t^2$ and $k(t) = 1$ so that

$$\Phi(s, p) = (4p)^{-1/2} e^{-sp^{1/2}}$$

and

$$\phi(s, u) = (4\pi u)^{-1/2} e^{-s^2/4u}.$$

From this the known relation

$$F(t^2) \overset{\mathscr{L}}{\underset{t\quad s}{\circ\!-\!\bullet}} \int_0^\infty (4\pi u)^{-1/2} e^{-s^2/4u} f(u)du$$

is obtained.

Special cases of $k(t)$ will sometimes simplify the image of $\Phi(s, u)$. If $k(t)=|g'(t)|K[g(t)]$, then

$$\Phi(s, u) \overset{\mathscr{L}}{\underset{u\;\;p}{\circ\!-\!\bullet}} K(p)\, e^{-s\,h(p)} .$$

If $k(t)=|g'(t)|[g(t)]^c$, then

$$\Phi(s, u) \overset{\mathscr{L}}{\underset{u\;\;p}{\circ\!-\!\bullet}} p^c\, e^{-s\,h(p)} .$$

In the proof of Theorem 1 it is noted that the only important property required of the kernel is that it be symmetric. Therefore consider the transformation

$$f(s)=\int_a^b K(s, t)\, F(t)\, dt$$

in which the integral is taken in the Lebesgue sense and in which the interval (a, b) may be unbounded. This transformation will be called the $\mathscr{T}$-transform and denoted by

$$F(t) \overset{\mathscr{T}}{\underset{t\;\;s}{\circ\!-\!\bullet}} f(s) ,$$

The following theorem is for this transformation with symmetric kernel.

THEOREM 2. *If*

(i) k, g, *and* $h=g^{-1}$ *are single-valued analytic functions, real on* (a, b), *and such that* $g(a)=a$ *and* $g(b)=b$ *(or* $g(a)=b$ *and* $g(b)=a$*);*

(ii) $F(t) \overset{\mathscr{T}}{\underset{t\;\;s}{\circ\!-\!\bullet}} f(s)$ *and this transformation integral converges for* $a<s<b$;

(iii) *there exists a function* $\Phi(s, u)$, $\Phi(s, u) \overset{\mathscr{T}}{\underset{u\;\;p}{\circ\!-\!\bullet}} \phi(s, p)$, *this transformation integral converges for* $a<s<b$, *and*

$$\phi(s, p)=K[s, h(p)]\, k[h(p)]\, |h'(p)| ;$$

(iv)

$$\int_a^b \left[\int_a^b |K(u, p)\, \Phi(s, u)\, F(p)|\, du\right] dp$$

converges for $s=s_0$; *and*

(v) $K(u, p)\equiv K(p, u)$; *then* $k(t)\, F[g(t)] \overset{\mathscr{T}}{\underset{t\;\;s}{\circ\!-\!\bullet}} \int_a^b \Phi(s, u)\, f(u)\, du$ *and this* *transformation integral converges for* $s=s_0$.

The proof follows in a manner similar to that of Theorem 1.

Formulas which hold provided $F(t)$ satisfies (ii) or (iv) of the theorem can be obtained for various transforms for specific $k(s)$ and $g(s)$ with the aid of tables [3, formulas 14.1(6), 8.12(10), 5.5(6)].

Formula 1. For the Stieltjes transformation $K(s, t)=(s+t)^{-1}$.

$$t^{b+1}F(at^2)\underset{t\quad s}{\overset{\mathscr{S}}{\circ\!-\!\bullet}}\int_0^\infty \frac{(u/a)^{b/2}}{2\pi a}\left[\frac{(u/a)\cos b\pi/2-s\sin b\pi/2}{s^2+u/a}\right]f(u)\,du$$

for a positive.

Formula 2. For the Hankel transformation $K(s, t)=J_\nu(st)(st)^{1/2}$

$$t^{-2}F(a/t)\underset{t\quad s}{\overset{\mathscr{H}}{\circ\!-\!\bullet}}a^{-1}\int_0^\infty \sqrt{aus}\,J_{2\nu}(2\sqrt{aus})f(u)\,du$$

for $-1/2<\nu$ and a positive.

The Laplace transformation will be considered in the next two formulas.

Formula 3.

$$(t+b/a)^d F(at^2+2bt)$$

$$\underset{t\quad s}{\overset{\mathscr{L}}{\circ\!-\!\bullet}}(1/2\pi)\,e^{bs/a}\int_0^\infty e^{-b^2s/a}\,e^{-s^2/8au}(\sqrt{2au})^{-d-1}D_d(s/\sqrt{2au})f(u)\,du$$

for a and b positive and in which $D_d(z)$ is the parabolic cylinder function. The range of permissible values of d will depend, according to (iv), on the particular function $F(u)$.

Formula 4.

$$t^{d-1}F(at^{-b})\underset{t\quad s}{\overset{\mathscr{L}}{\circ\!-\!\bullet}}b^{-1}\int_0^\infty (au)^{d/b}\phi[1/b,\ (d+b)/b;\ -s(au)^{1/b}]f(u)\,du$$

for a and b positive and in which $\phi(A, B; Z)$ is Wrights' function [4, vol. 3, § 18.1]. The range of permissible values of d will depend, according to (iv), on the particular function $F(u)$. In the special case $b=1$ the formula becomes

$$t^{d-1}F(a/t)\underset{t\quad s}{\overset{\mathscr{L}}{\circ\!-\!\bullet}}\int_0^\infty (\sqrt{au/s})^d\,J_d(2\sqrt{aus})f(u)\,du\ .$$

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NUMERICAL SOLUTION OF VIBRATION PROBLEMS IN TWO SPACE VARIABLES

S. D. CONTE

1. Introduction. The classical theory of vibrating plates leads to the following non-dimensional fourth order partial differential equation in two space variables $W(x, y, t)$ for the transverse vibrations:

$$\Delta\Delta W + W_{tt} = 0 , \tag{1}$$

where $\Delta\Delta$ is the biharmonic operator

$$\Delta\Delta = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} .$$

Solutions of this equation for two dimensional regions of arbitrary shape are of course not known, but even for those plate problems for which analytic solutions in series form for this equation are available, the series do not lend themselves easily to numerical calculations. Direct numerical solutions of this equation are therefore of considerable impotance. It is the purpose of this paper to present a new finite difference approximation to this equation which is stable for all values of the mesh ratios $\overline{\Delta t}/\overline{\Delta x}^2$ and $\overline{\Delta t}/\overline{\Delta y}^2$ and which involves an amount of work which is entirely feasible on large-scale digital computers. The method is a generalization of a method prepared by Douglas and Rachford [1] for solving the two dimensional diffusion equation.

2. The differential and difference equations. We consider first the specific problem of determining the transverse vibrations of a square homogeneous thin plate hinged at its boundaries and subjected to an arbitrary initial condition. The boundary value problem may be written

$$
(2)\quad
\begin{aligned}
&\text{a)}\quad \frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} + \frac{\partial^2 W}{\partial t^2} = 0 , && (x, y) \in R , \quad 0 \leqq t \leqq T ,\\
&\text{b)}\quad W(x, y, 0) = f(x, y) , && (x, y) \in R ,\\
&\text{c)}\quad W_t(x, y, 0) = 0 , && (x, y) \in R ,\\
&\text{d)}\quad W(x, y, t) = \frac{\partial^2 W}{\partial x^2}(x, y, t) = 0 , && \text{at } x = 0,\ 1 \text{ for } 0 < y < 1 ,\ t > 0 ,\\
&\text{e)}\quad W(x, y, t) = \frac{\partial^2 W}{\partial y^2}(x, y, t) = 0 , && \text{at } y = 0,\ 1 \text{ for } 0 < x < 1 ,\ t > 0 ,
\end{aligned}
$$

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where R is the open region $[0<x<1,\ 0<y<1]$. Letting $\Delta x=\Delta y=1/M$ we now lay a mesh over the region R and we introduce the following typical notation for difference operators

$$
(3)\qquad
\begin{aligned}
&w(i\Delta x, j\Delta y, n\Delta t)=w_{ijn}\ ,\\
&\Delta_t^2 w_{ijn}=(w_{i,j,n+1}-2w_{ijn}+w_{i,j,n+1})/\overline{\Delta t}^2\ ,\\
&\Delta_x^4 w_{ijn}=(w_{i+2,j,n}-4w_{i+1,j,n}+6w_{ijn}-4w_{i-1,j,n}+w_{i-2,j,n})/\overline{\Delta x}^4\ .
\end{aligned}
$$

We now approximate (2) by the following finite difference system:

$$
(4)\qquad
\begin{aligned}
&\text{a)}\quad \frac{1}{2}\Delta_x^4[w^*_{i,j,n+1}+w_{i,j,n-1}]+2\Delta_x^2\Delta_y^2 w_{ijn}+\Delta_y^4 w_{ijn}\\
&\qquad\qquad +\frac{w^*_{ijn+1}-2w_{ijn}+w_{i,j,n-1}}{\Delta t^2}=0\ ,\\
&\text{b)}\quad \frac{1}{2}\Delta_y^4[w_{i,j,n+1}+w_{ij,n-1}]=\Delta_y^4 w_{ijn}-\frac{w_{i,j,n+1}-w^*_{i,j,n+1}}{\Delta t^2}\ ,\\
&\qquad\qquad\qquad\qquad (i\Delta x, j\Delta y)\in R'\ ,\quad 0\leqq n\Delta t\leqq T\ ,\\
&\text{c)}\quad w_{i,j,0}=W_{ij0}=f_{ij}\ ,\qquad\qquad (i, j=1, 2, \cdots, M-1)\ ,\\
&\text{d)}\quad w_{i,j,1}=W_{i,j,0}\ ,\qquad\qquad (i, j=1, 2, \cdots, M-1)\ ,\\
&\text{e)}\quad \left\{\begin{aligned}&w_{0,j,n}=w_{M,j,n}=0\\ &w_{i+1,j,n}=-w_{i-1,j,n}\ (i=0, M)\end{aligned}\right\}\quad (j=1, \cdots, M-1\ ;\ 0\leqq n\Delta t\leqq T)\ ,\\
&\text{f)}\quad \left\{\begin{aligned}&w_{i,0,n}=w_{i,M,n}=0\\ &w_{i,j+1,n}=-w_{i,j-1,u}\ (j=0, M)\end{aligned}\right\}\quad (i=1, \cdots, M-1\ ;\ 0\leqq n\Delta t\leqq T)\ ,
\end{aligned}
$$

where R' is the set of lattice points $(i\Delta x,\ j\Delta y)$ in R and in condition e) and f) $w^*_{ijn}=w_{ijn}$.

Equation 4a) is implicit in x alone while equation 4b) is implicit in y alone. The numerical procedure consists of first solving equations 4a) to obtain $w^*_{ij,n+1}$. A system of $(M-1)$ equations in $(M-1)$ unknowns is obtained for the unknowns along a single line in the x-direction. The matrix of this system of equations has at most 5 non-zero elements in any one row (either on the main diagonal or on two adjacent diagonals). We shall call such matrices quidiagonal. These quidiagonal systems can be solved efficiently by an extension of an algorithm for solving tridiagonal matrices due to L. H. Thomas and involve about twice the amount of work as for tridiagonal matrices.

Use of equation 4a) above, however, is not sufficient to yield good values of w over a wide range in t because as will be shown the finite difference approximation is unstable. Equation 4b) then provides a corrective process which combined with 4a) does provide a stable, convergent process. Equation 4b) is implicit along lines parallel to the y-axis and again for rectangular regions yields $M-1$ systems of equations each involving $M-1$ unknowns. The matrices of these equations are again quidiagonal in form.

By eliminating $w^*_{ij,n+1}$ from equations 4a) and 4b) we obtain the following implicit finite difference equation

$$\frac{1}{2}\Delta_x^4[w_{ij,n+1}+w_{ij,n-1}]+2\Delta_x^2\Delta_y^2 w_{ijn}+\frac{1}{2}\Delta_y^4[w_{ij,n+1}+w_{ij,n-1}]$$
$$+\frac{w_{ij,n+1}-2w_{ijn}+w_{ij,n-1}}{\overline{\Delta t}^2}+\frac{1}{4}\overline{\Delta t}^2\Delta_x^4\Delta_y^4[w_{ij,n+1}-2w_{ijn}+w_{ij,n-1}]=0 , \tag{5}$$

which lends itself more readily to a stability and convergence analysis.

3. **Stability considerations.** Let $v(x, y, t)$ be the error due to round-off. Then since equation (5) is linear, it follows that v_{ijn} will satisfy the system

(6)

a) $$\frac{1}{2}\Delta_x^4[v_{ij,n+1}+v_{ij,n-1}]+2\Delta_x^2\Delta_y^2 v_{ijn}+\frac{1}{2}\Delta_y^4[v_{ij,n+1}+v_{ij,n-1}]$$
$$+\frac{v_{ij,n+1}-2v_{ijn}+v_{ij,n-1}}{\overline{\Delta t}^2}+\frac{1}{4}\overline{\Delta t}^2\Delta_x^4\Delta_y^4[v_{ij,n+1}-2v_{ijn}+v_{ij,n-1}]=0 ,$$

b) $v_{i,j,0}$ and $v_{i,j,1}$ arbitrary $\qquad (i, j=1, \cdots, M-1)$,

c) $$\left.\begin{cases} v_{0,j,n}=v_{M,j,n}=0 \\ v_{i+1,j,n}=-v_{i-1,j,n} \quad (i=0, M) \end{cases}\right\} \qquad (j=1, \cdots, M-1 ;\ 0\leqq n\Delta t\leqq T) ,$$

d) $$\left.\begin{cases} v_{i,0,n}=v_{i,M,n}=0 \\ v_{i,j+1,n}=-v_{i,j-1,n}=0 \quad (j=0, M) \end{cases}\right\} \qquad (i=1, \cdots, M-1 ;\ 0\leqq n\Delta t\leqq T) .$$

The eigenfunctions of (6) are of the form

$$v_{ijn}=a_n \sin \pi p x_i \sin \pi q y_j , \qquad p, q=1, \cdots, M-1 ,$$

where $x_i=i\Delta x$, $y_j=j\Delta y$. It is easily shown that, for example,

$$\overline{\Delta x}^4\Delta_x^4 v_{ijn+1}=16 \sin^4 \pi p\frac{\Delta x}{2}v_{ij,n+1} ,$$

$$\overline{\Delta x}^2\overline{\Delta y}^2\Delta_x^2\Delta_y^2 v_{ijn}=16 \sin^2 p\pi\frac{\Delta x}{2} \sin^2 q\pi\frac{\Delta y}{2}v_{ijn} .$$

Applying this to equation (6a) and rearranging we obtain the following recurrence relation in a_n:

$$a_{n+1}-2\alpha a_n+a_{n-1}=0 , \tag{7}$$

where

$$\alpha=\frac{1+\rho^2 s_p^4 s_q^4-2\rho s_p^2 s_q^2}{1+\rho^2 s_p^4 s_q^4+\rho(s_p^4+s_q^4)}=\frac{(1-\rho s_p^2 s_q^2)^2}{(1-\rho s_p^2 s_q^2)^2+\rho(s_p^2+s_q^2)^2} \tag{8}$$

and $s_p=\sin\frac{p\pi}{2M}$, $s_q=\sin\frac{q\pi}{2M}$, $\rho=8\frac{\overline{\Delta t^2}}{\Delta x^4}=8\frac{\overline{\Delta t^2}}{\Delta y^4}=8r^2$. The difference equation (6) will be stable provided that the roots of the characteristic equation

$$\xi^2-2\alpha\xi+1=0$$

corresponding to (7) are at most equal to one in absolute value. These roots are equal to one in absolute value if $|\alpha|\leqq 1$, a condition which follows at once from the definitions of s_p, s_q and ρ. Thus the finite difference system (4) is stable for all values of the mesh ratio ρ and for all values of p and q.

It should be pointed out that if (2) is replaced by an explicit finite difference approximation, a stability analysis leads to the requirement that

$$r=\frac{\Delta t}{\Delta x^2}=\frac{\Delta t}{\Delta y^2}\leqq 1/4 .$$

This restriction on the time step ordinarily leads to an amount of computing time which is not feasible even with the most modern computers. On the other hand a straightforward implicit finite difference approximation to (2), while simpler than (4) and also stable for all values of the mesh ratio, leads to a system of $(M-1)^2$ equations in $(M-1)^2$ unknowns which must be solved at each time step. Even a 20×20 interior grid leads to a system of 400 equations in 400 unknowns again involving an unreasonable amount of computing time.

Finally if one attempts to use 4a) without the corrective equation 4b) the same stability analysis given above leads to the characteristic equation

$$\xi^2-2\beta\xi+1=0$$

where

$$\beta=\frac{1-\rho(s_q^4+2s_p^2s_q^2)}{1+\rho s_p^4}$$

It is easily verified that for some values of p and q, $|\beta|>1$ and hence equation 4a) is not stable for all values of ρ.

4. Treatment of other boundary conditions. The stability analysis of § 3 depends upon the existence of a set of eigenfunctions of the difference operator given in (6a) which satisfy the boundary conditions (6c) and (6d). If the boundary conditions (2d) and (2e) corresponding to the difference conditions (6c) and (6d) change, the eigenfunctions of the system (6) will also change. Let us consider then the error equation (6a) with the boundary conditions (6c), (6d) replaced by the general homogeneous conditions:

$$L_m(v_{ijn})=0\ , \qquad (i, j)\in S^1\ .\ (m=1, 2, 3, 4)\ , \tag{9}$$

where S^1 is the set of boundary points affected by the conditions L_m.

Assume a set of eigenfunctions of (6a) of the form

$$v_{ijn}(p, q)=a_n\phi_{ij}(p, q)\ , \qquad p, q=1, \cdots, M-1\ .$$

Substituting into (6a) and rearranging, we obtain

$$a_{n+1}-2\alpha_{pq}a_n+a_{n-1}=0$$

where

$$\alpha_{pq}=\frac{-\overline{\Delta t}^2\Delta_x^2\Delta_y^2\phi_{ij}+\phi_{ij}+\frac{1}{4}\overline{\Delta t}^4\Delta_x^4\Delta_y^4\phi_{ij}}{\frac{1}{2}\overline{\Delta t}^2(\Delta_x^4\phi_{ij}+\Delta_y^4\phi_{ij})+1+\frac{1}{4}\overline{\Delta t}^4\Delta_x^4\Delta_y^4\phi_{ij}}=\frac{H[\phi_{ij}]}{K[\phi_{ij}]}$$

Now let H and K have a common set of eigenfunctions subject to the condition (9), i. e.

$$H[\phi_{ij}]=\lambda_{pq}\phi_{ij}\ , \quad L_m(\phi_{ij})=0\ ,$$

$$K[\phi_{ij}]=\beta_{pq}\phi_{ij}\ , \quad L_m(\phi_{ij})=0\ .$$

We then have

$$\alpha_{pq}=\frac{\lambda_{pq}}{\beta_{pq}}$$

and the condition for stability is simply that for all p and q

$$|\alpha_{pq}|\leqq 1\ .$$

Thus the stability analysis of § 3 can be applied for any boundary conditions for which the operators H and K have common eigenfunctions.

5. A mean square convergence theorem for the square region. For the problem considered in § 2 assume that the function $f(x, y)$, is sufficiently regular in the closed region $\bar{R}$ to guarantee the existence and boundedness of

$$\frac{\partial^6 w}{\partial x^6}\ , \quad \frac{\partial^6 w}{\partial y^6}\ , \quad \frac{\partial^6 w}{\partial y^4\partial t^2}\ , \frac{\partial^6 w}{\partial x^4\partial t^2}\ , \frac{\partial^4 w}{\partial t^2}$$

in $\bar{R}$. Then it can be shown following the usual series expansions that

$$\frac{1}{2}\Delta_x^4[w_{ij,n+1}+w_{ij,n-1}]+2\Delta_x^2\Delta_y^2 w_{ijn}+\frac{1}{2}\Delta_y^4[w_{ij,n+1}+w_{ij,n-1}] \tag{10}$$
$$+\frac{w_{ij,n+1}-2w_{ijn}+w_{ij,n-1}}{\overline{\Delta t}^2}=0(\overline{\Delta x}^2+\overline{\Delta t}^2)\,,$$

and moreover that

$$\overline{\Delta t}^2\Delta_x^4\Delta_y^4[w_{ij,n+1}-2w_{ijn}+w_{ij,n-1}]=0(\overline{\Delta t}^2)\,. \tag{11}$$

Hence the difference operator (5) approximates the differential equation (2) to terms which are $0(\overline{\Delta x}^2+\overline{\Delta t}^2)$. In the notation of [2] the elementary truncation error h_{ijn} is

$$h_{ijn}=0(\overline{\Delta x}^6+\overline{\Delta x}^4\overline{\Delta t}^2) \tag{12}$$

and by Theorem 1 of [2] we have

$$\| W_{ijn}-w_{ijn}\|=0\left(\frac{\overline{\Delta x}^6}{\overline{\Delta t}^2}+\overline{\Delta t}^4\right) \tag{13}$$

uniformly in n, where

$$\| W_{ijn}-w_{ijn}\|=\frac{1}{(M-1)}\left\{\sum_{i,j=1}^{M-1} | W_{ijn}-w_{ijn}|^2\right\}^{1/2} \tag{14}$$

It thus follows that if the boundary value problem (2) is sufficiently well defined in the sense that the derivatives mentioned above exist boundedly in the closed region $\bar{R}$, then the solution of (4) converges in the mean to the solution of (2) with errors given by (13) as Δx and Δt tend to zero.

The convergence proof given above holds for a rectangular region only. In practice one is usually interested in point-wise convergence rather than convergence in the mean square sense. Section 6 establishes point-wise convergence of the solution of the difference system to the solution of the differental system.

6. Point-wise convergence. A solution of the boundary value problem (2) can be given in series form

$$W(x, y, t)=\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} A_{pq}\sin p\pi x\sin q\pi y\cos (p^2+q^2)\pi^2 t\,. \tag{15}$$

The initial condition (2b) will be satisfied provided that A_{pq} are taken to be the Fourier coefficients of $f(x, y)$, i.e.

$$A_{pq}=4\int_0^1\int_0^1 f(x, y)\sin p\pi x\sin q\pi y\, dxdy\,. \tag{16}$$

The conditions on $f(x, y)$ are assumed to be such that the series (15) converges and is the unique solution of the boundary value problem (2). A solution $w(x, y, t)$ of the finite difference system consisting of (4a, b, e, f) can be obtained by separation of variables as follows:

$$w(x, y, t)=\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} B_{pq} \sin p\pi x \sin q\pi y \cos \frac{M^2 t}{r} \operatorname{arc\,cos} \frac{\lambda_1(p, q)}{\lambda_2(p, q)}, \tag{17}$$

where

$$\lambda_1(p, q)=(1-\rho s_p^2 s_q^2)^2 ,$$
$$\lambda_2(p, q)=(1-\rho s_p^2 s_q^2)^2+\rho(s_p^2+s_q^2)^2 ,$$
$$\Delta x=\Delta y=1/M ,$$

and B_{pq} are arbitrary constants. The series (17) satisfies the finite difference system (4) except for the initial condition 4c).

We will now show that it is possible to choose the coefficients B_{pq} so that the solution $w(x, y, t)$ of the difference system will converge to the solution $W(x, y, t)$ of the differential system as $M\to\infty$. We first define an integer $k(M)$ such that $k(M)<M^{1/5}$ and $\lim_{M\to\infty} k(M)=\infty$. We then choose the B_{pq} so that $B_{pq}=0$ for $p>k(M)$, $q>k(M)$ and the remaining B_{pq} so that for any $\varepsilon>0$ there exists an $M_1(\varepsilon)$ such that for $M>M_1$,

$$|B_{pq}-A_{pq}|<\varepsilon M^{-2/5} \text{ uniformly for } p, q=1, \cdots, k(M) . \tag{18}$$

One way of satisfying (18) for instance is to choose $B_{pq}=A_{pq}$ for $p, q=1, \cdots, k(M)$. An exact solution of the difference equation then is

$$w_M(x, y, t)=\sum_{p=1}^{k(M)}\sum_{q=1}^{k(M)} B_{pq} \sin p\pi x \sin q\pi y \cos \frac{M^2 t}{r} \operatorname{arc\,cos} \frac{\lambda_1}{\lambda_2} . \tag{19}$$

This solution satisfies the initial condition

$$w_M(x, y, 0)=\sum_{p=1}^{k}\sum_{q=1}^{k} B_{pq} \sin p\pi x \sin q\pi y \tag{20}$$

and of course does not satisfy the exact initial condition $w(x, y, 0)=f(x, y)$. However, it will satisfy this initial condition in the limit as $M\to\infty$.

LEMMA 1. *For any* $\rho>0$, $0\leqq z_1\leqq\frac{\pi}{2}M^{-4/5}$, $0\leqq z_2\leqq\frac{\pi}{2}M^{-4/5}$, *there exists an* $M_2(\rho)$ *such that for* $M>M_2$ *and for any* $\varepsilon>0$

$$\left|4r(z_1^2+z_2^2)-\operatorname{arc\,cos}\frac{\lambda_1}{\lambda_2}\right|\leqq\frac{r\varepsilon}{M^3} , \tag{21}$$

where

$$\lambda_1=(1-\rho \sin^2 z_1 \sin^2 z_2)^2 ,$$

$$\lambda_2=(1-\rho \sin^2 z_1 \sin^2 z_2)^2+\rho(\sin^2 z_1+\sin^2 z_2)^2 .$$

Proof. We first choose $M_3(\rho)$ such that $M>M_3$ and for all admissible z_1, z_2

$$1-\rho \sin^2 z_1 \sin^2 z_2>0 . \tag{22}$$

Let

$$F(z_1, z_2)=4r(z_1^2+z_2^2)-\text{arc cos}\frac{\lambda_1}{\lambda_2} .$$

It is obvious that $F(0, 0)=0$ and it can be shown by direct calculation that the partial derivatives of $F(z_1, z_2)$ up to and including those of order 3 all vanish at $z_1=0$, $z_2=0$. Thus in the Taylor series expansion of $F(z_1, z_2)$ the remainder term is

$$R(z_1, z_2)=a_{41}z_1^4+a_{42}z_1^3z_2+a_{43}z_1^2z_1^2+a_{44}z_1z_2^3+a_{45}z_2^4 ,$$

where the coefficients $a_{4i}(\bar{z}_1, \bar{z}_2)$, $i=(1, \cdots, 5)$, are related to the fourth derivatives of $F(z_1, z_2)$ and $0<\bar{z}_1<\frac{\pi}{2}M^{-4/5}$, $0\leqq\bar{z}_2<\frac{\pi}{2}M^{-4/5}$. Using the inequality (22) it is possible to show that the a_{4i} are bounded functions of ρ. Thus using the extreme limits of z_1, z_2 we have

$$|F(z_1, z_2)|\leqq|R(z_1, z_2)|\leqq A(\rho) \ . \ \frac{\pi^4}{2^4}\cdot M^{-16/5} ,$$

and hence it follows that there exists an $M_2(\rho)$ such that for $M>M_2$,

$$|F(z_1, z_2)|<\frac{\varepsilon r}{M^3} ,$$

as the lemma asserts.

Now multiplying (21) by $\dfrac{M^2t}{r}$ and putting $z_1=\dfrac{p\pi}{2M}$, $z_2=\dfrac{q\pi}{2M}$ we have

$$\left|\frac{M^2t}{r}\text{ arc cos }\frac{\lambda_1(p, q)}{\lambda_2(p, q)}-\pi^2(p^2+q^2)t\right|<\frac{\varepsilon t}{M} ,$$

and therefore

$$\left|\cos\frac{M^2t}{r}\text{ arc cos }\frac{\lambda_1(p, q)}{\lambda_2(p, q)}-\cos \pi^2(p^2+q^2)t\right|<\frac{\varepsilon t}{M} . \tag{23}$$

THEOREM 1 (THE CONVERGENCE THEOREM). *Under the assumptions*

a) $t>0$, $0<x<1$, $0<y<1$; $\rho>0$; ρ, t, x, y fixed;

b) $|A_{pq}|\leqq P$, P constant, for all $p, q=1, 2, \cdots, \infty$;

c) $k(M)<M^{1/5}$, $\lim_{M\to\infty} k(M)=\infty$;

d) $|B_{pq}-A_{pq}|<\varepsilon M^{-2/5}$, $1\leqq p, q\leqq k(M)$,

we have

$$\lim_{M\to\infty} w_M(x, y, t)=W(x, y, t)$$

or

$$\lim_{M\to\infty} \sum_{p=1}^{k(M)} \sum_{q=1}^{k(M)} B_{pq}(M) \sin p\pi x \sin q\pi y \cos \frac{M^2 t}{r} \operatorname{arc\,cos} \frac{\lambda_1(p, q)}{\lambda_2(p, q)}$$
$$=\sum_{p-1}^{\infty} \sum_{q=1}^{\infty} A_{pq} \sin p\pi x \sin q\pi y \cos (p^2+q^2)\pi^2 t .$$

Proof.

$$w_M(x, y, t)-W(x, y, t)=\sum_{p=1}^{k} \sum_{q=1}^{k} (B_{pq}-A_{pq}) \sin p\pi x \sin q\pi y \cos (p^2+q^2)\pi^2 t$$
$$+\sum_{p=1}^{k} \sum_{q=1}^{k} (B_{pq}-A_{pq}) \sin p\pi x \sin q\pi y\left[\cos \frac{M^2 t}{r} \operatorname{arc\,cos} \frac{\lambda_1}{\lambda_2}-\cos (p^2+q^2)\pi^2 t\right]$$
$$+\sum_{1}^{k} \sum_{1}^{k} A_{pq} \sin p\pi x \sin q\pi y\left[\cos \frac{M^2 t}{r} \operatorname{arc\,cos} \frac{\lambda_1}{\lambda_2}-\cos (p^2+q^2)\pi^2 t\right]$$
$$+\sum_{k+1}^{\infty} \sum_{k+1}^{\infty} A_{pq} \sin p\pi x \sin q\pi y \cos (p^2+q^2)\pi^2 t ,$$
$$=I_1+I_2+I_3+I_4 .$$

By conditions c) and d) above and Lemma 1,

$$|I_1|\leqq k^2(M)\cdot \varepsilon M^{-2/5}\leqq \varepsilon ,$$

$$|I_2|\leqq k^2(M)\cdot \varepsilon M^{-2/5} \frac{\varepsilon t}{M}\leqq \frac{\varepsilon^2}{M}<\varepsilon^2 t .$$

By condition b) and c) and Lemma 1,

$$|I_3|\leqq P\cdot k^2(M)\cdot \frac{\varepsilon t}{M}\leqq P\varepsilon t ,$$

and because the series for $w(x, y, t)$ converges there exists an M_4 such that for $M>M_4$

$$|I_4|<\varepsilon .$$

Thus for $M > \max(M_1, M_2, M_4)$,

$$|w_M(x, y, t) - W(x, y, t)| \leqq \varepsilon(2 + \varepsilon t + Pt) .$$

This establishes the convergence theorem.

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THE RAMO-WOOLDRIDGE CORPORATION

A PROPERTY OF DIFFERENTIAL FORMS IN THE CALCULUS OF VARIATIONS

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1. In the classical problems involving a simple integral

$$I_1 = \int L(t,\, q^i,\, \dot{q}^i)\, dt\ , \qquad i=1, \cdots, n\ , \tag{1}$$

one is led to the consideration of the Pfaffian form

$$\omega = L\, dt + \frac{\partial L}{\partial \dot{q}^i}\, \omega^i = \frac{\partial L}{\partial \dot{q}^i}\, dq^i - \left(\dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L\right) dt \tag{2}$$

where

$$\omega^i = dq^i - \dot{q}^i\, dt\ .$$

For example this form ω is the one which gives rise to the "relative integral invariant" of E. Cartan.

In a recent note [1] L. Auslander characterizes the form ω by a theorem equivalent to the following one.

THEOREM 1. *Among all semi-basic forms θ such that*

$$\theta \equiv L\, dt \bmod \omega^i \tag{3}$$

the form ω of (2) *is the only one satisfying the condition*

$$d\theta \equiv 0 \bmod \omega^i\ . \tag{4}$$

In this, a *semi-basic form* is a form for which the local expression contains only the differentials of t, q^i(not of $\dot{q}^i$). The integral I is defined over an arc $\bar{c}$ of a space $\mathscr{W}$ with local coordinates t, q^i, $\dot{q}^i$ satisfying the equations $\omega^i=0$: Therefore in (1) the form $L\,dt$ may be replaced by any θ satisfying (3).

Condition (4) is a special case of a congruence discovered by Lepage [5]. *The purpose of the present note is to give a natural reason for this congruence which goes beyond its nice algebraic expression.*

Let us observe that the space $\mathscr{W}$ is the manifold of 1-dimensional contact elements of a manifold $\mathscr{V}$ with local coordinates t, q^i. The map

$$(t,\, q^i,\, \dot{q}^i) \rightarrow (t,\, q^i)$$

is then the local expression of the natural projection $\pi\colon \mathscr{W} \rightarrow \mathscr{V}$. We

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remark that we do not integrate (1) on any arc $\bar{c}$ in $\mathscr{W}$ satisfying $\omega^i=0$ but on such an arc the projection c of which in $\mathscr{V}$ is regular.

2. Let U be the domain in $\mathscr{V}$ of the coordinates t, q^i; then the $t, q^i, \dot{q}^i$ are defined in an open subset $W \subset \mathscr{W}$ of projection $\pi(W)=U$. If we denote by L_i n real undeterminates, we have coordinates $t, q^i, \dot{q}^i, L_i$ in $W \times R^n$; we then define in this product the Pfaffian form

$$\Omega_W = L\,dt + L_i\omega^i . \tag{5}$$

Now, let us cover $\mathscr{W}$ with open sets $W, W', \cdots$; this way we get a family of products $W\times R^n$, $W'\times R^n$, $\cdots$ with forms $\Omega_W, \Omega_{W'}, \cdots$. Using fibre bundle techniques, one proves that over a non-empty intersection $W\cap W'$ the products $W\times R^n$ and $W'\times R^n$ can be glued together in such a way that the forms induced on $W\cap W'\times R^n$ coincide. This yields a fibre bundle $E(\mathscr{W}, R^n)$ over $\mathscr{W}$ as base, with fibre R^n. This bundle is covered by open subsets isomorphic with the products $W\times R^n$ and in which the $t, q^i, \dot{q}^i, L_i$ are local coordinates; there is also on E a global Pfaffian form Ω of local expression (5). Combining the projections $E\to\mathscr{W}$ and $\mathscr{W}\to\mathscr{V}$ we obtain a map $E\to\mathscr{V}$ locally defined by

$$(t, q^i, \dot{q}^i, L_i) \to (t, q^i) .$$

We want to characterize in E the extremal arcs c^* of $\int\Omega$ *which have a regular projection in* $\mathscr{V}$.

An extremal arc c^* of $\int\Omega$ has to satisfy the local equations

$$\frac{\partial(d\Omega)}{\partial(dt)} = \frac{\partial(d\Omega)}{\partial(\omega^i)} = \frac{\partial(d\Omega)}{\partial(d\dot{q}^i)} = \frac{\partial(d\Omega)}{\partial(dL_i)} = 0 .$$

We have

$$d\Omega = \frac{\partial L}{\partial \dot{q}^i}\omega^i\wedge dt + \left(\frac{\partial L}{\partial \dot{q}^i} - L_i\right) d\dot{q}^i \wedge dt + dL_i\wedge\omega^i .$$

These equations are therefore

$$\omega^i=0 , \qquad \left(\frac{\partial L}{\partial \dot{q}^i} - L_i\right) dt = 0 , \qquad \frac{\partial L}{\partial \dot{q}^i} dt - dL_i = 0 .$$

Since an arc c^* of regular projection in $\mathscr{V}$ cannot satisfy simultaneously $\omega^i=0$ and $dt=0$ it has to lie in the submanifold F of E locally characterized by

$$\frac{\partial L}{\partial \dot{q}^i} = L_i$$

or equivalently by condition (4).

THEOREM 2. *Every arc c^* in E for which $\int \Omega$ is stationary and the projection of which in $\mathscr{V}$ is regular necessarily lies in the submanifold F of E locally defined by the congruence* (4). *Furthermore the projection c of c^* in $\mathscr{V}$ extremizes in the classical sense the integral* (1). *Finally if c is a regular extremal are of* (1) *in $\mathscr{V}$ let c^* be the arc of F the projection $\bar{c}$ of which in $\mathscr{W}$ is the arc of tangent directions to c; then c^* extremizes $\int \Omega$.*

3. **The submanifold** F can be identified with $\mathscr{W}$ in an obvious way so that $\mathscr{W}$ can be considered as a submanifold of E. Then clearly Ω induces ω on $\mathscr{W}$.

THEOREM 3. *If the integral* (1) *is regular there exists a* (*one-to-one*) *correspondence between the regular extremal arcs c in $\mathscr{V}$ of* (1) *and the extremal arcs $\bar{c}$ of $\int \omega$ in $\mathscr{W}$ which have a regular projection in $\mathscr{V}$. Starting from an extremal c, the corresponding $\bar{c}$ is the arc the points of which are the tangent directions to c; starting from $\bar{c}$ the corresponding c is its projection in $\mathscr{V}$.*

In this statement, regularity of (1) means that the matrix $(\partial^2 L/\partial \dot{q}^i \partial \dot{q}^j)$ is everywhere non singular.

Theorem 2 and 3 give a complete justification of condition (4). Theorem 3 was actually proved by E. Cartan [2]. These theorems are special cases of similar theorems involving multiple integrals and even those in which the function L depends on higher order contact elements. Theorem 2 was first proved by the author [3], as well as the alluded generalizations.

Combining Theorems 2 and 3 yields the following.

THEOREM 4. *In the regular case, every arc $\bar{c}$ in $\mathscr{W}$ of regular projection in $\mathscr{V}$ which extremizes $\int \omega$ with respect to variations confined to $\mathscr{W}$ does also extremize $\int \Omega$ with respect to variations in the larger space E.*

4. **There is a last question** to be answered: why in Theorem 1 restrict oneself to semi-basic forms?

We can only add to $L.dt$ a linear combination of Pfaffian forms vanishing with ω^i; every such form is a linear combination of the ω^i

and is therefore semi-basic. Hence the restriction to semi-basic forms in Theorem 1 was actually redundant.

However, as mentioned above and as I have proved in various papers (e.g. [3, 4]), the above properties generalize to a multiple integral

$$I_p = \int L(t^\alpha, q^i, q^i_\alpha)\, dt\ , \tag{6}$$

$$dt = dt^1 \wedge \cdots \wedge dt^p, \qquad \alpha = 1, 2, \cdots, p\,; \qquad i = 1, 2, \cdots, n\ ,$$

to be integrated over a p-surface c defined by $q^i = q^i(t^\alpha)$ and where q^i_α stands for $\partial q^i/\partial t^\alpha$. Then $\mathscr{V}$ is of dimension $n+p$ and $\mathscr{W}$ (which is geometrically the manifold of p-dimensional contact elements of $\mathscr{V}$) is of dimension $n+p+np$. We can consider that we integrate (6) in $\mathscr{W}$ over a p-surface $\bar{c}$ of regular projection in $\mathscr{V}$ and solution of the Pfaffian equations

$$\omega^i = dq^i - \sum q^i_\alpha\, dt^\alpha = 0\ .$$

Such a p-surface $\bar{c}$ is formed of the contact elements of dimension p to a regular p-surface in $\mathscr{V}$ and will be called a *p-multiplicity.*

Now in (6) we can add to $L.dt$ any p-form vanishing on all p-multiplicities and all such forms are no longer semi-basic if $p>1$: for example $d\omega^i \wedge dt^3 \wedge \cdots \wedge dt^p$ is such one. Nevertheless, the semi-basic forms satisfying the Lepage congrences [5]:

$$\theta \equiv L\, dt \qquad \mathrm{mod}\ \omega^i\ , \tag{7}$$

$$d\theta \equiv 0 \qquad \mathrm{mod}\ \omega^i\ . \tag{8}$$

play an important role for a deeper reason which is actually a *transversality condition.* We briefly discuss this below referring the reader to my memoir [4] for further details.

5. Let $\mathscr{K}$ be a **p-dimensional** manifold and K a domain of $\mathscr{K}$ with regular boundary K. A map

$$c:\ K \to \mathscr{V}$$

is a domain of integration of (6); it gives rise canonically to a map

$$\bar{c}:\ K \to \mathscr{W}$$

such that for $k \in K$, $\bar{c}(k)$ is the contact element of dimension p to c at k. *A variation* (or *homotopy*) of c is a family of maps

$$c_t:\ K \to \mathscr{V}, \qquad t \in R\ , \qquad c_0 = c\ ;$$

this yields a variation of $\bar{c}$:

$$\bar{c}_t : K \to \mathscr{W} .$$

We also define $C: K \times R \to \mathscr{V}$, $\bar{C}: K \times R \to \mathscr{W}$ by

$$C(k, t) = c_t(k) , \qquad \bar{C}(k, t) = \bar{c}_t(k) .$$

The corresponding variation of $\int \theta$ is then

$$\Delta = \int_{\bar{c}_t} \theta - \int_{\bar{c}_0} \theta$$

which may be expressed as a sum of two terms:

$$\Delta = \int_{\bar{C}_{0t}} d\theta + \int_{\lambda_{0t}\bar{C}} \theta . \tag{9}$$

The domains of integration $\bar{C}_{0t}$ and $\lambda_{0t}\bar{C}$ are the restrictions of $\bar{C}$ to $K \times I_{0t}$ and $\dot{K} \times I_{0t}$ respectively (where $I_{0t} = [0, t] \subset R$). We say that the variation $\bar{C}$ is *transversal* to θ if this form vanishes on $\lambda\bar{C}$ (restriction of $\bar{C}$ to $\dot{K} \times R$). This being the case, the last integral (or boundary term) in (9) is zero.

Now the variations usually considered are those for which the restriction of C to $\dot{K}$ is constant (fixed boundary variations): for those, $\lambda\bar{C}$ has an everywhere non-regular projection in $\mathscr{V}$, so that every semi-basic form vanishes on $\lambda\bar{C}$. Therefore if we replace in (6) $L.dt$ by a semi-basic p-form θ satisfying (7), all variations with fixed boundary are transversal to it. This would of course not be the case, should we add to $L.dt$ a non-semi-basic p-form vanishing on all p-multiplicities.

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A NOTE ON ADDITIVE FUNCTIONS

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1. A real valued function $f(n)$, defined on the set of natural numbers, is called *additive* if $f(mn)=f(m)+f(n)$ whenever $(m, n)=1$, and *strongly additive* if also $f(p^\alpha)=f(p)$ for p prime and $\alpha=2, 3, \cdots$. We define

$$(1) \qquad A_n=\sum_{p<n} f(p)/p\ , \qquad B_n=\sum_{p<n} f^2(p)/p\ ,$$

and we assume throughout that

$$(2) \qquad B_n \to \infty\ , \qquad n \to \infty\ .$$

Additive functions for which $B_n=O(1)$ have already been discussed thoroughly in Erdös and Wintner [4]. They proved the following theorem:

Define

$$f'(p)=\begin{cases} 1 \text{ for } |f(p)|>1\ , \\ f(p) \text{ for } |f(p)|\leqq 1\ . \end{cases}$$

Then the additive function $f(n)$ possesses a distribution function if, and only if, the series

$$\sum_p f'(p)/p \qquad \text{and} \qquad \sum_p \{f'(p)\}^2/p$$

converge.

Moreover, it follows from a general result of P. Lévy [10] that this distribution function is continuous if, and only if, the series $\sum_{f(p)\neq 0} f(p)/p$ diverges. Surveys of this subject are given in Kac [7] and Kubilyus [9]. A comprehensive account is being prepared by H. N. Shapiro.

Our knowledge of functions subject to (2) is not as complete. Outstanding is the result of Erdös and Kac [3] which states that if

$$(3) \qquad f(p)=O(1)\ ,$$

the distribution of

$$\frac{f(m)-A_n}{B_n^{1/2}}\ , \qquad m\leqq n\ ,$$

is asymptotically Gaussian. In a recent note H. N. Shapiro [11] has shown that the theorem of Erdös and Kac remains true even when (3) is replaced by

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$$\lim_{n\to\infty} B_n^{-1} \sum_{\substack{p<n \\ |f(p)|>\varepsilon B_n^{1/2}}} f^2(p)/p = 0 \qquad \text{for every } \varepsilon>0\ . \tag{4}$$

Since (4) is essentially the Lindeberg condition which is necessary and sufficient for the central limit theorem to hold, one is led to conjecture that (4) is not only the sufficient but also the necessary condition for the truth of the theorem of Erdös and Kac. However, it seems very difficult to establish the necessity (see Kubilyus [8] and Tanaka [12]).

Associated with such questions about the distributions of additive arithmetic functions is a number of 'moment' problems, which, if solved, lead to results of independent interest. Thus, for example, the following result is suggested by, and includes, the theorem of Erdös and Kac.

THEOREM 1. *Let $f(m)$ be strongly additive and subject to* (2) *and*

$$f(p) = o(B_p^{1/2})\ . \tag{5}$$

Then we have for each fixed $k=1, 2, 3, \cdots$

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} (f(m)-A_n)^k}{nB_n^{k/2}} = (2\pi)^{-1/2}\int_{-\infty}^{\infty} \omega^k e^{-\omega^2/2}\, d\omega\ .$$

(For proofs see Delange [1], [2], Halberstam [5], [6].)

The purpose of the present communication is to indicate briefly a proof that Theorem 1 remains true even when (5) is replaced by the weaker pair of conditions (4) and

$$f(p) = O(B_p^{1/2})\ . \tag{5a}$$

That (5a) alone does not suffice can be seen readily from the case $f(p) = \log p$, which determines a very different kind of distribution. On the other hand, (4) alone would also be inadequate, as can be seen from the following example.

Let $p_1, p_2, \cdots, p_j, \cdots$ be an increasing sequence of primes with the property that the number of primes which belong to this sequence and do not exceed x is $o(\log\log x)$. Now take

$$f(p) = \begin{cases} (p_j)^{1/2} & \text{if } p = p_j\ , \\ 1, & \text{if } p \text{ does not belong to the sequence.} \end{cases}$$

Then $B_n \sim (\log\log n)$ and condition (4) is satisfied. However,

$$\sum_{m\leqq p_j} (f(m)-A_{p_j})^4 \geqq (f(p_j)-A_{p_j})^4 \sim p_j^2$$

whereas, if Theorem 1 were true in this case, we should have

$$\sum_{m \leqq p_j} (f(m) - A_{p_j})^4 \sim 3p_j(\log \log p_j)^2 .$$

The most general formulation of Theorem 1 remains an open question. The theorem shows, incidentally, that although the method of moments is in many ways more tractable for determining the distributions of given functions, it is not as wide in scope as the method evolved by Erdös and Kac.

2. **We suppose throughout this section** that (4) and (5a) hold. First of all, we rewrite (4) as

$$\lim_{n\to\infty} \phi(n, \varepsilon) = 0 \qquad \text{for every } \varepsilon > 0 , \tag{6}$$

where

$$\phi(n, \varepsilon) = B_n^{-1} \sum_{\substack{p<n \\ |f(p)| > \varepsilon B_n^{1/2}}} f^2(p)/p . \tag{7}$$

To simplify subsequent arithmetic we choose $\varepsilon < 1/2$ and keep it fixed; then we choose n so large that

$$\phi(n, \varepsilon) < \frac{1}{2}\varepsilon \tag{8}$$

as is possible by (6). We set

$$\alpha_n = n^{1/(3k)} \tag{9}$$

and observe that in view of (9) and the well-known relation

$$\sum_{p<y} p^{-1} = \log \log y + c + o(1) \tag{10}$$

where c is an absolute constant,[1]

$$\sum_{\alpha_n \leqq p < n} p^{-1} = O(1) . \tag{11}$$

We define

$$A_y^* = \sum_{\substack{p<y \\ |f(p)| \leqq \varepsilon B_n^{1/2}}} f(p)/p , \qquad B_y^* = \sum_{\substack{p<y \\ |f(p)| \leqq \varepsilon B_n^{1/2}}} f^2(p)/p \tag{12}$$

and

$$f^*(m) = \sum_{\substack{p<\alpha_n, p \mid m \\ |f(p)| \leqq \varepsilon B_n^{1/2}}} f(p) . \tag{13}$$

By (7) and (12)

---

[1] The constants implied by the use of the O-notation depend throughout on at most k.

$$B_n^* = B_n(1-\phi(n, \varepsilon))$$

and this combines with (11) to give

$$B_{\alpha_n}^* = B_n(1+O(\varepsilon^2+\phi(n, \varepsilon))) . \tag{14}$$

LEMMA 1. $A_n = A_{\alpha_n}^* + O(B_n^{1/2}\{\varepsilon + \varepsilon^{-1}\phi(n, \varepsilon)\})$.

Proof. By (1)

$$A_n = \sum_{\substack{p<\alpha_n \\ |f(p)|\leqq \varepsilon B_n^{1/2}}} f(p)/p + \sum_{\substack{\alpha_n \leqq p<n \\ |f(p)|\leqq \varepsilon B_n^{1/2}}} f(p)/p + \sum_{\substack{p<n \\ |f(p)|>\varepsilon B_n^{1/2}}} f(p)/p .$$

The first sum on the right is $A_{\alpha_n}^*$ by (12) with $y=\alpha_n$, the second sum is $O(\varepsilon B_n^{1/2})$ by (11), and the third is less than

$$\varepsilon^{-1}B_n^{-1/2} \sum_{\substack{p<n \\ |f(p)|>\varepsilon B_n^{1/2}}} f^2(p)/p = B_n^{1/2}\varepsilon^{-1}\phi(n, \varepsilon)$$

by (7). Hence the result.

LEMMA 2. *If* $r \leqq k$, *then*

$$\sum_{m=1}^{n} (f(m)-f^*(m))^{2r} = O(nB_n^r\{\varepsilon+\varepsilon^{-1}\phi(n, \varepsilon)\}) .$$

Proof. By (13) and the definition of $f(m)$

$$f(m)-f^*(m) = \sum_{\substack{p<n, p|m \\ |f(p)|>\varepsilon B_n^{1/2}}} f(p) + \sum_{\substack{\alpha_n \leqq p<n, p|m \\ |f(p)|\leqq \varepsilon B_n^{1/2}}} f(p) = \sum_{\substack{p|m \\ p\in \mathscr{E}_n}} f(p)$$

where $\mathscr{E}_n$ is the set of those primes less than n which satisfy either

$$\text{(i)} \quad |f(p)| > \varepsilon B_n^{1/2}$$

or

$$\text{(ii)} \quad |f(p)| \leqq \varepsilon B_n^{1/2}, \ p \geqq \alpha_n .$$

Then the sum of Lemma 2 is

$$O\Bigl(\sum_{\nu=1}^{2r} \sum_{\substack{r_1+\cdots+r_\nu=2r \\ r_1\geqq\cdots\geqq r_\nu\geqq 1}} \sum_{p_1,\cdots,p_\nu}'' |f^{r_1}(p_1)\cdots f^{r_\nu}(p_\nu)| \sum_{\substack{m=1 \\ (p_1\cdots p_\nu)|m}}^{n} 1\Bigr)$$

$$= O\Bigl(\sum_{\nu=1}^{2r} \{\max_{p\leqq n} |f(p)|^{2r-\nu}\} \sum_{p_1,\cdots,p_\nu}'' \Bigl[\frac{n}{p_1\cdots p_\nu}\Bigr] |f(p_1)\cdots f(p_\nu)|\Bigr)$$

where $\sum''$ indicates that the summation is carried out over all sets of distinct prime numbers $p_1, p_2, \cdots, p_\nu$ with $p_i \in \mathscr{E}$ $(i=1, 2, \cdots, \nu)$, and $[y]$ stands for the integer part of y. Using (5a), (i) and (ii) this expression is

$$O\Big(n\sum_{\nu=1}^{2r} B_n^{r-\frac{1}{2}\nu}\sum_{s=0}^{\nu}\Big\{\sum_{\substack{p<n\\ |f(p)|>\varepsilon B_n^{1/2}}}|f(p)|/p\Big\}^s\Big\{\sum_{\substack{\alpha_n\leqq p<n\\ |f(p)|\leqq\varepsilon B_n^{1/2}}}|f(p)|/p\Big\}^{\nu-s}\Big),$$

which, as in the proof of Lemma 1, becomes

$$O\Big(n\sum_{\nu=1}^{2r}B_n^{r-\frac{1}{2}\nu}\sum_{s=0}^{\nu}\{B_n^{1/2}(\varepsilon^{-1}\phi)\}^s\{B_n^{1/2}\varepsilon\}^{\nu-s}\Big)=O\Big(nB_n^r\sum_{\nu=1}^{2r}\sum_{s=0}^{\nu}(\varepsilon^{-1}\phi)^s\varepsilon^{\nu-s}\Big)$$
$$=O(nB_n^r\{\varepsilon^{-1}\phi+\varepsilon\})\ ;$$

here we have used the restrictions on the magnitudes of ε and ϕ imposed at the beginning of § 2 (see inequality (8)).

Next we set

$$M_k(n)=\sum_{m=1}^{n}(f(m)-A_n)^k\ ,\qquad M_r^*(n)=\sum_{m=1}^{n}(f^*(m)-A_{\alpha_n}^*)^r\ .$$

Then

$$M_k(n)=\sum_{m=1}^{n}\{(A_{\alpha_n}^*-A_n)+(f(m)-f^*(m))+(f^*(m)-A_{\alpha_n}^*)\}^k\ ,$$

so that by Lemmas 1 and 2 and Cauchy's inequality

$$M_k(n)-M_k^*(n)$$
$$=O\Big(\sum_{\substack{r_1+r_2+r_3=k\\ r_3\leqq k-1}}|A_n-A_{\alpha_n}^*|^{r_1}\sum_{m=1}^{n}|f(m)-f^*(m)|^{r_2}|f^*(m)-A_{\alpha_n}^*|^{r_3}\Big)$$
$$=O\Big(\sum_{\substack{r_1+r_2+r_3=k\\ r_3\leqq k-1}}B_n^{r_1/2}\{\varepsilon+\varepsilon^{-1}\phi\}^{r_1}\{\sum_{m=1}^{n}(f(m)-f^*(m))^{2r_2}\}^{1/2}\{M_{2r_3}^*(n)\}^{1/2}\Big)$$
$$=O\Big(n^{1/2}\sum_{r\leqq k-1}B_n^{(k-r)/2}\{\varepsilon+\varepsilon^{-1}\phi\}^{1/2}\{M_{2r}^*(n)\}^{1/2}\Big)\ .$$

But by the methods of Halberstam [5] or Delange [2] it is a straightforward matter to confirm that for n sufficiently large

$$M_l^*(n)=n(B_{\alpha_n}^*)^{l/2}(2\pi)^{-1/2}\int_{-\infty}^{\infty}\omega^l e^{-\omega^2/2}d\omega\{1+O(\varepsilon)\}\ ,\qquad l\leqq 2k\ ,$$

so that by (14) and (8)

$$M_l^*(n)=nB_n^{l/2}(2\pi)^{-1/2}\int_{-\infty}^{\infty}\omega^l e^{-\omega^2/2}d\omega\{1+O(\varepsilon)\}\ ,\qquad l\leqq 2k\ , \tag{15}$$

and, in particular

$$M_{2r}^*(n)=O(nB_n^r)\ ,\qquad r\leqq k\ .$$

Hence

$$M_k(n)-M_k^*(n)=O(nB_n^{k/2}\{\varepsilon+\varepsilon^{-1}\phi\}^{1/2})\ ;$$

now, whilst still keeping ε fixed, we let n tend to infinity, and obtain

$$\varlimsup_{n\to\infty}\left|\frac{M_k(n)}{nB_n^{k/2}}-\frac{M_k^*(n)}{nB_n^{k/2}}\right|=O(\varepsilon^{1/2})\,.$$

Thus, by (15) with $l=k$,

$$\varlimsup_{n\to\infty}\left|\frac{M_k(n)}{nB_n^{k/2}}-(2\pi)^{-1/2}\int_{-\infty}^{\infty}\omega^k e^{-\omega^2/2}d\omega\right|=O(\varepsilon^{1/2})\ .$$

Since the left side is entirely independent of ε, and yet the relation is true for every $\varepsilon<1/2$, we have now proved that

$$\lim_{n\to\infty}\frac{M_k(n)}{nB_n^{k/2}}=(2\pi)^{-1/2}\int_{-\infty}^{\infty}\omega^k e^{-\omega^2/2}d\omega$$

for every fixed $k=1, 2, 3, \cdots$.

This concludes the proof of Theorem 1 with condition (5) replaced by the pair of conditions (5a) and (4).

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CHARACTERISTIC DIRECTION FOR EQUATIONS OF MOTION OF NON-NEWTONIAN FLUIDS

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1. Introduction. According to the Reiner-Rivlin theory of non-Newtonian fluids,[1] the stress tensor t^i_j is given in terms of the rate of strain tensor d^i_j by relations of the form

$$t^i_j = -p\delta^i_j + \mathscr{F}_1 d^i_j + \mathscr{F}_2 d^i_k d^k_j , \tag{1}$$

where p is an arbitrary hydrostatic pressure, the $\mathscr{F}$'s are essentially arbitrary differentiable functions of

$$\mathrm{II} = -\frac{1}{2} d^i_j d^j_i , \qquad \mathrm{III} = \det d^i_j , \tag{2}$$

and d^i_j satisfies the incompressibility condition

$$d^i_i = 0 . \tag{3}$$

The tensors d^i_j and t^i_j are both symmetric.

It is known [2] that the characteristic directions of the corresponding equations of motion are the unit vectors ν_i satisfying

$$F(\nu_i) \equiv 2U^2 + 2UU^i_i + (U^i_i)^2 - U^i_j U^j_i = 0 , \tag{4}$$

where

$$U = \mathscr{F}_1 + \mathscr{F}_2 \mu^i \nu_i ,$$

$$U^i_j = \mathscr{F}_2 (d^i_j - \nu^i \mu_j) + 2(\mu^i - \nu^i \mu_k \nu^k)\left(\mu^m d_{mj} \frac{\partial \mathscr{F}_1}{\partial \mathrm{III}} - \mu_j \frac{\partial \mathscr{F}_1}{\partial \mathrm{II}} \right)$$

$$+ 2(d^i_m \mu^m - \nu^i \mu_m \mu^m)\left(\mu^n d_{nj} \frac{\partial \mathscr{F}_2}{\partial \mathrm{III}} - \mu_j \frac{\partial \mathscr{F}_2}{\partial \mathrm{II}} \right) ,$$

$$\mu_i = d_{ij} \nu^j .$$

Since $F(\nu_i)$ is a continuous function of ν_i on the compact set $\nu_i \nu^i = 1$, a necessary and sufficient condition that no real characteristic directions exist is that $F(\nu_i)$ be of one sign for all unit vectors. Using this fact, we obtain simpler necessary conditions which are shown to be sufficient when $\mathscr{F}_2 \equiv 0$.

2. Necessary conditions. Let d_1, d_2 and d_3 denote the eigenvalues of d^i_j. From (3),

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[1] This theory was proposed independently by Reiner [**4**] for compressible fluids, by Rivlin [**5**] for incompressible materials. We treat the latter case.

$$d_1+d_2+d_3=0 \,. \tag{5}$$

We restrict our attention to unit vectors ν_i which are perpendicular to an eigenvector of d^i_j and note that $F(\nu_i)$, being a continuous function of ν_i, must be of one sign for all unit vectors in order that no real characteristic directions exist. Given any unit vector ν_i perpendicular to an eigenvector e_i corresponding to d_3, we may introduce a rectangular Cartesian coordinate system such that, at a point, ν_i is parallel to the positive x^1-axis and e_i is parallel to the x^3-axis. Then

$$\begin{gathered}\nu_i=\delta_{i1}\,,\; d_{13}=d_{23}=d_{1i}d^i_3=d_{21}d^i_3=0\;,\\ 2d_{12}=(d_1-d_2)\sin 2\phi\;,\; d_{33}=d_3\;,\end{gathered}$$

where ϕ is the angle between ν_i and an eigenvector corresponding to d_1. Making these substitutions in $F(\nu_i)$, given by (4), we obtain, by a routine calculation,

$$\begin{aligned}F(\nu_i)=2[\mathscr{F}_1-\mathscr{F}_2 d_2]\Big\{\mathscr{F}_1-\mathscr{F}_2 d_3-\frac{1}{2}(d_1-d_2)^2\sin^2 2\phi\Big[\frac{\partial\mathscr{F}_1}{\partial \mathrm{II}}\\ -d_3\frac{\partial\mathscr{F}_2}{\partial \mathrm{II}}+d_3\frac{\partial\mathscr{F}_1}{\partial \mathrm{III}}-d_3^2\frac{\partial\mathscr{F}_2}{\partial \mathrm{III}}\Big]\Big\}\,,\end{aligned}\tag{6}$$

which must be of one sign for all real angles ϕ. This is clearly true if and only if it is of the same sign for $\phi=0$ and $\phi=\pi/4$. That is, either

$$[\mathscr{F}_1-\mathscr{F}_2 d_2][\mathscr{F}_1-\mathscr{F}_2 d_3]>0 \tag{7}$$

and

$$\begin{aligned}[\mathscr{F}_1-\mathscr{F}_2 d_2]\Big\{\mathscr{F}_1-\mathscr{F}_2 d_3-\frac{1}{2}(d_1-d_2)^2\Big[\frac{\partial\mathscr{F}_1}{\partial \mathrm{II}}\\ -d_3\frac{\partial\mathscr{F}_2}{\partial \mathrm{II}}+d_3\frac{\partial\mathscr{F}_1}{\partial \mathrm{II}}-d_3^2\frac{\partial\mathscr{F}_2}{\partial \mathrm{II}}\Big]\Big\}>0\,,\end{aligned}\tag{8}$$

or (7) and (8) hold simultaneously with the inequalities reversed. By similarly analyzing the cases where ν_i is perpendicular to eigenvectors of d^i_j corresponding to d_1 and d_2, we conclude that either

$$[\mathscr{F}_1-\mathscr{F}_2 d_i][\mathscr{F}_1-\mathscr{F}_2 d_j]>0 \qquad (i\neq j), \tag{9}$$

and

$$\begin{aligned}[\mathscr{F}_1-\mathscr{F}_2 d_j]\Big\{\mathscr{F}_1-\mathscr{F}_2 d_k-\frac{1}{2}(d_i-d_j)^2\Big[\frac{\partial\mathscr{F}_1}{\partial \mathrm{II}}\\ -d_k\frac{\partial\mathscr{F}_2}{\partial \mathrm{II}}+d_k\frac{\partial\mathscr{F}_1}{\partial \mathrm{III}}-d_k^2\frac{\partial\mathscr{F}_2}{\partial \mathrm{III}}\Big]\Big\}>0 \qquad (i,j,k\neq),\end{aligned}\tag{10}$$

or

$$[\mathscr{F}_1 - \mathscr{F}_2 d_i][\mathscr{F}_1 - \mathscr{F}_2 d_j] < 0 \qquad (i \neq j), \tag{11}$$

and (10) holds with the inequality reversed. Now (11) cannot hold for all i and j, so this possibility is ruled out. We thus have

THEOREM 1. *A necessary and sufficient condition that no real characteristic directions exist is that* $F(\nu_i) > 0$; *in order that there exist no real characteristic directions perpendicular to an eigenvector of* d^i_j, *it is necessary and sufficient that the inequalities* (9) *and* (10) *hold.*

For (9) and (10) to hold, it is necessary and sufficient that either

$$\mathscr{F}_1 - \mathscr{F}_2 d_i > 0 \tag{12}$$

and

$$\mathscr{F}_1 - \mathscr{F}_2 d_k - \frac{1}{2}(d_i - d_j)^2 \left[\frac{\partial \mathscr{F}_1}{\partial \mathrm{II}} - d_k \frac{\partial \mathscr{F}_2}{\partial \mathrm{II}} + d_k \frac{\partial \mathscr{F}_1}{\partial \mathrm{II}} - d_k^2 \frac{\partial \mathscr{F}_2}{\partial \mathrm{III}} \right] > 0 \qquad (i, j, k \neq), \tag{13}$$

or

$$\mathscr{F}_1 - \mathscr{F}_2 d_i < 0 \tag{14}$$

and

$$\mathscr{F}_1 - \mathscr{F}_2 d_k - \frac{1}{2}(d_i - d_j)^2 \left[\frac{\partial \mathscr{F}_1}{\partial \mathrm{II}} - d_k \frac{\partial \mathscr{F}_2}{\partial \mathrm{II}} + d_k \frac{\partial \mathscr{F}_1}{\partial \mathrm{III}} - d_k^2 \frac{\partial \mathscr{F}_2}{\partial \mathrm{III}} \right] < 0 \qquad (i, j, k \neq). \tag{15}$$

3. **Equivalent conditions.** Let t_i denote the eigenvalues of the stress tensor corresponding to the eigenvalue d_i of d_{mn} so that from (1),

$$t_i = -p + \mathscr{F}_1 d_i + \mathscr{F}_2 d_i^2 .$$

Using (5),

$$\begin{aligned} t_i - t_j &= [\mathscr{F}_1 + \mathscr{F}_2 (d_i + d_j)](d_i - d_j) \\ &= [\mathscr{F}_1 - \mathscr{F}_2 d_k](d_i - d_j) \qquad (i, j, k \neq). \end{aligned} \tag{16}$$

From (2) and (5),

$$\begin{aligned} \mathrm{II} &= -\frac{1}{2}(d_1^2 + d_2^2 + d_3^2) = -\frac{1}{4}(d_i - d_j)^2 - \frac{3}{4} d_k^2 , \\ \mathrm{III} &= d_1 d_2 d_3 = \frac{1}{4} d_k [d_k^2 - (d_i - d_j)^2] \qquad (i, j, k \neq). \end{aligned} \tag{17}$$

Using (16) and (17) to express $t_i - t_j$ as a function of $d_i - d_j$ and $d_k (i, j, k \neq)$, we calculate

$$(18)\quad \left.\frac{\partial(t_i-t_j)}{\partial(d_i-d_j)}\right|_{d_k=\text{const.}}$$

$$=\mathscr{F}_1-\mathscr{F}_2 d_k-\frac{1}{2}(d_i-d_j)^2\left[\frac{\partial\mathscr{F}_1}{\partial\mathrm{II}}-d_k\frac{\partial\mathscr{F}_2}{\partial\mathrm{II}}+d_k\frac{\partial\mathscr{F}_1}{\partial\mathrm{III}}-d_k^2\frac{\partial\mathscr{F}_2}{\partial\mathrm{III}}\right].$$

From (12), (13), (14), (15), (16), (18) and Theorem 1, we have

THEOREM 2. *When the eigenvalues of d^i_j are all unequal, a necessary and sufficient condition that there exist no real characteristic direction perpendicular to an eigenvector of d^i_j is that either*

$$(t_i-t_j)/(d_i-d_j)>0 \quad\text{and}\quad \partial(t_i-t_j)/\partial(d_i-d_j)|_{d_k=\text{const.}}>0\ ,$$

or

$$(t_i-t_j)/(d_i-d_j)<0 \quad\text{and}\quad \partial(t_i-t_j)/\partial(d_i-d_j)|_{d_k=\text{const.}}<0 \qquad (i,j,k\neq).$$

When (12) holds, the stress power Φ, given by

$$3\Phi=3t^i_jd^j_i=(t_1-t_2)(d_1-d_2)+(t_2-t_3)(d_2-d_3)+(t_3-t_1)(d_3-d_1)$$

is negative, a possibility which many writers exclude on thermodynamic grounds.

4. The case $\mathscr{F}_2\equiv 0$. When $\mathscr{F}_2\equiv 0$, $\mathscr{F}_1\neq 0$, the characteristic equation (4) has been shown [2] to reduce to

$$(19)\qquad G(\nu_i)\equiv\mathscr{F}_1+A^iB_i=0\ ,$$

where

$$A^i=2(\mu^i-\nu^i\mu_k\nu^k)\ ,$$

$$B_i=\mu^m d_{mi}\frac{\partial\mathscr{F}_1}{\partial\mathrm{III}}-\mu_i\frac{\partial\mathscr{F}_1}{\partial\mathrm{II}}\ .$$

In fact, $F(\nu_i)=2\mathscr{F}_1G(\nu_i)$. When $\mathscr{F}_2=0$, $\mathscr{F}_1=0$, every direction is characteristic, a case which we exclude. Using the Hamilton-Cayley theorem,

$$d^i_jd^j_kd^k_m=\mathrm{III}\delta^i_m-\mathrm{II}d^i_m\ ,$$

we can reduce (19) to the form

$$(20)\qquad G(\alpha,\beta)\equiv\mathscr{F}_1+2(\mathrm{III}-\mathrm{II}\alpha-\beta\alpha)\frac{\partial\mathscr{F}_1}{\partial\mathrm{III}}+2(\alpha^2-\beta)\frac{\partial\mathscr{F}_1}{\partial\mathrm{II}}=0\ ,$$

where

$$(21)\qquad \alpha=\mu_i\nu^i=d_{ij}\nu^i\nu^j\ ,\qquad \beta=\mu^i\mu_i=d^i_k d_{im}\nu^k\nu^m\ .$$

Now (21) is a mapping of the unit sphere $\nu_i\nu^i=1$ onto a region R in the $\alpha-\beta$ plane. The conditions

$$\frac{\partial G}{\partial\alpha}=-2(\mathrm{II}+\beta)\frac{\partial\mathscr{F}_1}{\partial\mathrm{III}}+4\alpha\frac{\partial\mathscr{F}_1}{\partial\mathrm{II}}=0\ ,$$

$$\frac{\partial G}{\partial\beta}=-2\alpha\frac{\partial\mathscr{F}_1}{\partial\mathrm{III}}-2\frac{\partial\mathscr{F}_1}{\partial\mathrm{II}}=0\ ,$$

$$\pm d^2G=\pm4\left[\frac{\partial\mathscr{F}_1}{\partial\mathrm{II}}d\alpha^2-\frac{\partial\mathscr{F}_1}{\partial\mathrm{III}}d\alpha d\beta\right]\geqq 0 \text{ for all } d\alpha, d\beta\ ,$$

must be satisfied at any interior point of R at which G is a maximum or minimum. These conditions cannot be satisfied unless $\partial\mathscr{F}_1/\partial\mathrm{II}=\partial\mathscr{F}_1/\partial\mathrm{III}=0$, in which case $G(\nu_i)$ is independent of ν_i, and $\mathscr{F}_1\neq0$ is then necessary and sufficient that there exist no real characteristics. From the implicit function theorem, values of ν_i corresponding to boundary points of R are such that the equations

$$d\alpha=2d_{ij}\nu^i d\nu^j\ ,\qquad d\beta=2d^i_k\, d_{im}\nu^k d\nu^m\ ,\qquad 0=\nu_i d\nu^i$$

do not admit a unique solution for $d\nu^i$ in terms of $d\alpha$ and $d\beta$. We thus have

THEOREM 3. *Maximum and minimum values of $G(\nu_i)$, hence of $F(\nu_i)$, hence of $F(\nu_i)$, occur only at values of ν_i such that the vectors ν_i, $d_{ij}\nu^j$ and $d^k_i d_{km}\nu^m$ are linearly dependent or, equivalently, at values such that the determinant D of these three vectors vanishes.*

Whatever be the unit vector ν_i, we can always choose rectangular Cartesian coordinates such that, at a point, $\nu_i=\delta_{i1}$, $d_{23}=0$. The condition $D=0$ then reduces to

$$0=\begin{vmatrix} 1 & 0 & 0 \\ d_{11} & d_{21} & d_{31} \\ d^2_{11}+d^2_{12}+d^2_{13} & d_{21}(d_{11}+d_{22}) & d_{31}(d_{11}+d_{33}) \end{vmatrix}$$
$$=d_{21}d_{31}(d_{33}-d_{22})\ .$$

If $d_{21}=0(d_{31}=0)$, $\delta_{i2}(\delta_{i3})$ is an eigenvector of d_{ij}. If $d_{21}d_{31}\neq0$, $d_{33}=d_{22}$, the vector with components $(0, d_{31}, -d_{21})$ is an eigenvector of d_{ij}, whence follows

THEOREM 4. *The vectors ν_i, $d_{ij}\nu^j$, $d^k_i d_{km}\nu^m$ can be linearly dependent only when ν_i is perpendicular to an eigenvector of d^i_j.*

Theorems 3 and 4 imply that, when $\mathscr{F}_2\equiv0$, we will have $F(\nu_i)>0$ for all unit vectors ν_i if and only if $F(\nu_i)>0$ for each unit vector ν_i which is perpendicular to an eigenvector of d^i_j. From Theorem 1, we then deduce

THEOREM 5. *When* $\mathscr{F}_2 \equiv 0$, *a necessary and sufficient condition that there exist no real characteristic directions is that the inequalities* (9) *and* (10) *hold.*

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APPLIED MATHEMATICS BRANCH, MECHANICS DIVISION, U. S. NAVAL RESEARCH LABORATORY.

ON TWO THEOREMS OF PHRAGMÉN-LINDELÖF FOR LINEAR ELLIPTIC AND PARABOLIC DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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1. Introduction. In Part I of this paper our main interest is to generalize to elliptic equations the following theorem of Phragmén-Lindelöf:

THEOREM 0. *If* $f(z) \to a$ *as* $z \to \infty$ *along two straight lines, and* $f(z)$ *is regular and bounded in the angle between them, then* $f(z) \to a$ *uniformly in the whole angle as* $z \to \infty$.

A generalization of the classic Phragmén-Lindelöf theorem to elliptic equations was given by Gilbarg [1] and Hopf [4]. A refined form of that classic theorem, due to the Nevanlinnas [5], [6; 42–44] and Heins [3], was generalized to elliptic equations by Serrin [8].

In generalizing Theorem 0 we shall make an extensive use of the Gilbarg-Hopf results.

In Part II we generalize to parabolic equations both the classic Phragmén-Lindelöf Theorem and Theorem 0.

In § 2, Theorem 0 is proved for elliptic equations defined in any 2-dimensional domains (Theorems 1, 2). The case $n>2$ is treated in § 3, for domains contained in a half space. In § 4 we consider the behavior of solutions in an angular neighborhood of the origin, and we obtain results similar to those of §§ 2, 3. In §§ 5, 6, generalizations to parabolic equations are given: Theorems 7, 9 extend the classic Phragmén-Lindelöf Theorem and Theorems 8, 10 extend Theorem 0.

The results in Part I are somewhat analogous with Theorems 2, 3, 3′ of Gilbarg-Serrin's paper [2]. The similarity appears both in the type of conditions imposed on the coefficients of the elliptic operator and in the assertions. It is however important to note that our results cannot be obtained by the Gilbarg-Serrin methods, since Harnack Inequalities which play an essential role in their paper, do not hold uniformly in open domains.

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Part I

2. Consider the differential operator

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} \qquad x=(x_1, \cdots, x_n) \tag{1}$$

defined in a domain D. In this and the following chapter D is supposed to be unbounded. We denote by ∂D the boundary of D, and by $\bar{D}$ the closure of D. We shall assume throughout Part I that L satisfies the following conditions ([1], [4]):

(i) $\sum_{i,j} |a_{ij}(x)|$ is bounded in D, and, for all $x \in D$, ξ_i real,

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geqq \alpha \sum_i \xi_i^2 \qquad \alpha > 0 ,$$

(ii) for all $x \in D$, $|x| = r$,

$$\sum_i |b_i(x)| \leqq p(r) , \tag{2}$$

where $p(r)$, defined for $0 < r < \infty$, is monotone decreasing and

$$\int_0^\infty p(r)\, dr < \infty .$$

Define $a_{ij}(\infty) = \lim a_{ij}(x)$ as $|x| \to \infty$ $(x \in D)$, whenever the limit exists. The matrix $(a_{ij}(x))$ is said to be *Dini continuous* at infinity, if there exists a monotone decreasing function $\varphi(r)$ with $\int^\infty r^{-1}\varphi(r)\, dr < \infty$, such that for $x \in D$, $|x| = r$,

$$\sum_{i,j} |a_{ij}(x) - a_{ij}(\infty)| \leqq \varphi(r) .$$

Let $u(x)$ be defined in D and belong to $C^2(D)$. In Theorems 1–6 the function $u(x)$ is also assumed to be continuous in $\bar{D}$. Denote

$$m(r) = \inf_{x \in D,\ |x|=r} u(x) , \qquad \mu(r) = \sup_{x \in D,\ |x|=r} |u(x)| .$$

Let K_β denote the n-dimensional cone with angular opening β, $0 < \beta \leqq 2\pi$, whose axis is the positive x_n-axis and whose vertex is at the origin.

LEMMA 1. *Suppose $D \subset K_\beta$, $n=2$. Assume that L satisfies* (i), (ii) *and that $(a_{ij}(x))$ is continuous at infinity with $a_{ij}(\infty) = \delta_{ij}$. If $Lu(x) \leqq 0$ in the open set $D_{r_0} = D \cap |x| > r_0$, $u(x) \geqq 0$ on ∂D_{r_0} and for some $\gamma' < \gamma = \pi/\beta$,*

$$\lim_{k\to\infty} r_k^{-\gamma'} m(r_k) = 0 \qquad (r_k \to \infty \text{ as } k \to \infty) ,$$

and if r_0 *is sufficiently large (depending only on* L, β *and* γ'), *then* $u(x) \geqq 0$ *in* D_{r_0}.

By $u(x) \geqq 0$ on ∂G we mean: $\liminf u(x) \geqq 0$ as x tends to ∂G $(x \in G)$.

Proof. Following the Gilbarg-Hopf method, it is enough to prove the existence of functions $v_R(x)$, $r_0 < R < \infty$, with the following properties:

$$(3) \qquad \begin{aligned} &v_R(x) \geqq 0 \quad \text{if } |x| \leqq R, \quad x \in \partial D_{r_0} \\ &v_R(x) = 1 \quad \text{if } |x| = R, \quad x \in D_{r_0}, \end{aligned}$$

$$(4) \qquad Lv_R(x) \leqq 0 \quad \text{if } |x| < R, \quad x \in D_{r_0},$$

$$(5) \qquad \text{for every } x \in D_{r_0}, \quad R^{\gamma'} v_R(x) \text{ is bounded as } R \to \infty .$$

Denote by $h(x_1', x_2')$ the harmonic function defined in the semicircle C': $x_1'^2 + x_2'^2 < 1$, $x_2' > 0$, which takes the value 0 on the diameter and the value 1 on the rest of the boundary. The transformation $z' = z^\delta$, where $\gamma' < \delta < \gamma$, $z' = x_2' + ix_1'$, $z = x_2 + ix_1$, maps $S = K_\beta \cap |x| < 1$ onto a domain $S' \subset C'$. The function $k(x_1, x_2) = h(x_1', x_2')$ is harmonic in S and takes boundary values $\geqq 0$ on the radii and the value 1 on the rest of the boundary. We shall find $v_R(x)$ in the form $v_R(x) = f_R\left(k\left(\frac{x}{R}\right)\right)$.

If we show, in addition to $Lf_R \leqq 0$, that

$$(6) \qquad f_R(0) = 0, \quad f_R(1) = 1, \quad 0 \leqq f_R(k) \leqq 1 \quad \text{if} \quad 0 \leqq k \leqq 1, \quad \text{and}$$

$$(7) \qquad f_R(k) = 0(k^{\gamma'/\delta}) \text{ uniformly in } R, \quad \text{as } k \to 0,$$

then (3), (4), (5) follow. Note, in proving (5), that $R^\delta k\left(\frac{x}{R}\right)$ is bounded as $R \to \infty$. The construction of f_R proceeds as in Hopf's proof [4], except for the facts that property d) p. 421 and the inequality

$$(8) \qquad \sum_{i,j} \frac{|h_{ij}''(x)|}{|h'(x)|^2} < C \qquad 0 < |x| < 1$$

do not hold for the corresponding k.

The image of K_β under the mapping $z' = z^\delta$ is a 2-dimensional cone $K_{\pi-\varepsilon}'$ $(\varepsilon > 0)$ with opening $\pi - \varepsilon$ and $S' \subset K_{\pi-\varepsilon}'$. From Hopf's proof it is clear that instead of satisfying d), it is enough for k to satisfy:

d') along each equipotential arc $k(x) = \text{const.}$,

$$|k'(x)| = \left(\sum \left(\frac{\partial k(x)}{\partial x_i}\right)^2\right)^{1/2} \geqq H \frac{\partial k}{\partial x_2}$$

on the axis of x_2 (say at $\tilde{x}$), $H > 0$. Since the equipotential arcs of $k(x)$ is S correspond to equipotental arcs of $h(x')$ in S', we have

$$|k'(x)|=|h'(x')|\left|\frac{dz'}{dz}\right| \geqq \frac{\partial h(\tilde{x}')}{\partial x_2'}\delta|x|^{\delta-1}=\frac{\partial h(\tilde{x}')}{\partial x_2'}\delta|x'|^{(\delta-1)/\delta}$$
$$\geqq H\frac{\partial h(\tilde{x}')}{\partial x_2'}\delta|\tilde{x}'|^{(\delta-1)/\delta}=H\frac{\partial k(\tilde{x})}{\partial x_2},$$

where $\tilde{x}'$ is the image of $\tilde{x}$ and $H>0$. Here, in the case $\delta<1$, we used the inequality $|x'|<H_1|\tilde{x}'|$ $(H_1>0)$, noting that $S'\subset K'_{\pi-\varepsilon}$.

The estimation of $\sum a_{ij}(x)k''_{ij}(\xi)$ in Lk (see [4; p. 423]) has to be modified, since (8) does not hold for k. Defining

$$\varepsilon_{ij}(x)=a_{ij}(x)-\delta_{ij}, \qquad \varepsilon(r)=\sup_{x\in D,\ |x|=r}\sum|\varepsilon_{ij}(x)|, \tag{9}$$

and using the harmonicity of k, we get

$$I=|k'(\xi)|^{-2}|\sum a_{ij}(x)k''_{ij}(\xi)|\leqq AC+\sum|\varepsilon_{ij}(x)|\frac{|\delta-1|\,|\xi|^{\delta-2}}{\delta|h'(\xi')|\,|\xi|^{2(\delta-1)}}$$
$$\leqq AC+\frac{B\varepsilon(r)}{2|\xi|^{\delta}},$$

where A and B are constants, and $|\xi|<1$.

Using the inequality $2|\xi'|\geqq h(\xi')$ ([1; p. 414]), we obtain

$$I\leqq AC+B\varepsilon(r)k^{-1}.$$

Define r_0 to be such that if $r>r_0$ then $B\varepsilon(r)<1-\gamma'/\delta$. Then, the last inequality for I shows that Hopf's method can be applied to prove that $Lf_R\leqq 0$, provided that f_R satisfy:

$$\frac{f''(k)}{f'(k)}=-AC-\frac{1-\gamma'/\delta}{k}-\frac{P(x_2)}{H(\partial k(\tilde{x})/\partial x_2)}, \qquad f'(k)>0, \tag{10}$$

where $\tilde{x}=(0, x_2)$ (k is a monotone function of x_2).
Solving (10) we obtain,

$$f'_R(k)=Ek^{\gamma'/\delta-1}\exp(-ACk-P(x_2)), \qquad f_R(0)=0, \tag{11}$$

where

$$E^{-1}=\int_0^1 k^{\gamma'/\delta-1}\exp(-AC-P(x_2))\,dk, \qquad P(s)=H^{-1}\int_0^s p(t)\,dt.$$

The verification of (6), (7) is immediate and the proof is thereby completed.

LEMMA 2. *Suppose $D\subset K_\beta$, $n=2$. Assume that L satisfies* (i), (ii) *and that $(a_{ij}(x))$ is continuous at infinity with $a_{ij}(\infty)=\delta_{ij}$. If r_0 is sufficiently large, then there exists a function $w(x)$, defined in D_{r_0}, and having the following properties:*

(a) $w(x) \geqq 0$ *if* $x \in \partial D_{r_0}$,

(b) $w(x)=1$ *if* $x \in D$, $|x|=r_0$,

(c) $Lw(x) \leqq 0$ *if* $x \in D_{r_0}$, *and*

(d) $w(x) \to 0$ *uniformly in* D_{r_0} *as* $|x| \to \infty$.

Proof. To prove the lemma, define $\tilde{v}(x')=\frac{2}{\pi}\vartheta(x')$, where $\vartheta(x')$ is the polar angle of the point x' with $(-r_0', r_0')$ as a pole. Define also $v(x)=\tilde{v}(x')$, where x' is the image of x under the mapping $z'=z^{\gamma}$, where $\gamma=\pi/\beta$, $z'=x_2'+ix_1'$, $z=x_2+ix_1$. We try to find w in the form $w=f(v)$.
(c) implies that

$$f''(v)\sum_{i,j} a_{ij}(x)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}+f'(v)\Big(\sum_{i,j} a_{ij}(x)\frac{\partial^2 v}{\partial x_i\partial x_j}+\sum_i b_i(x)\frac{\partial v}{\partial x_i}\Big)\leqq 0 . \tag{12}$$

Using the harmonicity of $v(x)$ we conclude, after some calculations (see [1; p. 414]), that (12) is a consequence of the inequalities:

$$\frac{f''(v)}{f'(v)}<-A_1\varepsilon(|x|)\frac{|z'|}{r_0'}-A_2|x|p(|x|)\frac{|z'|}{r_0'} , \qquad f'(v)>0 , \tag{13}$$

where A_1, A_2 are proper constants and $\varepsilon(r)$ is defined by (g).
Taking r_0 to be such that $2A_1\varepsilon(r)+2A_2rp(r)<1-\delta$ $(0<\delta<1)$ if $r>r_0$ (note that $rp(r) \to 0$), and using the elementary inequalities

$$|z'| \leqq r_0' \operatorname{ctg} \vartheta'/2 \leqq 2r_0'/\tilde{v}(x') ,$$

we conclude that if $f(v)$ satisfies:

$$f''(v)/f'(v)=-(1-\delta)/v , \qquad f'(v)>0 , \tag{14}$$

then (13) follows. Solving (14) we find that the function $f(v)=v^{\delta}$ satisfies (a)–(d).

THEOREM 1. *Suppose* $D \subset K_\beta$, $n=2$, *and assume that* L *satisfies* (i), (ii) *and that* $(a_{ij}(x))$ *is continuous at infinity with* $a_{ij}(\infty)=\delta_{ij}$. *If* $Lu(x)=0$ *in* D, *and, for some* η,

$$\lim_{r\to\infty}\frac{\mu(r)}{r^{\pi/\beta-\eta}}=0 \qquad (\eta>0 \textit{ if } \beta\neq\pi , \quad \eta=0 \textit{ if } \beta=\pi) , \tag{15}$$

and if $u(x) \to 0$ *on* ∂D *as* $|x| \to \infty$, *then* $u(x) \to 0$ *uniformly in* D *as* $|x| \to \infty$.

Proof. Given $\varepsilon>0$, there exists $r_0>0$ such that $-\varepsilon<u(x)<\varepsilon$ for

$x \in \partial D$, $|x| \geqq r_0$. Denoting $M_0 = \max_{|x|=r_0} |u(x)|$, we can apply Lemma 1 (in the case $\beta = \pi$ we apply the Gilbarg-Hopf theorem) to the function $v(x) = u(x) + M_0 w(x) + \varepsilon$ in the open set D_{r_0}. We get $v(x) \geqq 0$ in D_{r_0}. Taking r_1 to be such that $M_0 w(x) < \varepsilon$ in D_{r_1}, we conclude that $u(x) > -2\varepsilon$ in D_{r_1}. Similarly we get $u(x) < 2\varepsilon$ in D_{r_1} and the theorem is proved.

REMARK. Using a proper linear transformation we conclude that the assumption $a_{ij}(\infty) = \delta_{ij}$, can be dismissed if in (15) β is replaced by β', where β' is the angular opening of the image of K_β under the linear transformation. The continuity assumption of the $a_{ij}(x)$ at infinity can be replaced by the weaker assumption that the oscillation of the $a_{ij}(x)$ near infinity is sufficiently small.

We can reduce the case $0 < \beta \leqq 2\pi$ to the case $\beta = \pi$ by the conformal mapping $z' = z^{\pi/\beta}$, where $z = x_2 + ix_1$, $z' = x_2' + ix_1'$. Applying Theorem 1, we get the following theorem after some calculation.

THEOREM 2. *Let $D \subset K_\beta$, $n = 2$, and assume that L satisfies* (i), (ii), *that $(a_{ij}(x))$ is Dini continuous at infinity with $a_{ij}(\infty) = \delta_{ij}$, and that $r^{1-\gamma} p(r)$ $(\gamma = \pi/\beta)$ is monotone decreasing. If $Lu(x) = 0$ in D, and*

$$\lim_{r \to \infty} \frac{\mu(r)}{r^{\pi/\beta}} = 0 , \tag{16}$$

and if $u(x) \to 0$ on ∂D as $|x| \to \infty$, then $u(x) \to 0$ uniformly in D as $|x| \to \infty$.

As in Theorem 1, the restriction $a_{ij}(\infty) = \delta_{ij}$ can be dismissed, but then in (16) and in $r^{1-\gamma} p(r)$, β should be replaced by β'.

In analogue with Theorem 2, one can formulate an extension of the Gilbarg-Hopf theorem to the case $0 < \beta \leqq 2\pi$. Serrin's results [8] can also be extended to domains $D \subset K_\beta$ $(0 < \beta \leqq 2\pi)$ such that the image of D under the mapping $z' = z^{\pi/\beta}$ contains a half plane $x_2' > c$. In particular we have the following.

If $Lu \leqq 0$ in D and $u \geqq 0$ on ∂D, then $\lim_{r \to \infty} r^{-\pi/\beta} m(r)$ exists and is $\leqq 0$.

3. In this section we consider the case $n \geqq 3$.

LEMMA 3. *Suppose $D \subset K_\beta$, $\dfrac{\pi}{3} \leqq \beta < \pi$, $n \geqq 3$. Assume that L satisfies* (i), (ii) *and that $(a_{ij}(x))$ is continuous at infinity with $a_{ij}(\infty) = \delta_{ij}$. If $Lu(x) \leqq 0$ in D_{r_0}, $u(x) \geqq 0$ on ∂D_{r_0}, and, for some $\gamma' < \gamma = \pi/\beta$,*

$$\lim_{k \to \infty} r_k^{-\gamma'} m(r_k) = 0 \qquad (r_k \to \infty \text{ as } k \to \infty) ,$$

and if r_0 *is sufficiently large, then* $u(x) \geqq 0$ *in* D_{r_0}.

Proof. The proof proceeds as in Lemma 1, if (following Hopf [4]), we define

$$K(x) = k(\rho, x_n) , \qquad \rho = \sqrt{x_1^2 + \cdots + x_{n-1}^2} = \sqrt{r^2 - x_n^2} , \qquad 0 < r < 1 ,$$

where k is the function defined in the proof of Lemma 1. The only essential difference will be in estimating $\sum a_{ij}(x) K''_{ij}(\xi)$. Clearly,

$$\sum K''_{ii}(x) = (n-2) \frac{1}{\rho} \frac{\partial k}{\partial \rho} ,$$

and

$$\sum |K''_{ij}(x)| \leqq A_3 \sum |k''_{ij}| + A_4 \frac{1}{\rho} \left| \frac{\partial k}{\partial \rho} \right| \qquad (|x| < 1 , \ A_3 > 0 , \ A_4 > 0) .$$

If we show that

$$J \equiv \frac{1}{\rho} \frac{\partial k}{\partial \rho} \Big/ |k'|^2 \leqq B_1 \qquad \text{and} \quad |J| \leqq B_1 + \frac{B_2}{k} , \tag{17}$$

where B_1 and B_2 are positive constants, then we can proceed as in the proof of Lemma 1, and the proof of Lemma 3 will be completed.

To prove the first part of (17), we write J in the form

$$J = \frac{|z|^{\delta-1} \sin \delta\vartheta}{\sin \vartheta} \frac{1}{\rho'} \frac{|z|^{\delta-1} \cos (\delta-1)\vartheta}{|h'(z')|^2 \delta |z|^{2(\delta-1)}} \frac{\partial h}{\partial \rho'}$$
$$- \frac{1}{|z| \sin \vartheta} \frac{|z|^{\delta-1} \sin (\delta-1)\vartheta}{|h'(z')|^2 \delta |z|^{2(\delta-1)}} \frac{\partial h}{\partial x'_n} = J_1 + J_2$$

where J_1 is the first term and $z' = z^\delta$, $z = x_n + i\rho$, $z' = x'_n + i\rho'$, $\rho = |z| \sin \vartheta$, etc.. Since $\dfrac{1}{\rho'} \dfrac{\partial h}{\partial \rho'}$ is bounded near $\rho' = 0$, and since $|h'(z')|$ is bounded from below by a positive constant, we get $|J_1| \leqq B_1$.

Since $\dfrac{\partial h(z')}{\partial x'_n} \geqq 0$ and $\dfrac{\sin (\delta-1)\vartheta}{\sin \vartheta} \geqq 0$ if $1 < \delta < 3$ (since $1 < \gamma \leqq 3$ we can take $1 < \delta < 3$), it follows that $J_2 \leqq 0$ and consequently, $J \leqq B_1$.

The second part of (17) follows from noting that $|J_2| \leqq \dfrac{B_2}{2|z|^\delta} \leqq \dfrac{B_2}{k}$.

LEMMA 4. *Lemma 2 is true also in the case* $n \geqq 3$.

Proof. The function $t(x) = r_0^{n-2} |x|^{2-n}$ satisfies (a), (b) and (d). We shall find $w(x)$ in the form $f(t)$. Condition (c) implies that

$$(18)\qquad f''(t)\sum_{i,j} a_{ij}(x)\frac{(n-2)^2 x_i x_j r_0^{n-2}}{|x|^{2n}} + f'(t)\Big(\sum_{i,j} a_{ij}(x)\frac{n(n-2)x_i x_j}{|x|^{n+2}} - \sum_i a_{ij}(x)\frac{n-2}{|x|^n} - \sum_i b_i(x)\frac{(n-2)x_i}{|x|^n}\Big) \leqq 0 \,.$$

By our assumptions, $\sum |a_{ij}(x)-\delta_{ij}| \leqq \varepsilon(|x|) \to 0$ as $|x| \to 0$. Using the harmonicity of $|x|^{2-n}$, we find that if $f(t)$ satisfies

$$(19)\qquad f''(t)/f'(t) < -(B_1\varepsilon(|x|)+B_2|x|p(|x|))/t \,, \qquad f'(t)>0 \,,$$

where B_1 and B_2 are proper constants, then (18) follows. Now, if r_0 is such that $B_1\varepsilon(r)+B_2 rp(r) < 1-\delta$ $(0<\delta<1)$ for $r>r_0$, and if

$$(20)\qquad f''(t)/f'(t) = -(1-\delta)t^{-1} \,, \qquad f'(t)>0 \,,$$

then (19) follows. Solving (20) we get the function $f(t)=t^\delta$, which satisfies (a)–(d).

With Lemmas 2 and 3 at hand, we can use the argument used in proving Theorem 1 and thus get the following.

THEOREM 3. *Suppose* $D\subset K_\beta$, $\frac{\pi}{3} \leqq \beta \leqq \pi$, $n \geqq 3$. *Assume that* L *satisfies* (i), (ii) *and that* $a_{ij}(x))$ *is continuous at infinity with* $a_{ij}(\infty)=\delta_{ij}$. *If* $Lu(x)=0$ *in* D, *and for some* η,

$$\lim_{r\to\infty} \frac{\mu(r)}{r^{\pi/\beta-\eta}} = 0 \qquad (\eta>0 \textit{ if } \beta\neq\pi \,,\ \eta=0 \textit{ if } \beta=\pi) \,,$$

and if $u(x)\to 0$ *on* ∂D *as* $|x|\to\infty$, *then* $u(x)\to 0$ *uniformly in* D *as* $|x|\to\infty$.

REMARKS. (a) The remark which follows Theorem 1, applies also to Theorem 3.

(b) If we assume in Theorem 3, that $u(x)=0(r^{2-n+\delta})$, $\delta>0$ on ∂D then the same holds in D. This follows by applying the maximum principle to functions of the form $u(x)\pm Ar^{2-n+\delta}\pm\varepsilon$, where A is a proper fixed constant and $\varepsilon>0$ (compare [2; 324–325]).

4. Let D belong to the half space $x_n>0$ and denote by C_r the open set $D\cap|x|<r$. We shall consider the behavior of solutions near $x=0$; it is therefore assumed that $0\in\overline{D}$.

We first observe that the construction of $w(x)$ in Lemma 4, can be easily modified to derive functions $w_r(x)$ defined in $C'_r=C_{r_0}\cap|x|>r$ for all $0<r<r_0$, and having the following properties:

(a) $w_r(x) \geqq 0$ if $x \in \partial C'_r$,

(b) $w_r(x)=1$ if $x \in C_{r_0}$, $|x|=r$,

(c) $Lw_r(x) \leqq 0$ in C'_r, and

(d) there exists δ $(0<\delta<1)$ depending on r_0 ($\delta \to 1$ as $r_0 \to 0$), such that

$$\lim_{r\to 0} r^{\delta(2-n)} w_r(x)=0 \qquad \text{if} \quad x \in C_{r_0} ;$$

here, r_0 is assumed to be sufficiently small, and, $(a_{ij}(x))$ is assumed to be continuous at $x=0$ with $a_{ij}(0)=\delta_{ij}$.

With the aid of $w_r(x)$ we can prove an analogue of the Gilbarg-Hopf theorem.

If $Lu \leqq 0$ in C_{r_0}, $u \geqq 0$ on ∂C_{r_0} and

$$\lim_{r\to 0} r^{\delta(n-2)} m(r)=0 \qquad (0<\delta<1) ,$$

and if r_0 is sufficiently small (depending on δ), then $u \geqq 0$ in C_{r_0}.

We can now use the method used in proving Theorem 1, noting that the role that $w(x)$ played in that proof is now given to the function $f_{r_0}\left(h\left(\frac{x}{r_0}\right)\right)$ of Gilbarg-Hopf. The following theorem is thus proved.

THEOREM 4. *Let D belong to the half space $x_n>0$, $n \geqq 3$. Assume that L satisfies* (i), (ii) *and that $(a_{ij}(x))$ is continuous at $x=0$. If $Lu(x) =0$ in D, and, for some positive ε,*

$$\lim_{r\to 0} r^{n-2-\varepsilon}\mu(r)=0 ,$$

and if $u(x) \to 0$ on ∂D as $|x| \to 0$, then $u(x) \to 0$ uniformly in D as $|x| \to 0$.

The continuity assumption on the $a_{ij}(x)$ at $x=0$, can be weakened.

The case $n=2$ can be treated in a similar manner. Note that now, instead of modifying Lemma 4, we rather modify Lemma 2 and thus obtain $w_r(x)$ in the form $\left(\frac{2}{\pi}\vartheta(x'_1, x'_2)\right)^{\delta}$, where (x'_1, x'_2) is the image of (x_1, x_2) under the mapping $z'=z^{\pi/\beta}$. We have the following.

THEOREM 5. *Let $D \subset K_\beta$, $n=2$, and assume that L satisfies* (i), (ii) *and that $(a_{ij}(x))$ is continuous at $x=0$ with $a_{ij}(0)=\delta_{ij}$. If $Lu(x)=0$ in D, and, for some positive ε,*

$$\lim_{r\to 0} r^{\pi/\beta-\varepsilon}\mu(r)=0 ,$$

and if $u(x) \to 0$ *on* ∂D *as* $|x| \to 0$, *then* $u(x) \to 0$ *uniformly in* D *as* $|x| \to 0$.

Another way to treat the case $n=2$, is to reduce it to Theorem 1, using the mapping $z' = z^{-\pi/\beta}$. We thus get the following.

THEOREM 6. *Let* $D \subset K_\beta$, $n=2$, *and assume that* L *satisfies* (i), (ii) *and that* $(a_{ij}(x))$ *is Dini continuous at* $x=0$ *with* $a_{ij}(0)=\delta_{ij}$. *Assume further that* $r^{1+\pi/\beta}p(r)$ *is monotone increasing. If* $Lu(x)=0$ *in* D *and*

$$\lim_{r\to 0} r^{\pi/\beta}\mu(r)=0 ,$$

and if $u(x) \to 0$ *on* ∂D *as* $|x| \to 0$, *then* $u(x) \to 0$ *uniformly in* D *as* $|x| \to 0$.

By using the same mapping $z' = z^{-\pi/\beta}$, we can derive theorems analogous with the Gilbarg-Hopf ([1], [4]) and Serrin's ([8]) theorems, provided that L satisfies the assumptions of Theorem 6.

In the case $n \geqq 3$, $\beta \leqq \pi$, such theorems can also be obtained, by using the transformation $x_i' = x_i/|x|^n$ $(i=1, \cdots, n)$.

PART II

5. Let $x=(x_1, \cdots, x_n)$ **and denote** $X=(x, t)$, $|X|=(|x|^2+t^2)^{1/2}$. Consider the operator

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(X) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(X) \frac{\partial u}{\partial x_i} - \frac{\partial u}{\partial t} , \tag{1}$$

defined in an unbounded domain D. We shall assume that L satisfies the following conditions:

(i) $\sum_{i,j} |a_{ij}(X)|$ is bounded in D, and, for all $X \in D$, ξ_i real,

$$\sum_{i,j} a_{ij}(X)\xi_i\xi_j \geqq \alpha \sum_i \xi_i^2 \qquad (\alpha > 0) ,$$

(ii) for all $X \in D$, $|X|=R$,

$$|\sum_i x_i b_i(X)| \leqq p(R) , \tag{2}$$

where $p(R)$ $(0<R<\infty)$ is bounded and $p(R) \to 0$ as $R \to \infty$.

Beside the functions $m(R)$, $\mu(R)$ defined in Part I, we introduce the functions

$$m'(R) = \inf_{X \in T_R} u(X) , \qquad \mu'(R) = \sup_{X \in T_R} |u(X)| ,$$

where $T_R \equiv D \cap |x|^2 + |t| = R$.

Let K_β denote the cone with angular opening β, whose axis is the positive t-axis and whose vertex is in the origin. In what follows, $u(X)$ is assumed to belong to $C^2(D)$. In Theorems 8, 10 $u(X)$ is also assumed to be continuous in $\bar{D}$.

THEOREM 7. *Let D belong to the half space $t>0$, and assume that L satisfies* (i), (ii). *If $u(X)\geqq 0$ on ∂D, $Lu(X)\leqq 0$ in D, and if*

$$\lim_{k\to\infty}\frac{m(R_k)}{R_k^2}=0 \qquad (R_k\to\infty \text{ as } k\to\infty), \tag{3}$$

then $u(X)\geqq 0$ in D.

Proof. The function $v_R(X)=(|x|^2+(t+K)^2)/R^2$ $(K>0)$ has the following properties:

(a) $v_R(X)\geqq 0$ if $X\in\partial D$, $|X|\leqq R$,

(b) $v_R(X)\geqq 1$ if $X\in D$, $|X|=R$,

(c) $Lv_R(X)<0$ in $C_R=D\cap|X|<R$, if K is sufficiently large, and

(d) $R^2v_R(X)$ is bounded, for every X, as $R\to\infty$.

The function $\tilde{u}(X)=u(X)-\sigma(R)v_R(X)$, where $\sigma(R)=\min(0, m(R))$, is nonnegative on ∂C_R and $Lu(X)\leqq 0$ in C_R. Applying the (weak) minimum principle [7], we conclude that $\tilde{u}(X)\geqq 0$ in C_R. Taking $R=R_k\to\infty$ and using (3), we get $u(X)\geqq 0$.

REMARK. It is clear that the same proof holds under weaker assumptions on L: (ii) may be replaced by $\sum x_ib_i(X)\leqq H$, where H is a constant, and in (i), the boundedness of $\sum|a_{ij}(X)|$ in D may be replaced by the boundedness of $\sum a_{ii}(X)$ in D and the boundedness of $\sum|a_{ij}(X)|$ in each C_R.

LEMMA 5. *Let D belong to the half space $t>0$, and assume that L satisfies* (i), (ii). *If R_0 is sufficiently large, then there exists a function $w(X)$ defined in $D_{R_0}=D\cap|X|>R_0$, and having the following properties*:

(a) $w(X)\geqq 0$ *if* $X\in\partial D_{R_0}$,

(b) $w(X)\geqq 1$ *if* $X\in D$, $|X|=R_0$,

(c) $Lw(X)\leqq 0$ *in* D_{R_0}, *and*

(d) $w(X)\to 0$ *uniformly in* D_{R_0} *as* $|X|\to\infty$.

Proof. Define

$$w(X)=\frac{C}{(t+1)^{\varepsilon}}\exp\left(\frac{-H|x|^2}{t+1}\right) \qquad (C>0,\ \varepsilon>0,\ H>0)\ .$$

Since $W(X)>0$ if $|X|=R_0$, $t\geqq 0$, we can choose C such that (b) is satisfied. Since (a) and (d) are also satisfied, it remains to verify (c).

$$Lw=w\left\{\sum a_{ij}\frac{4H^2x_ix_j}{(t+1)^2}-\sum a_{ii}\frac{2H}{t+1}-\sum x_ib_i\frac{2H}{t+1}+\frac{\varepsilon}{t+1}-\frac{H|x|^2}{(t+1)^2}\right\}\ ;$$

consequently, if

$$(4)\qquad 4H\sum a_{ij}x_ix_j\leqq |x|^2\ ,\qquad 2H\sum a_{ii}+2H\sum x_ib_i\geqq\varepsilon\ ,$$

then $Lw\leqq 0$. Obviously we can choose H and ε such that (4) is satisfied.

With Theorem 7 and Lemma 5 at hand, we can now proceed as in the proof of Theorem 1 and get the following.

THEOREM 8. *Let D belong to the half space $t>0$, and assume that L satisfies* (i), (ii). *If $Lu(X)=0$ in D and*

$$(5)\qquad \lim_{R\to\infty}\frac{\mu(R)}{R^2}=0\ ,$$

and if $u(X)\to 0$ on ∂D as $|X|\to\infty$, then $u(X)\to 0$ uniformly in D as $|X|\to\infty$.

Theorems 7, 8 are not true for domains D in the half space $t<0$. As an example take D to be the whole half space $t<0$, and take $u(x,t)=t^{1/m}$, where m is an odd positive integer. Then

$$u=0\quad\text{on}\quad t=0\ ,\qquad Lu=-\frac{1}{m}t^{1/m-1}<0\quad\text{if}\quad t<0\ ,$$

$$\lim_{R\to\infty}\frac{\mu(R)}{R^{\varepsilon}}=0\quad\text{if}\quad\frac{1}{m}<\varepsilon\ ,$$

but $u(X)<0$ if $t<0$, and $\lim u(X)$ does not exist as $|X|\to\infty$, $t\leqq 0$.

6. THEOREM 9. *Let $D\subset K_\beta$, $0<\beta<2\pi$, and assume that L satisfies* (i), (ii). *If $Lu(X)\leqq 0$ in D, $u(X)\geqq 0$ on ∂D, and if*

$$(6)\qquad \lim_{k\to\infty}\frac{m'(R_k)}{R_k^2}=0\qquad (R_k\to\infty\ \text{as}\ k\to\infty)\ ,$$

then $u(X)\geqq 0$ in D.

Taking $v_R(X)=2(|x|^2+Bt+C)/R^2$ (B and C are proper constants), we proceed as in the proof of Theorem 7. Details will be omitted. The remark that follows Theorem 7 applies also to Theorem 9.

Lemma 5 can also be generalized to the case $D\subset K_\beta$, $0<\beta<2\pi$. Indeed, the function $w(X)$ may be defined as follows:

$$w(X)=\begin{cases} \dfrac{C}{(t+R_0)^\varepsilon}\exp\left(-\dfrac{H|x|^2}{t+R_0}\right) & \text{if} \quad t>-R_0 \\ 0 \quad \text{if} \quad t\leqq -R_0\,. \end{cases}$$

Proceeding as in § 5, we get the following theorem.

THEOREM 10. *Let $D\subset K_\beta$, $0<\beta<2\pi$, and assume that L satisfies* (i), (ii). *If $Lu(X)=0$ in D and*

$$\lim_{R\to\infty}\frac{\mu'(R)}{R^2}=0\,, \tag{7}$$

and if $u(X)\to 0$ on ∂D as $|X|\to\infty$, then $u(X)\to 0$ uniformly in D as $|X|\to\infty$.

Note that (7) can be replaced by the stronger assumption

$$\lim_{R\to\infty}\frac{\mu(R)}{R}=0\,. \tag{7'}$$

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ADDITIVE FUNCTIONALS OF A MARKOV PROCESS

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1. Introduction. We are concerned with functionals of the form $L=\int_0^t V[x(\tau)]d\tau$ where $x(t)$ is a temporally homogeneous Markov process in a locally compact Hausdorff space, X, and V is a non-negative measurable function on X. In studying the distribution of this functional various authors (e.g. [1], [3], and [7] have considered the following function

$$r(t, x, A)=E\{e^{-uL}\,|\,x(0)=x;x(t)\in A\}\,p(t, x, A) \tag{1.1}$$

where $p(t, x, A)$ is the transition probability function of $x(t)$. If one can determine r then one can in essence determine the distribution of L since $(u>0)$

$$r(t, x, A)=\int_0^\infty e^{-u\lambda}d_\lambda P[L\leqq\lambda|x(0)=x;x(t)\in A] \qquad p(t, x, A)\ .$$

Formally it is quite easy to see that if p satisfies an equation of diffusion type

$$\frac{\partial p}{\partial t}=\Omega p \tag{1.2}$$

that r should satisfy the equation

$$\frac{\partial r}{\partial t}=(\Omega-uV)r\ . \tag{1.3}$$

If $x(t)$ is the Wiener process in E^N and V satisfies a Lipschitz condition of order $\alpha>0$ Rosenblatt [12] has given a rigorous derivation of (1.3). In this paper we use the theory of semi-groups to give a meaning to (1.3) for a wide class of processes without assuming any smoothness conditions on V. Rosenblatt's result does not follow from ours since our results only imply that r is a "weak" solution of (1.3). However, for many applications (e.g. [10]) this is all that is really required.

Because of certain difficulties connected with the definition of the conditional expectation in (1.1) we define r directly and prove that if $p(t, x, A)>0$ then $\dfrac{r(t, x, A)}{p(t, x, A)}$ is the appropriate conditional expectation. Since we intend to apply analytic methods it is necessary to investigate the dependence of r on its various variables. This is done in § 2.

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Beginning in § 3 we assume that $p(t, x, A)$ has a density $f(t, x, y)$ with respect to a Radon measure m and we show (§§ 4 and 5) that if $U_t\varphi(x) = \int\varphi(y)p(t, x, dy)$ has infinitesimal generator Ω on $L_2(m)$ then $T_t\varphi(x) = \int\varphi(y)r(t, x, dy)$ has infinitesimal generator $\Omega - uV$ if V is bounded, subject to certain regularity conditions on f. If V is unbounded our results are less complete and are contained in Theorem 5.2. In the sequel we will suppress the parameter u.

We use throughout this paper the function space approach to stochastic processes. We also make use of certain elementary facts about integration in locally compact spaces. The reader is referred to [2], [4], and [5] for the basic facts required. In a future paper we plan to study the spectral properties of the operators defined here. In that paper X will be an open subset of an N dimensional Euclidean space.

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2. A class of integrals over a function space. Let X be a locally compact Hausdorff space and $\mathfrak{B}(X)$ the Borel sets of X; that is, the smallest σ-algebra of subsets of X containing the compact sets of X. Let $\mathfrak{X}$ be the set of all functions from $[0 \leqq t < \infty]$ to X which are right continuous; that is, $x(t) \to x(t_0)$ if $t \downarrow t_0$. Let $p(t, x, A)$ be a transition probability function defined for $t > 0$, $x \in X$, and $A \in \mathfrak{B}(X)$, such that given an arbitrary probability measure μ on $\mathfrak{B}(X)$ there exists a Markov process $x_\mu(t)$ with paths which are right continuous and which has μ as its initial distribution and $p(t, x, A)$ as its transition probability. In other words, if $\mathfrak{B}(\mathfrak{X})$ is the σ-algebra of subsets of $\mathfrak{X}$ generated by sets of the form

$$\Lambda = \{x(\cdot) | x(t_j) \in A_j ; j = 0, 1, \cdots, n ; A_j \in \mathfrak{B}(X) ; 0 = t_0 < t_1 < \cdots < t_n\}$$

then there exists a countably additive probability measure, P_μ, on $\mathfrak{B}(\mathfrak{X})$ such that

$$(2.1) \qquad P_\mu(\Lambda) = \int_{A_0}\int_{A_1} \cdots \int_{A_n} \mu(dx_0)p(t_1, x_0, dx_1)p(t_2 - t_1, x_1, dx_2) \cdot p(t_n - t_{n-1}, x_{n-1}, dx_n) .$$

If μ assigns mass one to a single point, x, we write P_x for P_μ.

We assume that

(P_1) $p(\cdot, \cdot, A)$ is jointly measurable[1] in (t, x) for each $A \in \mathfrak{B}(X)$. We also pick a fixed μ, and $x(t)$ will always denote the processes having μ as

[1] Measurability conditions in t are understood to be with respect to the ordinary Borel sets of $[0 \leqq t < \infty]$.

its initial distribution. Clearly (once we have established Theorem 2.1)

$$P[\Lambda \mid x(0)=x]=P_x(\Lambda)\ , \qquad (\Lambda \in \mathfrak{B}(\mathfrak{X}))\ . \tag{2.2}$$

If $A \in \mathfrak{B}(X)$ we define

$$A_t=\{x(\cdot)|x(\cdot) \in \mathfrak{X}\ ;\ x(t) \in A\} \in \mathfrak{B}(\mathfrak{X})\ . \tag{2.3}$$

If $\Lambda \in \mathfrak{B}(\mathfrak{X})$ and $A \in \mathfrak{B}(X)$ we define for $t>0$

$$P(\Lambda\,;\,x\,;\,t,\,A)=P_x[\Lambda \cap A_t]\ . \tag{2.4}$$

It is evident that $P(\cdot\,;\,x\,;\,t,\,A)$ is a finite measure on $\mathfrak{B}(\mathfrak{X})$ for fixed x, t, A and that $P(\Lambda\,;\,x\,;\,t,\,\cdot)$ is a finite measure on $\mathfrak{B}(X)$ for fixed Λ, x, t. It is easy to see that if t and A are such that $p(t, x, A)>0$ for all x, then (again assuming Theorem 2.1)

$$P[\Lambda \mid x(0)=x\,;\,x(t) \in A]=\frac{P[\Lambda\,;\,x\,;\,t,\,A]}{p(t,\,x,\,A)}\ . \tag{2.5}$$

THEOREM 2.1. *$P[\Lambda\,;\,\cdot\,;\,\cdot,\,A]$ is a measurable function of (t, x) for fixed Λ, A.*

Proof. Let A be fixed and suppose

$$\Lambda=\{x(\cdot)\,|\,x(t_j) \in A_j\,;\,j=1,\,\cdots,\,n\}$$

then $P[\Lambda\,;\,x\,;\,t,\,A]=P_x[\Lambda \cap A_t]$ which is measurable in (t, x) in view of (2.1) and (P_1). Hence $P[\Lambda\,;\,x\,;\,t,\,A]$ is measurable in (t, x) for Λ's which are finite disjoint unions of sets of the above form. But the measurability of $P[\Lambda\,;\,x\,;\,t,\,A]$ is preserved under monotone limits of Λ's and hence $P[\Lambda\,;\,x\,;\,t,\,A]$ is measurable for all $\Lambda \in \mathfrak{B}(\mathfrak{X})$. See [8].

The following lemmas will be of use in the sequel.

LEMMA 2.1. *Let $(Y, \mathfrak{G})$ and $(Z, \mathfrak{H})$ be measurable spaces and let $m(A, B)$ be defined for $A \in \mathfrak{G}$ and $B \in \mathfrak{H}$. Suppose that $m(\cdot\,, B)$ is a measure on $(Y, \mathfrak{G})$ for each fixed $B \in \mathfrak{H}$ and that $m(A, \cdot)$ is a measure on $(Z, \mathfrak{H})$ for each fixed $A \in \mathfrak{G}$. Let $f \geqq 0$ be a measurable function on $(Y, \mathfrak{G})$ then*

$$q(B)=\int f(y)m(dy,\,B) \tag{2.6}$$

is a measure on $(Z, \mathfrak{H})$.

Proof. The only thing that requires proof is that q is countably additive. Let $\{f_n\}$ be a sequence of simple functions such that $f_n \geqq 0$ and $f_n \uparrow f$. Clearly

$$q_n(B)=\int f_n(y)m(dy,\,B)$$

are measures and $q_n(B)\uparrow q(B)$ for each $B\in\mathfrak{H}$. Let $B=\bigcup_{j=1}^{\infty}B_j$ where the B_j's are disjoint. Put $B^{(k)}=\bigcup_{j=1}^{k}B_j$, then $\lim_n\lim_k q_n(B^{(k)})=q(B)$. Since $q_n(B^{(k)})$ is increasing in both n and k we can interchange the limits obtaining

$$q(B)=\lim_k\lim_n q_n(B^{(k)})=\lim_k q(B^{(k)})=\sum_{j=1}^{\infty}q(B_j)\ .$$

LEMMA 2.2. *Let $(Y,\mathfrak{G})$ be a measurable space and let $f(t,y)$ be an X valued function defined for $t\geqq 0$ and $y\in Y$. If $f(\cdot,y)$ is right continuous for each $y\in Y$ and $f(t,\cdot)$ is $\mathfrak{G}$-measurable for each t then $f(t,y)$ is jointly $\mathfrak{B}\times\mathfrak{G}$ measurable.* ($\mathfrak{B}$ is the σ-algebra of ordinary Borel sets.)

Proof. Define $g_n(t,y)=f((j+1)/n,y)$ if $j/n<t\leqq(j+1)/n$ for $j=0,1,2,\cdots$ and $n=1,2,\cdots$. Let $B\in\mathfrak{B}(X)$ and define $G_{jn}=f((j+1)/n,\cdot)^{-1}(B)$, then since $f(t,\cdot)$ is $\mathfrak{G}$-measurable $G_{jn}\in\mathfrak{G}$. Let $A_{jn}=\{t|j/n<t\leqq(j+1)/n\}\in\mathfrak{B}$, then

$$g_n^{-1}(B)=\bigcup_{j=0}^{\infty}A_{jn}\times G_{jn}$$

which is in $\mathfrak{B}\times\mathfrak{G}$. Hence g_n is jointly $\mathfrak{B}\times\mathfrak{G}$ measurable for each n, but $g_n(t,x)\to f(t,x)$ as $n\to\infty$ and thus f is $\mathfrak{B}\times\mathfrak{G}$ measurable.

If $\Phi\ [x(\cdot)]$ is a complex valued measurable[2] functional on $\mathfrak{X}$ we write $r[\Phi;t,x,A]$ for the integral of Φ over $\mathfrak{X}$ with respect to the measure $P[\cdot\,;x;t,A]$, provided the integral exists.

THEOREM 2.2. *If $\Phi\geqq 0$ is a measurable functional on $\mathfrak{X}$ then $r[\Phi;t,x,A]$ is a measure on $\mathfrak{B}(X)$ for fixed (t,x) and is measurable in (t,x) for fixed A.*

Proof. This is an immediate consequence of Lemma 2.1 and Theorem 2.1.

Let φ be a complex valued measurable function on X, then for each $t>0$ we define a measurable functional, φ_t, on $\mathfrak{X}$ as follows: $\varphi_t[x(\cdot)]=\varphi[x(t)]$. Also if Φ is a measurable functional on $\mathfrak{X}$ we denote its integral over $\mathfrak{X}$ with respect to the measure P_x by $E\{\Phi[x(\cdot)]|x(0)=x\}$.

THEOREM 2.3. *Let $\Phi\geqq 0$ be a measurable functional on $\mathfrak{X}$ and φ a complex valued measurable function on X; then*

$$\int\varphi(y)r[\Phi;t,x,dy]=E\{\Phi\cdot\varphi_t|x(0)=x\}\ , \tag{2.7}$$

provided either integral exists.

[2] Measurability of real or complex valued functions always means Borel measurability.

Proof. Suppose $\varphi=I_A$ where I_A denotes the characteristic function A, then the left hand side of (2.7) is $r[\Phi; t, x, A]$. Now if $\Phi=I_\Lambda$ then

$$r[I_\Lambda; t, x, A]=P[\Lambda; x; t, A] .$$

But

$$(I_A)_t[x(\cdot)]=I_A[x(t)]=I_{A_t}[x(\cdot)] ,$$

where $A_t=\{x(\cdot)\,|\,x(t)\in A]$. Thus

$$E\{I_\Lambda\cdot(I_A)_t\,|\,x(0)=x\}=P_x[\Lambda\cap A_t]=P[\Lambda; x; t, A] .$$

Let Φ_n be a sequence of simple functionals such that $\Phi_n\uparrow\Phi$, then $\Phi_n\cdot(I_A)_t$ is a sequence of simple functionals increasing to $\Phi\cdot(I_A)_t$. Therefore

$$E\{\Phi_n\cdot(I_A)_t\,|\,x(0)=x\}\uparrow E\{\Phi\cdot(I_A)_t\,|\,x(0)=x\} .$$

On the other hand $r(\Phi_n; t, x, A)\uparrow r(\Phi; t, x, A)$ by the monotone convergence theorem and since

$$E\{\Phi_n\cdot(I_A)_t\,|\,x(0)=x\}=r[\Phi_n; t, x, A]$$

it follows that if either of the integrals in (2.7) is finite the other is also and they are equal in the case $\varphi=I_A$.

If $\varphi\geqq 0$ let φ_n be a sequence of simple functions increasing to φ then if either of the integrals in (2.7) exists we have equality for each φ_n and by monotone convergence for φ. The result for a general φ now follows in the usual manner.

For each $t\geqq 0$ let $x_t(\tau)=x(t+\tau)$ for all $\tau\geqq 0$, then we define a map, S_t, from $\mathfrak{X}$ into $\mathfrak{X}$ by $S_tx(\cdot)=x_t(\cdot)$. Clearly S_t is a measurable transformation of $\mathfrak{X}$ into $\mathfrak{X}$. If Φ is a measurable functional we define $S_t\Phi[x(\cdot)]=\Phi[S_tx(\cdot)]$.

THEOREM 2.4. *Let Φ be a functional measurable with respect to*[3] $\mathfrak{B}_t$ *and Ψ be measurable with respect to $\mathfrak{B}_s$ such that $0\leqq\Phi\leqq M$ and $0\leqq\Psi\leqq M$, then*

$$\int r[\Phi; t, x, dy]r[\Psi; s, y, A]=r[\Phi\cdot S_t\Psi; t+s, x, A] . \tag{2.8}$$

Proof. Since Φ and Ψ are non-negative and bounded it is clear that the integral in question exists. If $\Phi=I_F$ and $\Psi=I_G$ with $F\in\mathfrak{B}_t$ and $G\in\mathfrak{B}_s$ then

$$S_tI_G[x(\cdot)]=I_G[S_tx(\cdot)]=I_{S_t^{-1}G} ,$$

thus to prove (2.8) for I_F and I_G we must show that

[3] $\mathfrak{B}[t_1, t_2]$ denotes the σ-algebra of subsets of $\mathfrak{X}$ generated by sets of the form $\{x(\cdot)\,|\,x(\tau_j)\in A_j;\ t_1\leqq\tau_j\leqq t_2\}$, and $\mathfrak{B}_t=\mathfrak{B}[0, t]$.

$$\text{(2.9)} \qquad \int P[F; x; t, dy] P[G; y; s, A] = P[F \cap S_t^{-1} G; x; t+s, A] .$$

We first consider the case in which

$$F = \{x(\cdot) \mid x(t_j) \in A_j;\ j=1, \cdots, n;\ t_j < t\}$$
$$G = \{x(\cdot) \mid x(t'_k) \in B_k;\ k=1, \cdots, m;\ t'_k < s\} .$$

In this case

$$S_t^{-1} G = \{x(\cdot) \mid x(t+t'_k) \in B_k;\ k=1, 2, \cdots, m\} ,$$

thus

$$\begin{aligned}
&\int P[F; x; t, dy] P[G; y; s, A] \\
&= \iint_{A_1} \cdots \int_{A_n} p(t_1, x, dx_1) p(t_2 - t_1, x_1, dx_2) \cdots p(t - t_n, x_n, dy) \\
&\qquad \cdot \int_{B_1} \cdots \int_{B_m} p(t'_1, y, dy_1) \cdots p(s - t'_m, y_m, A) \\
&= \int_{A_1} \cdots \int_{A_n} \int_{B_1} \cdots \int_{B_m} p(t_1, x, dx_1) \cdots p(t + t'_1 - t_n, x_n, dy_1) \\
&\qquad \cdot p(t + t'_2 - (t + t'_1), y_1, dy_2) \cdots p(t + s - (t + t'_m), y_m, A) \\
&= P[F \cap S_t^{-1} G; x; t+s, A] .
\end{aligned}$$

If $t = t_n$, or $s = t'_m$, or both, it is necessary to make only minor changes in the above argument.

This equality clearly extends to finite disjoint unions of such F's and G's and since S_t^{-1} is a σ-homomorphism it extends to monotone limits of such G's. Thus (2.9) holds for each F in the algebra of sets generated by sets of the given form and for each $G \in \mathfrak{B}_s$. For fixed $G \in \mathfrak{B}_s$ the left hand side of (2.9) is a measure in F by Lemma 2.1, hence (2.9) holds under monotone limits of such F's and thus finally (2.9) holds for all F and G in the appropriate σ-algebras.

Let Φ_n and Ψ_n be sequences of simple functionals increasing to Φ and Ψ, then by monotone convergence

$$\int r[\Phi_n; t, x, dy] r[\Psi; s, y, A] = r[\Phi_n \cdot S_t \Psi; t+s, x, A] .$$

Applying an argument similar to that used in the proof of Lemma 2.1 the equality (2.8) results. (This also follows from Theorem 2.3.)

We conclude this section with the following theorem which is easily proved using standard approximation techniques.

THEOREM 2.5. *Let $\Phi\ (t, x(\cdot)) \geqq 0$ be jointly measurable in t and $x(\cdot)$ then $r[\Phi(t, x(\cdot));\ t, x, A]$ is jointly measurable in (t, x).*

3. Additive functionals. For each pair (t_1, t_2) with $0 \leqq t_1 < t_2$ let $L[t_1, t_2; x(\cdot)]$ be a functional (L may be $+\infty$) on $\mathfrak{X}$ which is measurable with respect to $\mathfrak{B}[t_1, t_2]$ and which is jointly measurable in t_1, t_2, and $x(\cdot)$. We further assume that for $t_1 < t < t_2$ and each $x(\cdot) \in \mathfrak{X}$ we have

$$L[t_1, t_2; x(\cdot)] = L[t_1, t; x(\cdot)] + L[t, t_2; x(\cdot)]\,; \tag{3.1}$$

and that

$$S_t L[t_1, t_2; x(\cdot)] = L[t_1 + t, t_2 + t; x(\cdot)]\,. \tag{3.2}$$

Such a functional will be called an additive functional on $\mathfrak{X}$ (See [1]).

THEOREM 3.1. *Let $V \geqq 0$ be a measurable function on X, then*

$$L[t_1, t_2; x(\cdot)] = \int_{t_1}^{t_2} V[x(\tau)]d\tau$$

is an additive functional on $\mathfrak{X}$.

Proof. Define $F(t, x(\cdot)) = x(t)$ then F is measurable in $x(\cdot)$ for fixed t and right continuous in t for fixed $x(\cdot)$. Thus by Lemma 2.2 F is jointly measurable in t and $x(\cdot)$. Since $V[x(t)] = V[F(t, x(\cdot))]$ is the composition of measurable transformations $V[x(\tau)]$ is jointly measurable in τ and $x(\cdot)$, and therefore (a simple argument using Lemma 2.2 shows that) $\int_{t_1}^{t_2} V[x(\tau)]d\tau$ is jointly measurable in t_1, t_2, and $x(\cdot)$. The other properties that L must satisfy are obvious.

We suppose that $L[t_1, t_2; x(\cdot)] \geqq -M$ where $M > 0$ is independent of t_1, t_2, and $x(\cdot)$. We define

$$r(t, x, A) = r[e^{-L[0,t;x(\cdot)]}; t, x, A]\,. \tag{3.3}$$

Theorems 2.2 and 2.5 imply that $r(t, x, A)$ is a measure on $\mathfrak{B}(X)$ for fixed (t, x) and is jointly measurable in (t, x) for fixed $A \in \mathfrak{B}(X)$. Moreover the fact that

$$0 \leqq r(t, x, A) \leqq e^M p(t, x, A) \tag{3.4}$$

is a simple consequence of our definitions.

THEOREM 3.2. $r(t+s, x, A) = \int r(t, x, dy) r(s, y, A)\,.$

Proof. This is a corollary of Theorem 2.4 once we observe that

$$S_t e^{-L[0,s;x(\cdot)]} = e^{-L[0,s;S_t x(\cdot)]} = e^{-S_t L[0,s;x(\cdot)]} = e^{-L[t,t+s;x(\cdot)]}\,,$$

and therefore

$$e^{-L[0,t;x(\cdot)]}\cdot S_t e^{-L[0,s;x(\cdot)]}=e^{-L[0,t+s;x(\cdot)]}\ .$$

At this point we assume that there exists a Radon measure, m, on $\mathfrak{B}(X)$ such that $p(t, x, A)$ has a density $f(t, x, y)\geqq 0$ with respect to m for $t>0$; that is

$$p(t, x, A)=\int_A f(t, x, y)m(dy)\ , \qquad t>0\ . \tag{3.5}$$

We assume that f is jointly measurable in $t, x,$ and y, but we do not assume that m is finite. We introduce the following conditions on $f(t, x, y)$:

(P_2) $\int f(t, x, y)m(dx)\leqq ke^{\alpha t}$ where k and α are positive constants independent of y and t.

(P_3) Given $\varepsilon>0$ and a compact set $A\subset X$ there exists a compact set B such that

$$\int_{x\notin B} f(t, x, y)m(dx)<\varepsilon \text{ for } y\in A \text{ and } t\leqq 1\ .$$

We define operators on appropriate function spaces as follows:

$$(T_t\varphi)(x)=\int\varphi(y)r(t, x, dy) \tag{3.6}$$

$$(U_t\varphi)(x)=\int\varphi(y)p(t, x, dy)=\int\varphi(y)f(t, x, y)m(dy)\ . \tag{3.7}$$

THEOREM 3.3. *If $f(t, x, y)$ satisfies* (P_2) *then $\{T_t; t>0\}$ and $\{U_t; t>0\}$ are semi-groups of bounded operators on $L_2(m)$.*

Note. All Borel sets are m-measurable [4; 5].

Proof. From (3.4) we obtain

$$\begin{aligned}|T_t\varphi(x)|&\leqq\int|\varphi(y)|r(t, x, dy)\\ &\leqq e^M\int|\varphi(y)|p(t, x, dy)=e^M U_t|\varphi|(x)\end{aligned}$$

and thus it will suffice to prove that U_t is a bounded operator on $L_2(m)$ for each $t>0$. But

$$\begin{aligned}|U_t\varphi(x)|^2&=|\int f(t, x, y)\varphi(y)m(dy)|^2\\ &\leqq\int f(t, x, y)\,|\varphi(y)|^2 m(dy)\ ,\end{aligned}$$

and therefore

$$\int |U_t\varphi(x)|^2 m(dx) \leqq \int m(dx) \int f(t, x, y) |\varphi(y)|^2 m(dy)$$
$$\leqq ke^{\alpha t} \cdot \|\varphi\|^2 .$$

Thus $\|U_t\|^2 \leqq ke^{\alpha t}$ and $\|T_t\|^2 \leqq ke^{2M+\alpha t}$. The fact that $\{T_t; t>0\}$ and $\{U_t; t>0\}$ are semi-groups now follows from Theorem 3.2 and the fact that $p(t, x, A)$ satisfies the Chapman-Kolmogorov equation.

THEOREM 3.4. *If $f(t, x, y)$ satisfies* (P_2) *and* (P_3) *and* $\lim_{t\to 0} L[0, t; x(\cdot)]=0$ *for all $x(\cdot) \in \mathfrak{X}$ then the semi-groups $\{U_t; t>0\}$ and $\{T_t; t>0\}$ are strongly continuous*[4] *on $L_2(m)$.*

Proof. We prove the theorem for $\{T_t; t>0\}$ the results for $\{U_t; t>0\}$ being a special case (take $L\equiv 0$). We must show that $\|T_t\varphi-\varphi\|\to 0$ as $t\to 0$ for all $\varphi \in L_2(m)$. Since $\|T_t\|$ is uniformly bounded for $t\leqq 1$ it will be sufficient to show that $\|T_t\varphi-\varphi\|\to 0$ as $t\to 0$ for φ continuous with compact support, such functions being dense in $L_2(m)$ since m is a Radon measure, [2]. We first show that $T_t\varphi(x)\to\varphi(x)$ pointwise as $t\to 0$ if φ is continuous with compact support. According to Theorem 2.3

$$T_t\varphi(x) = \int \varphi(y) r(t, x, dy) = E\{e^{-L[0,t;\, x(\cdot)]} \cdot \varphi(x(t)) | x(0)=x\} .$$

Using the right continuity of $x(\cdot)$ and our assumption on L we see that

$$e^{-L[0,t;\, x(\cdot)]}\, \varphi[x(t)] \to \varphi[x(0)]$$

boundedly as $t \downarrow 0$ and hence by the bounded convergence theorem

$$T_t\varphi(x) \to E\{\varphi[x(0)] \,|\, x(0)=x\} = \varphi(x) \qquad \text{as} \quad t \downarrow 0 .$$

Let A be the support of φ, then if B is compact and $B\supset A$ we have

$$\|T_t\varphi - \varphi\|^2 = \int_B |T_t\varphi(x) - \varphi(x)|^2 m(dx) + \int_{x\notin B} |T_t\varphi(x)-\varphi(x)|^2 m(dx)$$
$$= I_1 + I_2 .$$

But

$$|T_t\varphi(x)| \leqq \int |\varphi(y)| r(t, x, dy) \leqq \sup_{x\in A} |\varphi(x)| \cdot e^M ,$$

hence $I_1 \to 0$ since B is compact. Now since $B\supset A$ we have

$$I_2 \leqq \int_{x\notin B} |T_t\varphi(x)|^2 m(dx) \leqq e^M \int_A |\varphi(y)|^2 \int_{x\notin B} f(t, x, y) m(dy) m(dx) ,$$

[4] By the strong continuity of a semi-group $\{T_t; t>0\}$ we will always mean strong continuity for $t\geqq 0$ where T_0 is the identity.

and so, if B is chosen properly, using (P_3), we see that I_2 is small. This completes the proof of the fact that $\{T_t; t>0\}$ is strongly continuous on $L_2(m)$.

4. The Darling-Siegert equations. In [3] Darling and Siegert showed that $r(t, x, A)$ has to satisfy two integral equations. We give a derivation of these equations based on the material of § 2. We assume that $p(t, x, A)$ satisfies (P_1) and that

$$L[t_1, t_2; x(\cdot)]=\int_{t_1}^{t_2} V[x(\tau)]d\tau$$

where V is a bounded, non-negative, measurable function on X. The formal outline of the derivation given below is exactly that of Darling and Siegert.

We begin with the following identities which are easily verified (f measurable, non-negative, and bounded)

$$\exp\left[-\int_0^t f(\tau)d\tau\right]=1-\int_0^t f(s)\exp\left[-\int_s^t f(\tau)d\tau\right]ds \tag{4.1}$$

$$\exp\left[-\int_0^t f(\tau)d\tau\right]=1-\int_0^t f(s)\exp\left[-\int_0^s f(\tau)d\tau\right]ds\ . \tag{4.2}$$

Also using Theorem 2.4 we have

$$\begin{aligned} &r\left[V[x(s)]\exp\left(-\int_s^t V[x(\tau)]d\tau\right);\ t,\ x,\ A\right] \\ &=r\left[V[x(s)]\cdot S_s\exp\left(-\int_0^{t-s} V[x(\tau)]d\tau\right);\ (t-s)+s,\ x,\ A\right] \\ &=\int r[V[x(s)];\ s,\ x,\ dy]r[\exp\left(-\int_0^{t-s} V[x(\tau)]d\tau\right);\ t-s,\ y,\ A] \\ &=\int V(y)p(s,\ x,\ dy)r(t-s,\ y,\ A) \end{aligned} \tag{4.3}$$

provided we show that

$$\int f(y)r[V[x(s)];\ s,\ x,\ dy]=\int f(y)V(y)p(s,x,\ dy) \tag{4.4}$$

for measurable, bounded $f\geqq 0$. Suppose $f=I_A$ and $V=I_B$ then

$$\begin{aligned} &\int f(y)r[V[x(s)];\ s,\ x,\ dy]=P[I_B[x(s)];\ x;\ s,\ A] \\ &=P_x[B_s\cap A_s]=P(s,\ x,\ A\cap B)=\int f(y)V(y)p(s,\ x,\ dy)\ . \end{aligned}$$

The standard approximation technique now yields the desired result (4.4).

Putting $f(\tau)=V[x(\tau)]$ in (4.1) and applying (4.3) we obtain (the interchange in the order of integration is valid since

$$V[x(s)]\cdot\exp\left(-\int_s^t V[x(\tau)]\,d\tau\right)$$

is bounded and jointly measurable in s and $x(\cdot)$)

$$r(t,x,A)=p(t,x,A)-\int_0^t ds\int V(y)r(t-s,y,A)p(s,x,dy)\ . \tag{4.5}$$

In a similar manner using (4.2) we find

$$r(t,x,A)=p(t,x,A)-\int_0^t ds\int V(y)p(t-s,y,A)r(s,x,dy)\,; \tag{4.6}$$

and these are the Darling-Siegert equations. In deriving (4.6) one needs the relation

$$r[V[x(0)]\,;t,y,A]=V(y)p(t,y,A) \tag{4.7}$$

which is obtained in much the same manner as (4.4).

Taking Laplace transforms in (4.5) and (4.6) yields (the necessary interchange of order of integration is again justified since the integrand is bounded and jointly measurable in its variables)

$$\hat{r}(\lambda,x,A)=p(\lambda,x,A)-\int V(y)\hat{r}(\lambda,y,A)\hat{p}(\lambda,x,dy) \tag{4.8}$$

$$\hat{r}(\lambda,x,A)=\hat{p}(\lambda,x,A)-\int V(y)\hat{p}(\lambda,y,A)\hat{r}(\lambda,x,dy) \tag{4.9}$$

where $\hat{r}$ and $\hat{p}$ are the Laplace transforms of r and p.

5. The infinitesimal generators. Let Ω and Ω' be the infinitesimal generators of $\{U_t;t>0)$ and $\{T_t;t>0\}$ respectively. We assume in this section that (P_1), (P_2), and (P_3) are satisfied. It then follows, since the semi-groups involved are strongly continuous on $L_2(m)$, that Ω and Ω' are closed densely defined operators on $L_2(m)$. See [6] and [9].

We assume that

$$L[t_1,t_2\,;x(\cdot)]=\int_{t_1}^{t_2} V[x(\tau)]d\tau \tag{5.1}$$

where V is a non-negative measurable function on X. Note that in this case $M=0$.

THEOREM 5.1. *If V is bounded then $\Omega'=\Omega-V$.*

Proof. Let J_λ be the resolvent of $\{T_t;t>0\}$ then for $\lambda>\alpha$ we have

$$J_\lambda\varphi(x)=\int_0^\infty e^{-\lambda t}T_t\varphi(x)dt\ ,$$

and thus

$$|J_\lambda\varphi(x)|^2\leqq\frac{1}{\lambda}\int_0^\infty e^{-\lambda t}|T_t\varphi(x)|^2dt\ .$$

Applying the Fubini theorem we see that $J_\lambda\varphi(x)$ exists for almost all $x(\lambda>\alpha)$ and is in $L_2(m)$, moreover for $\lambda>\alpha$ we have

$$\|J_\lambda\|^2\leqq\frac{k}{\lambda(\lambda-\alpha)}\ . \tag{5.2}$$

In view of the above facts we can write

$$J_\lambda\varphi(x)=\int\varphi(y)\hat{r}(\lambda,\, x,\, dy)\ . \tag{5.3}$$

From the general theory of semi-groups, [6] and [9], we know that for $\lambda>\alpha$ the range of J_λ is independent of λ and is, in fact, the domain of Ω', which we denote by $D_{\Omega'}$. In addition it is known that

$$(\lambda-\Omega')J_\lambda\varphi=\varphi \qquad \text{for all} \quad \varphi\in L_2(m)\,; \tag{5.4}$$

$$J_\lambda(\lambda-\Omega')\varphi=\varphi \qquad \text{for all} \quad \varphi\in D_{\Omega'}\ . \tag{5.5}$$

Let I_λ be the resolvent of $\{U_t;\, t>0\}$ and then in a similar manner we have

$$I_\lambda\varphi(x)=\int\varphi(y)\hat{p}(\lambda,\, x,\, dy)=\int\varphi(y)\hat{f}(\lambda,\, x,\, y)m(dy)\ . \tag{5.7}$$

From (4.8) we see that

$$\begin{aligned}J_\lambda\varphi(x)&=I_\lambda\varphi(x)-\int\varphi(z)\int V(y)\hat{r}(\lambda,\, y,\, dz)\hat{p}(\lambda,\, x,\, dy)\\&=I_\lambda\varphi(x)-I_\lambda[V\cdot J_\lambda\varphi](x)\\&=I_\lambda[\varphi-V\cdot J_\lambda\varphi](x)\ .\end{aligned}$$

The above steps are justified since $V\cdot J_\lambda\varphi\in L_2(m)$ under our assumption that V is bounded. Thus $D_{\Omega'}\subset D_\Omega$ and conversely using (4.9) $D_\Omega\subset D_{\Omega'}$, that is, $D_\Omega=D_{\Omega'}$. Now

$$\begin{aligned}(\lambda-\Omega)J_\lambda\varphi&=(\lambda-\Omega)I_\lambda[\varphi-VJ_\lambda\varphi]\\&=\phi-VJ_\lambda\varphi\ ,\end{aligned}$$

or equivalently,

$$[\lambda-(\Omega-V)]J_\lambda\varphi=\varphi \qquad \text{for all} \quad \varphi\in L_2\ .$$

Thus $\Omega - V$ is an extension of Ω', but since V is bounded the domain of $\Omega - V$ is $D_\Omega = D_{\Omega'}$. Hence $\Omega' = \Omega - V$.

COROLLARY. *If V is bounded and $f(t, x, y) = f(t, y, x)$ then Ω and Ω' are self-adjoint operators.*

Proof. Since $f(t, x, y) = f(t, y, x)$ each U_t is a bounded self-adjoint operator and hence Ω is also self-adjoint, although not necessarily bounded. The boundedness of V implies that V considered as an operator on $L_2(m)$ is bounded and self-adjoint, therefore $\Omega - V$ is self-adjoint, [11]. Thus $\Omega' = \Omega - V$ is a self-adjoint operator which in turn implies that each T_t is a bounded self-adjoint operator.

If V is not bounded our results are much less complete (V is no longer a bounded operator on $L_2(m)$ and one runs into the usual " domain problems "). It is natural to try to approximate V by bounded functions and then use a limiting procedure. Accordingly we define

$$V_N(x) = \begin{cases} V(x) & \text{if} \quad V(x) \leqq N, \\ N & \text{if} \quad V(x) \geqq N \end{cases} \tag{5.8}$$

and it is evident that each V_N is measurable and bounded. Let

$$D_V = \{\varphi | \varphi \in L_2(m);\ V \cdot \varphi \in L_2(m)\};$$

that is, D_V is the domain of V considered as an operator on $L_2(m)$. We are, of course, assuming that $f(t, x, y)$ satisfies (P_1), (P_2), and (P_3).

THEOREM 5.2. *If V is non-negative and measurable then $D_\Omega \cap D_V \subset D_{\Omega'}$ and if $\varphi \in D_\Omega \cap D_V$ then $\Omega' \varphi = (\Omega - V)\varphi$.*

Proof. We define

$$r_N(t, x, A) = r[e^{-\int_0^t V_N[x(\tau)]d\tau};\ t, x, A]$$

and

$$T_t^{(N)} \varphi(x) = \int \varphi(y) r_N(t, x, dy).$$

For each N we know that $\{T_t^{(N)}; t > 0\}$ is a strongly continuous semi-group of bounded operators on $L_2(m)$ whose infinitesimal generator is $\Omega - V_N$. Since $V_N \uparrow V$ we have by monotone convergence that

$$r_N(t, x, A) \downarrow r(t, x, A). \tag{5.9}$$

We first show that for each $t > 0$ and all $\varphi \in L_2(m)$

$$\| T_t^{(N)} \varphi - T_t \varphi \| \to 0 \qquad \text{as } N \to \infty. \tag{5.10}$$

Since $\|T_t^{(N)}\| \leqq ke^{\alpha t}$ it will suffice to prove (5.10) for φ continuous with compact support. Let $\mu_N(A) = r_N(t, x, A) - r(t, x, A) \geqq 0$, then $\mu_N(A) \downarrow 0$ for each fixed A and is a measure on $\mathfrak{B}(X)$ for each fixed N. It is clear that

$$|T_t^{(N)}\varphi(x) - T_t\varphi(x)| \leqq \int |\varphi(y)|\mu_N(dy) .$$

Let φ_j be a sequence of simple functions decreasing to $|\varphi|$, then since $\int \varphi_j(y)\mu_N(dy)$ is decreasing in both N and j we can interchange the limits obtaining $|T_t^{(N)}\varphi(x) - T_t\varphi(x)| \to 0$ pointwise as $N \to \infty$ at least if φ is continuous with compact support. If the support of φ is A then (5.10) follows exactly as in the proof of Theorem 3.4 since

$$\int_{x \notin B} |T_t^{(N)}\varphi(x)|^2 m(dx) \leqq \int_A |\varphi(y)|^2 \int_{x \notin B} f(t, x, y) m(dx) m(dy)$$

for compact B. Thus (5.10) is established.

We prove next that $D_\Omega \cap D_V \subset D_{\Omega'}$. Let $J_\lambda^{(N)}$ and J_λ be the resolvents of $\{T_t^{(N)}; t > 0\}$ and $\{T_t; t > 0\}$ respectively. Since $\|T_t^{(N)}\| \leqq ke^{\alpha t}$ and $T_t^{(N)}\varphi \to T_t\varphi$ it follows that $J_\lambda^{(N)}\varphi \to J_\lambda\varphi$ for each $\varphi \in L_2(m)$ and $\lambda > \alpha$. Choose a $\lambda > \alpha$ and let it be fixed for the remainder of the present proof. If $\varphi \in D_\Omega \cap D_V$ then $\varphi \in D_{\Omega - V_N}$ for each N, hence there exist $\psi_N \in L_2(m)$ such that $\varphi = J_\lambda^{(N)}\psi_N$. Moreover $[\lambda - (\Omega - V_N)]\varphi = \psi_N$ or $\psi_N = \lambda\varphi - \Omega\varphi + V_N\varphi$. Clearly $V_N\varphi \to V\varphi$ pointwise and since $|V_N\varphi| \leqq |V\varphi|$ it follows that $\|V_N\varphi - V\varphi\| \to 0$. Thus $\psi_N \to \lambda\varphi - \Omega\varphi + V\varphi = \psi$ as $N \to \infty$ in $L_2(m)$. But

$$\|J_\lambda^{(N)}\psi_N - J_\lambda\psi\| \leqq \|J_\lambda^{(N)}\psi_N - J_\lambda^{(N)}\psi\| + \|J_\lambda^{(N)}\psi - J_\lambda\psi\|$$

and therefore $J_\lambda^{(N)}\psi_N \to J_\lambda\psi$ as $N \to \infty$ since $\|J_\lambda^{(N)}\|$ is uniformly bounded in N. However, $\varphi = J_\lambda^{(N)}\psi_N$ for all N and hence $\varphi = J_\lambda\psi$ which implies that $\varphi \in D_{\Omega'}$.

Since $\varphi = J_\lambda\psi$ where $\psi = \lambda\varphi - \Omega\varphi + V\varphi$ we see that $(\lambda - \Omega')\varphi = \psi = \lambda\varphi - \Omega\varphi + V\varphi$ or equivalently that $\Omega'\varphi = (\Omega - V)\varphi$ for $\varphi \in D_\Omega \cap D_V$. This completes the proof of Theorem 5.2.

COROLLARY. *If Ω is self-adjoint (that is, $f(t, x, y) = f(t, y, x)$) then Ω' is self-adjoint. Let $E_N(\lambda)$ denote the spectral resolution of $\Omega - V_N$ and $E(\lambda)$ the spectral resolution of Ω', then $E_N(\lambda)\varphi \to E(\lambda)\varphi$ for all $\varphi \in L_2(m)$ provided that λ is a continuity point of $E(\lambda)$.*

Proof. We use the same notation as in the proof of Theorem 5.2. From the corollary to Theorem 5.1 it follows that each $T_t^{(N)}$ is self-adjoint and T_t being the strong limit of self-adjoint operators is self-adjoint for each $t>0$. Hence the infinitesimal generator, Ω', of $\{T_t; t>0\}$ is self-adjoint. The strong continuity of $\{T_t; t>0\}$ implies that $T_t\varphi=0$ if and only if $\varphi=0$. A similar statement holds for $T_t^{(N)}$. Under these circumstances $E_N^{(\lambda)}=F_N(e^\lambda)$ and $E(\lambda)=F(e^\lambda)$ where F_N and F are the spectral resolutions of $T_1^{(N)}$ and T_1 respectively. See [11]. Thus if we show that $F_N(\lambda)\varphi\to F(\lambda)\varphi$ at all continuity points of F we will have proved the corollary. Since $T_1^{(N)}\varphi\to T_1\varphi$ this follows from a theorem of Rellich (See [11], p. 366).

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SEATTLE WASHINGTON

(γ, k)-SUMMABILITY OF SERIES

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1. Introduction. Let $\gamma_k(x)$ denote the (C, k) mean of $\cos x$, so that

$$\gamma_0(x) = \cos x \,, \tag{1.1}$$

and

$$\begin{aligned} \gamma_k(x) &= \frac{k}{x^k}\int_0^x (x-u)^{k-1}\cos u\,du \,, \quad (k>0)\,, \\ &= k\int_0^1 (1-t)^{k-1}\cos xt\,dt \,, \\ &= \Gamma(k+1)\frac{C_k(x)}{x^k} \,, \end{aligned} \tag{1.2}$$

where $C_k(x)$, the kth fractional integral of $\cos x$, is commonly known as Young's function [6, p. 564].

We shall say that the infinite series $\sum_0^\infty a_n$ is summable (γ, k) if

$$\text{(i)} \quad \sum_0^\infty a_n \gamma_k(nt) \text{ converges for } 0<t<A$$

and

$$\text{(ii)} \quad \lim_{t\to 0}\sum_0^\infty a_n\gamma_k(nt) = S \,, \quad \text{where } S \text{ is finite.}$$

We see that $(\gamma, 1) \equiv (R, 1)$ and $(\gamma, 2) \equiv (R, 2)$, where $(R, 1)$ and $(R, 2)$ are the well known Riemann summability methods. Hence the (γ, k)-summability methods constitute, in a sense, an extension of $(R, 1)$ and $(R, 2)$ summability methods to (R, k) methods where k may be non-integral. But this extension is not linked with the ideas which lie at the root of the Riemann summability methods, that is, taking generalised symmetric derivatives of repeatedly integrated Fourier series, so that the equivalence of (γ, k) and (R, k) for $k=1, 2$ may be considered to be somewhat accidental, and the extension artificial. However, the (γ, k) methods are also connected with certain aspects of the summability problems of Fourier series. For, let $\sum_0^\infty A_n(x)$ be the Fourier series of a periodic and Lebesgue integrable function $f(x)$ and let

$$\phi(t) = \frac{1}{2}\{f(x+t)+f(x-t)\} \,.$$

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Then, by some well known theorems (see, for example, [1].) the problem of Cesàro summability of $\sum A_n(x)$ is connected with Cesàro continuity of $\phi(t)$ at $t=0$, $\phi(t)$ being said to be (C, k) continuous at $t=0$ if $k\int_0^1 (1-y)^{k-1}\phi(ty)\,dy$ exists for $0<t<A$ and tends to a finite limit as t tends to zero. On the other hand, under certain conditions (e.g., if $k\geqq 1$) we have

$$k\int_0^1 (1-y)^{k-1}\phi(ty)\ dy=\sum_0^\infty A_n(x)\gamma_k(nt)\,.$$

Thus the nature of the connexion between (γ, k) and Cesàro summability methods, when the series in question is a Fourier series, is immediately apparent.

Some known theorems which may be interpreted as results on (γ, k) methods are stated below. δ denotes any arbitrary positive number.

(A_1) *If a series is summable* $(\gamma, 1)$ *then it is summable* $(C, 1+\delta)$.

See Zygmund [11].

(A_2) *If a series is summable* $(\gamma, 2)$ *then it is summable* $(C, 2+\delta)$.

See Kuttner [7].

(A_3) *If a Fourier-Lebesgue series is summable* (γ, k), $k\geqq 1$, *then it is summable,* $(C, k+\delta)$.

See Bosanquet [1] and Paley [8].

Neither Bosanquet nor Paley actually states any such result, but if $\alpha\geqq 1$, then Bosanquet's Theorem 1 as well as Paley's Theorem 1 can be restated in the present form.

(B_1) *If a series is summable* $(C, -\delta)$ *then it is summable* $(\gamma, 1)$.

See Hardy and Littlewood [4].

(B_2) *If a series is summable* $(C, 1-\delta)$ *then it is summable* $(\gamma, 2)$.

See Bosanquet [1] and Verblunsky [10].

(B_3) *If* $\sum_0^\infty a_n/n^2$ *is convergent and* $\sum_0^\infty a_n$ *is summable* (C, k), $k\geqq -1$, *then* $\sum_0^\infty a_n$ *is summable* $(\gamma, k+1+\delta)$.

See Bosanquet [1].

In view of the above results the question naturally arises whether we can remove the restrictions (a) that in (A_3) the series in question is a Fourier series and $k \geqq 1$ and (b) that in (B_3) $\sum a_n/n^2$ is convergent, and obtain the general results

(A'_3) *(γ, k) implies $(C, k+\delta)$, $k \geqq 0$,*

and

(B'_3) *(C, k) implies $(\gamma, k+1+\delta)$, $k \geqq -1$.*

But so far as (B_3) is concerned, we may state here that the convergence of $\sum a_n/n^2$ is essential for the truth of the conclusion, because we shall prove a result (Lemma 5) which implies that if $\sum a_n$ is summable (γ, k), $k \geqq 2$, then $\sum a_n/n^2$ is convergent.

In this paper we shall obtain the following results of the type of (A'_3):

(i) *If k is zero or a positive integer, then (γ, k) implies $(C, k+\delta)$.*

(ii) *If $[k] \geqq 4$, then (γ, k) implies $(C, k+\delta)$.*

The question of the truth of (A'_3) for fractional values of k less than 4 is still open.

2. Lemmas.

LEMMA 1. *If $k>0$, then for large positive values of x,*

$$\gamma_k(x)=\frac{A}{x^2}+O\left(\frac{1}{x^3}\right)+\frac{B\cos\left(x-\frac{k\pi}{2}\right)}{x^k}+O\left(\frac{1}{x^{k+1}}\right),$$

where A and B are non-zero constants; the asymptotic formulae for the derivatives of $\gamma_k(x)$ are obtained by formal differentiation of this formula.

This result is familiar. See, for example, Bosanquet [1].

LEMMA 2. *Let $f(x)$ be periodic with period 2π and Lebesque integrable, and let $\sum (a_n \cos nx + b_n \sin nx)$ be its Fourier series. Set*

$$\phi(t)=\frac{1}{2}\{f(x+t)+f(x-t)\},$$

$$\psi(t)=\frac{1}{2}\{f(x+t)-f(x-t)\},$$

$$g(t)=\begin{cases}\phi(t)/t^r & \textit{if } r \textit{ is an even integer}\\ \psi(t)/t^r & \textit{if } r \textit{ is an odd integer.}\end{cases}$$

If $g(t)$ is integrable in the Cesàro-Lebesgue sense in $(0,\pi)$ and its Fourier series is summable (C,k) at $t=0$, then $\sum\left(\frac{d}{dx}\right)^r(a_n \cos nx+b_n \sin nx)$ is summable $(C,k+r)$.

For this result see Bosanquet [2, Theorem 2].

LEMMA 3. *Let $f(t)$ be an even periodic function with period 2π. If $f(t)\varepsilon C_\lambda L$ where $0<\lambda<1$ and $f(t)=o(1)(C,\lambda+1)$ as $t\to 0$, then the $(C_\lambda L)$-Fourier series of $f(t)$ at $t=0$ is summable (C,k) for every $k>\lambda+1$.*

For this result see Sargent [9, Theorem 4].

LEMMA 4. *If $\sum_0^\infty a_n\gamma_k(nt)$ is convergent for $\alpha<t<\beta$, then*

$$\text{(i)}\quad a_n=o(n^k) \quad \textit{if} \quad 0<k\leqq 2\,,$$

$$\text{(ii)}\quad a_n=o(n^2) \quad \textit{if} \quad k\geqq 2\,.$$

Proof Case 1. $k=0$. The hypothesis implies that $\lim_{n\to\infty} a_n \cos nt=0$ for $\alpha<t<\beta$, which, by the Cantor-Lebesgue theorem, implies that $a_n=o(1)$.

Case 2. $k>0$. The hypothesis implies that $\lim_{n\to\infty} a_n\gamma_k(nt)=0$ for $\alpha<t<\beta$, which, on account of Lemma 1, implies that

$$\lim_{n\to\infty} a_n\left\{\frac{A}{n^2t^2}+O\left(\frac{1}{n^3t^3}\right)+\frac{B\cos\left(nt-k\frac{\pi}{2}\right)}{n^kt^k}+O\left(\frac{1}{n^{k+1}t^{k+1}}\right)\right\}=0 \quad \text{for } \alpha<t<\beta\,.$$

If $0<k\leqq 2$, we write this as

$$\lim_{n\to\infty}\frac{a_n}{n^k}\left\{\frac{A}{n^{2-k}t^2}+O\left(\frac{1}{n^{3-k}t^3}\right)+\frac{B\cos\left(nt-k\frac{\pi}{2}\right)}{t^k}+O\left(\frac{1}{nt^{k+1}}\right)\right\}=0\,,$$

and if $k>2$, we write

$$\lim_{n\to\infty}\frac{a_n}{n^2}\left\{\frac{A}{t^2}+O\left(\frac{1}{nt^3}\right)+\frac{B\cos\left(nt-k\frac{\pi}{2}\right)}{n^{k-2}t^k}+O\left(\frac{1}{n^{k-1}t^{k+1}}\right)\right\}=0\,.$$

The result, for $0<k\leqq 2$, is now obtained by a slight modification of the

usual proof of the Cantor-Lebesgue theorem, whereas, for $k>2$ we get the result by noticing that the expression within brackets tends to a non-zero limit as n tends to infinity.

LEMMA 5. *If* $k\geqq 2$ *and* $\sum_0^\infty a_n\gamma_k(nt)$ *is convergent for* $\alpha<t<\beta$, *then* $\sum_1^\infty a_n/n^2$ is convergent.

Proof. Since Kuttner [7] has proved the result for $k=2$, we assume that $k>2$. We also assume without any loss of generality that $\alpha>0$. Now suppose that (α_0, β_0) is a subinterval of (α, β). Since

$$C_k(nt)=\frac{(nt)^{k-2}}{\Gamma(k-1)}-C_{k-2}(nt)\ , \tag{2.1}$$

therefore

$$\begin{aligned}&\int_{\alpha_0}^{\beta_0} C_k(nt)(\beta_0-t)(t-\alpha_0)\,dt\\ &\quad=\int_{\alpha_0}^{\beta_0}\frac{n^{k-2}t^{k-2}}{\Gamma(k-1)}(\beta_0-t)(t-\alpha_0)\,dt-\int_{\alpha_0}^{\beta_0} C_{k-2}(nt)(\beta_0-t)(t-\alpha_0)\,dt\\ &\quad=n^{k-2}\phi(\alpha_0, \beta_0)-\int_{\alpha_0}^{\beta_0} C_{k-2}(nt)(\beta_0-t)(t-\alpha_0)\,dt\ ,\end{aligned}$$

where

$$\phi(\alpha_0, \beta_0)=\frac{1}{\Gamma(k-1)}\int_{\alpha_0}^{\beta_0} t^{k-2}(\beta_0-t)(t-\alpha_0)\,dt$$

is positive.

Hence, integrating by parts twice, we have

$$\begin{aligned}&\int_{\alpha_0}^{\beta_0} C_k(nt)(\beta_0-t)(t-\alpha_0)\,dt \\ &\quad=n^{k-2}\phi(\alpha_0, \beta_0)-\frac{1}{n^2}[(\beta_0-\alpha_0)\,\{C_k(n\beta_0)+C_k(n\alpha_0)\}]+\frac{2}{n^2}\int_{\alpha_0}^{\beta_0} C_k(nt)\,dt\ .\end{aligned} \tag{2.2}$$

Now, since $k>2$,

$$C_k(nt)=\frac{n^k t^k}{\Gamma(k+1)}\gamma_k(nt)=O(n^{k-2})\ ,$$

for any fixed t. Hence, from (2.2) we get

$$\int_{\alpha_0}^{\beta_0} C_k(nt)(\beta_0-t)(t-\alpha_0)\,dt=n^{k-2}\phi(\alpha_0, \beta_0)+O(n^{k-4})\ . \tag{2.3}$$

Therefore, if p and q are two positive integers and $q>p$,

$$(2.4)\quad \int_{\alpha_0}^{\beta_0}\left\{\sum_{n=p}^{q}\frac{a_n}{n^k}C_k(nt)(\beta_0-t)(t-\alpha_0)\right\}dt=\sum_{n=p}^{q}\int_{\alpha_0}^{\beta_0}\frac{a_n}{n^k}C_k(nt)(\beta_0-t)(t-\alpha_0)\,dt$$

$$=\phi(\alpha_0,\beta_0)\sum_{n=p}^{q}\frac{a_n}{n^2}+\sum_{n=p}^{q}O(n^{-4})a_n \quad \text{by (2.3)}$$

$$=\phi(\alpha_0,\beta_0)\sum_{n=p}^{q}\frac{a_n}{n^2}+\sum_{n=p}^{q}o\left(\frac{1}{n^2}\right) \quad \text{by Lemma 4.}$$

If possible, let the lemma be false. Then we can find a positive number ε such that, for an infinity of pairs of integers (p_i, q_i), $q_i>p_i$, we have

$$(2.5)\qquad \left|\sum_{n=p_i}^{q_i}\frac{a_n}{n^2}\right|>2\varepsilon\ .$$

Again, if p_0 is sufficiently large, then

$$(2.6)\qquad \sum_{n=p_0}^{q_0}o\left(\frac{1}{n^2}\right)<\varepsilon\phi(\alpha_0,\beta_0)\ .$$

From (2.4), (2.5), (2.6) it follows that

$$(2.7)\qquad \left|\int_{\alpha_0}^{\beta_0}\left\{\sum_{n=p_0}^{q_0}\frac{a_n}{n^k}C_k(nt)(\beta_0-t)(t-\alpha_0)\right\}dt\right|>\varepsilon\phi(\alpha_0,\beta_0)\ .$$

But the quantity on the left hand side of (2.7)

$$(2.8)\qquad \leqq\left\{\underset{\alpha_0\leqq t\leqq\beta_0}{\text{l.u.b.}}\left|\sum_{n=p_0}^{q_0}\frac{a_n}{n^k}C_h(nt)\right|\right\}\int_{\alpha_0}^{\beta_0}(\beta_0-t)(t-\alpha_0)\,dt\ .$$

From (2.7) and (2.8).

$$\begin{aligned}\underset{\alpha_0\leqq t\leqq\beta_0}{\text{l.u.b.}}\left|\sum_{n=p_0}^{q_0}\frac{a_n}{n^k}C_k(nt)\right| &> \varepsilon\phi(\alpha_0,\beta_0)\Big/\int_{\alpha_0}^{\beta_0}(\beta_0-t)(t-\alpha_0)\,dt\\ &> \varepsilon\alpha_0^{k-2}/\Gamma(k-1)\\ &\geqq \varepsilon\alpha^{k-2}/\Gamma(k-1)\\ &> M\varepsilon\ ,\end{aligned}$$

where M is a positive constant independent of the subinterval (α_0, β_0) Hence $\left|\sum_{n=p_0}^{q_0}\frac{a_n}{n^k}C_k(nt)\right|<M\varepsilon$ at some point in (α_0, β_0) and therefore throughout a subinterval (α_1, β_1) of (α_0, β_0), since $\sum_{n=p_0}^{q_0}\frac{a_n}{n^k}C_k(nt)$ is a continuous function of t.

If, in the above argument, we now replace (α_0, β_0) by (α_1, β_1), (p_0, q_0) by (p_1, q_1) where $p_1>q_0$, we will reach the conclusion that

$$\left|\sum_{n=p_1}^{q_1} \frac{a_n}{n^k} C_k(nt)\right| > M\varepsilon$$

throughout a subinterval (α_2, β_2) of (α_1, β_1). We can thus determine a sequence of pairs of integers (p_i, q_i), tending to infinity with i, and a corresponding sequence of intervals (α_i, β_i) such that $\alpha_{i+1} \geqq \alpha_i$, $\beta_{i+1} \leqq \beta_i$ and

$$\left|\sum_{n=p_i}^{q_i} \frac{a_n}{n^k} C_k(nt)\right| > M\varepsilon$$

throughout (α_i, β_i). Therefore, there is at least one point t_0 common to all these intervals such that the infinite series $\sum \frac{a_n}{n^k} C_k(nt)$ diverges for $t = t_0$. This contradicts the hypothesis of the lemma.

3. Theorems.

THEOREM 1. *If* $\sum_0^\infty a_n$ *is summable* (γ, k) *where* k *is zero or a positive integer, then* $\sum_0^\infty a_n$ *is summable* $(C, k+\delta)$, $\delta > 0$.

Proof. Case 1. $k > 0$. As we have already noted in the introduction that the result is known to be true for $k=1$ and $k=2$ we take k to be an integer greater than 2 and assume that $\sum_0^\infty a_n$ is summable (γ, k) to S.

Suppose k is an even integer. Then by (1.2) and repeated application of (2.1), we find that, if $n \geqq 1$ and $t \neq 0$,

$$\gamma_k(nt) = \left(\frac{A_1}{n^2t^2} + \frac{A_2}{n^4t^4} + \cdots + \frac{A_{k/2}}{n^kt^k}\right) + \frac{R \cos nt}{n^kt^k},$$

where R, A_1, A_2, etc. are some constants. Therefore

$$\sum_{n=0}^\infty a_n \gamma_k(nt) = a_0 + \sum_{n=1}^\infty a_n \left\{\left(\frac{A_1}{n^2t^2} + \frac{A_2}{n^4t^4} + \cdots + \frac{A_{k/2}}{n^kt^k}\right) + \frac{R \cos nt}{n^kt^k}\right\}, \tag{3.1}$$

for $t \neq 0$. Since, by Lemmas 4 and 5, $\sum_1^\infty a_n/n^2$, $\sum_1^\infty a_n/n^4$, etc. converge respectively to S_1, S_2, etc. say, it follows from (3.1) that $\sum_1^\infty \frac{a_n}{n^k} \cos nt$ is covergent for $0 < t < A$. It is also convergent for $t = 0$.

From (3.1)

$$a_0 + \frac{R}{t^k} \sum_1^\infty \frac{a_n}{n^k} \cos nt = \sum_0^\infty a_n \gamma_k(nt) - \left(\frac{A_1S_1}{t^2} + \frac{A_2S_2}{t^4} + \cdots + \frac{A_{k/2}S_{k/2}}{t^k}\right). \tag{3.2}$$

By suitably altering $k/2+1$ terms of the series $\sum a_n$ and working with the resulting series, say $\sum a'_n$, we can simultaneously have $S=a_0=a'_0$, $S_1=S_2=\cdots=S_{k/2}=0$, so that by (3.2),

$$\sum_1^\infty \frac{a'_n}{n^k}\cos nt=o(t^k) \text{ as } t\to 0 . \tag{3.3}$$

Again, $\frac{a'_n}{n^k}=o(n^{2-k})=o(n^{-1})$, so that $\sum_1^\infty \frac{a'_n}{n^k}\cos nt$ is a Fourier series converging to a function $f(t)$, say, in a neighbourhood of the origin. Since $f(t)=o(t^k)$ for small t, it follows that the kth symmetric generalised derivative of $f(t)$ exists at $t=0$, and is equal to zero there. Hence, by virtue of well known results in the theory of Fourier series, we can immediately conclude that $(C, k+\delta)\sum_1^\infty a'_n=0$, so that $(C, k+\delta)\sum_0^\infty a_n=S$. The proof, when k is an odd integer, is similar.

Case 2. $k=0$. We are given that $\sum_0^\infty a_n\cos nt$ converges to a function $f(t)$ for $0<t<A$ and $\lim_{t\to 0} f(t)$ is a finite number S. Therefore $f(t)$ is bounded in some interval $0<t<\eta$.

Let $\sum_0^\infty b_n\cos nt$ be the Fourier series of an even periodic function $\lambda(t)$ defined as follows.

$$\lambda(t)=\begin{cases}1 & \text{for } 0\leqq t\leqq \eta'<\eta\\ 0 & \text{for } \eta\leqq t\leqq \pi .\end{cases}$$

Moreover, let $\lambda(t)$ change smoothly as t increases from η' to η, so that $\lambda'''(t)$ exists and is continuous. Hence $b_n=o(1/n^3)$. (See [**5** Theorem 40]).

If $\sum_0^\infty c_n\cos nt$ is the formal product of $\sum_0^\infty a_n\cos nt$ and $\sum_0^\infty b_n\cos nt$, then it follows from Rajchman's theory of formal multiplication [**12**, section 11.42] that $\sum_0^\infty c_n\cos nt$ converges to $f(t)$ in $0<t\leqq\eta'$, to $\lambda(t)f(t)$ in $\eta'\leqq t\leqq\eta$, and to zero in $\eta\leqq t\leqq\pi$. Hence it follows that $\sum_0^\infty c_n\cos nt$ is a Fourier series [**12**, Theorem 11.33], and therefore $\sum_0^\infty c_n$ is summable (c, δ) for $\delta>0$, because $\lim_{t\to 0} f(t)=S$. Consequently, $\sum_0^\infty a_n$ is also summable (c, δ) (See [**12**, section 11.42].

THEOREM 2. *Let* $\sum_0^\infty a_n$ *be summable* (γ, k) *where* $k\geqq 1$ *and let* $\sum_1^\infty \frac{a_n}{n^{[k]-1}}\cos nx$ *(when* $[k]$ *is odd) or* $\sum_1^\infty \frac{a_n}{n^{[k]-1}}\sin nx$ *(when* $[k]$ *is even) be* *a Fourier series. Then* $\sum_0^\infty a_n$ *is summable* $(c, k+\delta)$, $\delta>0$.

Proof. Since, in Theorem 1, we have already proved the result for integral k under more general conditions, we assume k to be non-integral. We also take $[k]$ to be an odd integer. The proof, when $[k]$ is even, is similar.

By making $\dfrac{[k]-1}{2}$ applications of (2.1), using Lemma 5, and arguing as in the deduction of (3.3), we get

$$\sum_1^\infty \frac{a'_n}{n^k} C_{k-[k]+1}(nt)=o(t^k) , \tag{3.4}$$

where $\sum_0^\infty a'_n$ differs from $\sum_0^\infty a_n$ in a finite number of terms only, $\sum_1^\infty a'_n/n^2=0$, $\sum_1^\infty a'_n/n^4=0$, etc., and $\sum_0^\infty a'_n$ is summable (γ, k) to $a'_0=a_0$. Let $\sum_1^\infty \dfrac{a'_n}{n^{[k]-1}} \cos nx$ be the Fourier series of an even function $\phi(x)\varepsilon L$. Then it can be easily shown that

$$\begin{aligned}\phi_{k-[k]+1}(t)&=\sum_1^\infty \frac{a'_n}{n^k}C_{k-[k]+1}(nt) \\ &=o(t^k) \quad \text{by (3.4)} .\end{aligned} \tag{3.5}$$

Again, $\phi(t)\in L$ obviously implies that $\phi(t)$ is Cesàro-Lebesgue integrable $C_\lambda L$ for any $\lambda \geqq 0$ so that

$$\phi(t)\in C_{k-[k]}L \tag{3.6}$$

From (3.5) and (3.6), we have (See [3, Theorem 2])

$$\frac{\phi(t)}{t^{[k]-1}}\in C_{k-[k]}L$$

and

$$\frac{\phi(t)}{t^{[k]-1}}=o(1)(C, k-[k]+1) \quad \text{as} \quad t\to 0 .$$

Hence, in view of Lemma 3, we conclude that the (Cesàro-Lebesgue) Fourier series of $\dfrac{\phi(t)}{t^{[k]-1}}$ is summable $(C, k-[k]+1+\delta)$ at $t=0$ for any $\delta>0$. Now it follows from Lemma 2, where we take $r=[k]-1$, that $\sum_1^\infty a'_n$ is summable $(C, k+\delta)$. Hence $\sum_0^\infty a_n$ is also summable $(C, k+\delta)$.

COROLLARY. *If* $\sum_{0}^{\infty} a_n$ *is summable* (γ, k), $k \geqq 4$, *then* $\sum_{0}^{\infty} a_n$ *is summable* $(C, k+\delta)$, $\delta > 0$.

The corollary follows immediately from the theorem because $a_n = o(n^2)$ by Lemma 4.

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THE TAUBERIAN THEOREM FOR GROUP ALGEBRAS OF VECTOR-VALUED FUNCTIONS

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1. Introduction. The object of this paper is to prove the ideal-theoretic version of Wiener's tauberian theorem for algebras which we will call *group algebras of vector-valued functions*. These algebras are defined as follows. Let $G=\{a, b, \cdots\}$ denote a locally compact abelian group and let $X=\{x, y, \cdots\}$ represent a complex commutative Banach algebra. Our group algebra $B=B(G, X)$ consists of the set of all measurable absolutely integrable functions defined over G with values in X. Of course we must identify functions which differ on sets of Haar measure 0. As norm for an element $f \in B$ we take

$$\|f\|_B=\int_G |f(a)|_X \, da \; .$$

(Hereafter, we will omit an indication of the domain of integration if the integral is taken over the entire group G.) The space $B(G, X)$ is known to be complete in the given norm [4]. Further, we introduce into B the following operations

$$(f+g)(a)=f(a)+g(a) \; , \qquad (\lambda f)(a)=\lambda f(a)$$

where λ is a complex number, and

$$(f*g)(a)=\int f(b)g(a-b) \, db$$

where the integral is taken in the sense of Bochner [1, 4] with respect to Haar measure db. The algebra $B(G, X)$ thus becomes, as is easily shown, a complex commutative Banach algebra which specializes into the classical group algebra $L(G)$ if X is chosen as the complex numbers. It is these algebras $B(G, X)$ which will be the object of our study.

The tauberian theorem for $B(G, X)$ will be proved by appealing to a theorem in the general theory of Banach algebras (see [5], p. 85 corollary, or [6], Theorem 38.) This latter result might be designated as the "general tauberian theorem." It says that if a complex commutative B-algebra Y is semi-simple, regular, and is such that the set of $y \in Y$ with $\phi_M(y)$ having compact support in $\mathfrak{M}(Y)$ is dense in Y, then every proper closed ideal in Y is contained in a regular maximal ideal.

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Here $\mathfrak{M}(Y)$ denotes the space (in the usual weak topology) of regular maximal ideals in Y and ϕ_M represents the canonical homomorphism from Y onto the complex numbers associated with an $M \in \mathfrak{M}(Y)$. It will be taken as known that the classical group algebra $L(G)$ satisfies the hypotheses of this general tauberian theorem. This amounts, then, to assuming the tauberian theorem in the case of $L(G)$. It will also be assumed, but only in the final theorem of the paper, that the range space X meets the conditions of the general tauberian theorem. It is clear, therefore, that the proof of the tauberian theorem for $B(G, X)$ found here, does not yield a new proof in the case of $L(G)$. However, this paper does provide, it is hoped, an interesting application of the general tauberian theorem in the case of our generalized algebras.

2. Proof of the theorem. It is important to know the form of the most general multiplicative linear functional in $B(G, X)$. This is determined in Lemma 1 which requires the following preliminary observations.

The convolution $f * g$ of a function $f \in L(G)$ with a function $g \in B(G, X)$ results, as in easily seen, in a function contained in $B(G, X)$. Suppose $\{j_W\}$ is an approximate identity for $L(G)$; that is, for each neighborhood W of the identity 0 in G, j_W is some (numerical) non-negative function vanishing off W such that $\int j_W(a)\,da=1$. Then for every $f \in L(G)$ we have $j_W * f \to f$ as $W \to 0$. (Of course, convergence is here understood in the sense of directed systems.) But $\{j_W\}$ acts, also, as an approximate identity in $B(G, X)$, that is, $j_W * g \to g$ in B-norm for every $g \in B$. This can be shown, just as in the case of $L(G)$, by noting that functions in B are continuous in B-norm [4], i.e., for any $\varepsilon > 0$ there is a neighborhood W_ε of 0 in G such that $\|f(a-b)-f(a)\|_B < \varepsilon$ if $b \in W_\varepsilon$.

The approximate identity will be of service to us in proving Lemma 1 which we now state.

LEMMA 1. *Let $\hat{G}=\{\hat{a}, \hat{b}, \cdots\}$ denote the dual group of G in the usual Pontrjagin topology. Define the "Fourier transform" of $f \in B$ as*

$$\hat{f}(M, \hat{a}) = \int \phi_M f(a)\,(a, \hat{a})\,da\ .$$

The Fourier transform evaluated at a fixed $(M, \hat{a}) \in \mathfrak{M}(X) \times \hat{G}$ is a non-zero, continuous multiplicative linear functional in B and, further, all such functionals are of this type, that is, if μ is a non-zero, continuous multiplicative linear functional in B, then there is some $(M, \hat{a})$ such that $\mu(f)=\hat{f}(M, \hat{a})$ for every $f \in B$.

Proof. That the Fourier transform, at a fixed $(M, \hat{a})$, is a multiplicative functional is easily shown. We, consequently, turn to the second half of the lemma. Choose a function $f \in B$ such that $\mu(f) \neq 0$ and let $\{j_W\}$ be an approximate identity. For every $x \in X$, $\lim_{W \to 0} \mu(j_W x)$ exists. (Here, $j_W x$ denotes the function $(j_W x)(a) = j_W(a) \cdot x$. Of course, $j_W x \in B$.) For

$$\mu(j_W x * f) = \mu(j_W x)\mu(f) = \mu[(j_W * f)x] \to \mu(fx) \qquad \text{as} \quad W \to 0$$

because $(j_W * f)x \to fx$. Hence $\mu(j_W x)$ necessarily converges to a limit independent of the approximate identity $\{j_W\}$, namely $\mu(fx)/\mu(f)$. This limit is likewise independent of the $f \in B$ with $\mu(f) \neq 0$, for if $g \in B$ is such that $\mu(g) \neq 0$, then

$$\mu(fx)\,\mu(g) = \mu[(f * g)x] = \mu(gx * f) = \mu(gx)\,\mu(f)$$

so that $\mu(fx)/\mu(f) = \mu(gx)/\mu(g)$. We will denote the limit of $\mu(j_W x)$ by $\phi_\mu(x)$ for $x \in X$.

Suppose, temporarily, that X possesses an identity e. Then ϕ_μ is certainly not zero. For $\phi_\mu(e) = \mu(fe)/\mu(f) = \mu(f)/\mu(f) = 1$. Further, ϕ_μ is easily seen to be additive and homogeneous, that is,

$$\phi_\mu(\lambda_1 x + \lambda_2 y) = \lambda_1 \phi_\mu(x) + \lambda_2 \phi_\mu(y)$$

for all $x, y \in X$ and complex numbers λ_1, λ_2.

$$\phi_\mu(xy) = \frac{\mu(f * f \cdot xy)}{\mu(f * f)} = \frac{\mu(fx)}{\mu(f)} \cdot \frac{\mu(fy)}{\mu(f)} = \phi_\mu(x)\phi_\mu(y) ,$$

so that ϕ_μ is multiplicative. Therefore, as is well known, there is some $M \in \mathfrak{M}(X)$ (depending on μ) such that $\phi_\mu(x) = \phi_M(x)$.

Still assuming that X has an e (which we may take of norm 1), let $g \in L(G)$, $x \in X$. Then

$$\mu(j_W * gx) = \mu(j_W x * ge) = \mu(j_W x)\mu(ge) \to \phi_M(x)\mu(ge) .$$

But

$$\mu(j_W * gx) = \mu[(j_W * g)x] \to \mu(gx)$$

so that $\mu(gx) = \phi_M(x)\mu(ge)$ for any $g \in L(G)$ and any $x \in X$. Since $Le = \{ge \in B \mid g \in L(G)\}$ is isometrically isomorphic with $L(G)$ and since μ is a continuous multiplicative linear functional on $Le \subset B$ (not identically zero on Le, because linear combinations of functions gx with $g \in L(G)$, $x \in X$ are dense in B [1, 4]) there is an $\hat{a} \in \hat{G}$ (depending on μ) such that $\mu(ge) = \int g(a)(a, \hat{a})\, da$ for all $g \in L(G)$.

Suppose, now, that f is any function in B. Then, because the simple functions are dense in $B(G, X)$ as we observed above, there exists a sequence $g_n \in B$ such that $g_n \to f$ and $\mu(g_n) = \hat{g}_n(M, \hat{a}) \to \hat{f}(M, \hat{a})$ and so $\mu(f) = \hat{f}(M, \hat{a})$.

We now remove the restriction that X possess an identity. If X lacks an e, then we imbed X, isometrically and isomorphically, in a Banach algebra X' with unit e in such a way that maximal ideals in X' are the regular maximal ideals in X and X itself. This is done in the usual well-known manner. The homomorphisms of X' onto the complex numbers are ϕ_M ($M \in \mathfrak{M}(X)$) and the additional functional ϕ_X, where $\phi_X(x + \lambda e) = \lambda$ for $x \in X$, λ a complex number. By what we have already proved, the non-zero multiplicative functionals in $B(G, X')$ are of the form $\hat{f}(M, \hat{a})$ and the additional functionals $\hat{f}(X, \hat{a})$. These latter functionals, namely, $\int \phi_X f(a)(a, \hat{a})\, da$ are, however, all identically zero in $B(G, X)$ and thus the lemma is established.

The following lemma gives a topological characterization of the space of regular maximal ideals $\mathfrak{M}(B)$ in $B(G, X)$. For a similar result and proof see [2].

LEMMA 2. *The space $\mathfrak{M}(B)$ of regular maximal ideals in B, topologized in the weak topology, is homeomorphic with $\mathfrak{M}(X) \times \hat{G}$, that is the topological product of $\mathfrak{M}(X)$ and $\hat{G}$.*

Proof. There is a $1-1$ correspondence between the points of $\mathfrak{M}(B)$ and those of $\mathfrak{M}(X) \times \hat{G}$. To see this, suppose $(M, \hat{a}) \neq (N, \hat{b})$. If $\hat{a} \neq \hat{b}$ and $M = N$, take $x \notin M$ and find an $f \in L(G)$ such that

$$\hat{f}(\hat{a}) = \int f(a)(a, \hat{a})\, da \neq \hat{f}(\hat{b}) \ .$$

Then

$$\widehat{fx}(M, \hat{a}) = \hat{f}(\hat{a})\phi_M(x) \neq \widehat{fx}(N, \hat{b}) \ .$$

If $\hat{a} \neq \hat{b}$ and $M \neq N$ or if $\hat{a} = \hat{b}$ and $M \neq N$, then we may proceed in the same way to construct a function fx with $f \in L(G)$, $x \in X$ such that the Fourier transform of fx separates the points $(M, \hat{a})$, $(N, \hat{b})$. No two points in $\mathfrak{M}(X) \times \hat{G}$ give rise to the same regular maximal ideal in $\mathfrak{M}(B)$.

The topology of $\mathfrak{M}(B)$ is precisely that induced by the family $\mathfrak{F} = \{\hat{f}(M, \hat{a}) | f \in B\}$ of functions defined on $\mathfrak{M}(X) \times \hat{G}$. We must show that this topology is identical with the product topology of $\mathfrak{M}(X) \times \hat{G}$. This will be done by showing that the $\mathfrak{F}$-topology of $\mathfrak{M}(X) \times \hat{G}$ is identical with that induced by another family of functions $\mathfrak{F}\subset\mathfrak{J}$ defined on $\mathfrak{M}(X)\times\hat{G}$. Then the proof will be completed by showing that this $\mathfrak{F}$-topology is identical with the product topology of $\mathfrak{M}(X)\times\hat{G}$.

First we must define $\mathfrak{F}$. For each positive integer n and each choice $f_1, f_2, \cdots, f_n \in L(G)$; $x_1, x_2, \cdots, x_n \in X$, there is a function $\hat{h}$ defined on $\mathfrak{M}(X)\times\hat{G}$ by $\hat{h}(M, \hat{a})=\sum_{i=1}^{n}\widehat{f_i x_i}(M, \hat{a})$, Let $\mathfrak{F}$ be the family of all functions $\hat{h}$ so defined. Clearly $\mathfrak{F}\subset\mathfrak{J}$. But $\mathfrak{F}$ is also dense in $\mathfrak{J}$ in the uniform norm. For, suppose $f\in B$. Then we can find $f_i\in L(G)$, $x_i\in X$, such that $\left\|f-\sum_{i=1}^{n} f_i x_i\right\|_B<\varepsilon$. Hence

$$\left|\hat{f}(M, \hat{a})-\sum_{i=1}^{n}\widehat{f_i x_i}(M, \hat{a})\right|=\left|\int\left[\phi_M f(a)-\sum_{i=1}^{n} f_i(a)\phi_M(x_i)\right](a, \hat{a})\,da\right|<\varepsilon\,.$$

Therefore, $\sup\left|\hat{f}(M, \hat{a})-\sum_{i=1}^{n}\widehat{f_i x_i}(M, \hat{a})\right|\leqq\varepsilon$ where the sup is taken over $\mathfrak{M}(X)\times\hat{G}$. This shows $\mathfrak{F}$ is dense in $\mathfrak{J}$ in the sup-norm and it is easy to see, from this, that the $\mathfrak{F}$- and $\mathfrak{J}$-topologies on $\mathfrak{M}(X)\times\hat{G}$ are identical.

It remains to show that the $\mathfrak{F}$-topology on $\mathfrak{M}(X)\times\hat{G}$ is the same as the product topology. To do this we first develop a few properties of $\mathfrak{F}$.

(i) *The functions in $\mathfrak{F}$ separate the points of $\mathfrak{M}(X)\times\hat{G}$* as we saw in the beginning of this proof.

(ii) *Functions in $\mathfrak{F}$ are continuous over $\mathfrak{M}(X)\times\hat{G}$ in the product topology.* For, if $f\in L(G)$, $x\in X$, $(M_0, \hat{a}_0)$ is a fixed point of $\mathfrak{M}(X)\times\hat{G}$, and $\varepsilon>0$, then

$$\begin{aligned}
&|\widehat{fx}(M, \hat{a})-\widehat{fx}(M_0, \hat{a}_0)|\\
&\qquad\leqq|\hat{f}(\hat{a})\phi_M(x)-\hat{f}(\hat{a})\phi_{M_0}(x)|+|\hat{f}(\hat{a})\phi_{M_0}(x)-\hat{f}(\hat{a}_0)\phi_{M_0}(x)|\\
&\qquad=|\hat{f}(\hat{a})|\cdot|\phi_M(x)-\phi_{M_0}(x)|+|\phi_{M_0}(x)|\cdot|\hat{f}(\hat{a})-\hat{f}(\hat{a}_0)|\\
&\qquad\leqq|f|_L\cdot|\phi_M(x)-\phi_{M_0}(x)|+|x|\cdot|\hat{f}(\hat{a})-\hat{f}(\hat{a}_0)|\\
&\qquad<|f|_L\cdot(\varepsilon/2|f|_L)+|x|\cdot(\varepsilon/2|x|)=\varepsilon
\end{aligned}$$

if $(M, \hat{a})\in U(M_0)\times\hat{U}(\hat{a}_0)$ where $\hat{U}(M_0)$, $\hat{U}(\hat{a}_0)$ are neighborhoods of M_0, $\hat{a}_0$ in $\mathfrak{M}(X)$ and $\hat{G}$, respectively, such that $|\phi_M(x)-\phi_{M_0}(x)|<\varepsilon/2|f|_L$ for $M\in U(M_0)$ and $|\hat{f}(\hat{a})-\hat{f}(\hat{a}_0)|<\varepsilon/2|x|$ for $\hat{a}\in\hat{U}(\hat{a}_0)$. Since $\mathfrak{F}$ consists of finite linear combinations of $\widehat{fx}(f\in L(G), x\in X)$, each function in $\mathfrak{F}$ is continuous in the product topology of $\mathfrak{M}(X)\times\hat{G}$.

(iii) Let $(M_0, \hat{a}_0)\in\mathfrak{M}(X)\times\hat{G}$. Choose $f\in L(G)$ such that $\hat{f}(\hat{a}_0)\neq 0$

and let $x \in X$ be such that $x \notin M_0$. Then $\widehat{fx}(M_0, \hat{a}_0) \neq 0$, so that *not all functions in* $\mathfrak{F}$ *vanish at a fixed point in* $\mathfrak{M}(X) \times \hat{G}$.

(iv) *Each function in* $\mathfrak{F}$ *vanishes at infinity in* $\mathfrak{M}(X) \times \hat{G}$. For, suppose $\varepsilon > 0$ is given. If $\sum_{i=1}^{n} \widehat{f_i x_i}(M, \hat{a}) \in \mathfrak{F}$, then

$$\left| \sum_{i=1}^{n} \widehat{f_i x_i}(M, \hat{a}) \right| = \left| \sum_{i=1}^{n} \hat{f}_i(\hat{a}) \phi_M(x_i) \right| \leqq \varepsilon$$

if

$$(M, \hat{a}) \notin \left(\bigcup_{i=1}^{n} \mathfrak{C}_i \right) \times \left(\bigcup_{i=1}^{n} \hat{C}_i \right) \equiv \Gamma$$

where $|\hat{f}_i(\hat{a})| < \delta$, $|\phi_M(x_i)| < \delta$ if $\hat{a} \notin \hat{C}_i \subset G$ and $M \notin \mathfrak{C}_i \subset \mathfrak{M}(X)$. Here, $\delta < \min(\sqrt{\varepsilon/n}, \varepsilon/nK_1, \varepsilon/nK_2)$ with $K_1 = \sup_{1 \leqq i \leqq n} |x_i|$ and $K_2 = \sup_{1 \leqq i \leqq n} \sup_{\hat{a} \in \hat{G}} |\hat{f}_i(\hat{a})|$; $\hat{C}_i$ and $\mathfrak{C}_i$ are compact sets which exist because each $\hat{f}_i$ and each x_i vanish at ∞ in $\hat{G}$ and $\mathfrak{M}(X)$, respectively. Γ is compact in $\mathfrak{M}(X) \times \hat{G}$ so that each function in $\mathfrak{F}$ vanishes at ∞.

We now appeal to a result in general point-set topology (see [5] p. 12) which states: If $\mathfrak{G}$ is a family of complex-valued continuous functions vanishing at infinity on a locally compact space S, separating the points of S and not all vanishing at any point of S, then the weak topology induced on S by $\mathfrak{G}$ is identical with the given topology of S. We take $S = \mathfrak{M}(X) \times \hat{G}$ and $\mathfrak{G} = \mathfrak{F}$. This finishes the proof.

The next lemma deals with the radical and regularity in $B(G, X)$. Following this we conclude with the tauberian theorem.

LEMMA 3. (i) *The radical of* B *consists of those functions* $f \in B$ *with values in the radical of* X *a.e.*

(ii) *If* X *is regular, then* $B(G, X)$ *is regular.*

Proof. Necessity (i). Suppose f takes values in the radical $\mathfrak{R} = \bigcap_{M \in \mathfrak{M}(X)} M$ of X a.e. Then $\phi_M f = 0$ a.e. for each $M \in \mathfrak{M}(X)$ and thus $\hat{f}(M, \hat{a}) = 0$ for each $(M, \hat{a}) \in \mathfrak{M}(X) \times \hat{G}$. This means f is in the radical of B.

Sufficiency (i). Suppose that f is in the radical of B. We must show that f takes values in the radical $\mathfrak{R}$ of X, a.e. We have $\hat{f}(M, \hat{a}) = 0$ for all $(M, \hat{a}) \in \mathfrak{M}(X) \times \hat{G}$, that is $\int \phi_M f(a)(a, \hat{a})\, da = 0$ for all $(M, \hat{a})$. Since $\phi_M f$ is in $L(G)$ and since $L(G)$ is semi-simple, we have $\phi_M f = 0$ a.e. for each $M \in \mathfrak{M}(X)$.

Let $\{j_W\}$ be an approximate identity for $L(G)$ consisting of bounded functions vanishing outside neighborhoods W of the identity in G. Since f is continuous in B-norm, it follows that the functions $j_W * f$ from G to X are continuous. Consequently, the functions $j_W * f$ take values in $\mathfrak{N}$ *everywhere* over G since $\mathfrak{N}$ is closed in X. Choose a sequence $\{j_{W_n}\}$ from $\{j_W\}$ such that $j_{W_n} * f \to f$ in B-norm. Then, as is known, there is a subsequence of the $j_{W_n} * f$ converging to f pointwise a.e. in X-norm. Since $\mathfrak{N}$ is closed, f takes values in $\mathfrak{N}$ a.e.

Proof of (ii). Suppose X is a regular algebra. We wish to show that, given any point $(M_0, \hat{a}_0) \in \mathfrak{M}(X) \times \hat{G}$ and any open set $\mathfrak{O}$ containing $(M_0, \hat{a}_0)$, there is a function $g \in B(G, X)$ such that $\hat{g}(M_0, \hat{a}_0) = 1$ and $\hat{g}(M, \hat{a}) = 0$ if $(M, \hat{a}) \notin \mathfrak{O}$. By Lemma 2, the open sets of $\mathfrak{M}(B)$ are of the form $\bigcup_{i \in \Omega} (\hat{O}_i \times \mathfrak{N}_i)$ where the $\hat{O}_i$ are open in $\hat{G}$ and the $\mathfrak{N}_i$ are open in $\mathfrak{M}(X)$. Suppose our $\mathfrak{O}$ equals $\bigcup_{i \in \Omega} (\hat{O}_i \times \mathfrak{N}_i)$; then $(M_0, \hat{a}_0) \in \hat{O}_{i_0} \times \mathfrak{N}_{i_0}$ for some $i_0 \in \Omega$, that is, $a_0 \in \hat{O}_{i_0}$ and $M_0 \in \mathfrak{N}_{i_0}$. We can find a function $f \in L(G)$ such that $\hat{f}(\hat{a}_0) = 1$ and $\hat{f}(\hat{a}) = 0$ if $\hat{a} \notin \hat{O}_{i_0}$. This follows from the regularity of the group algebra $L(G)$. Since X is regular, by hypothesis, there is an $x \in X$ such that $\phi_{M_0}(x) = 1$ and $\phi_M(x) = 0$ if $M \notin \mathfrak{N}_{i_0}$. We will show that the g, above, can be taken to be fx. Firstly, $\widehat{fx}(M_0, \hat{a}_0) = 1$. Now, suppose $(M, \hat{a}) \notin \mathfrak{O}$. Then $(M, \hat{a}) \notin \hat{O}_{i_0} \times \mathfrak{N}_{i_0}$ so that $\hat{a} \notin \hat{O}_{i_0}$ or $M \notin \mathfrak{N}_{i_0}$. In either case, $\widehat{fx}(M, \hat{a}) = 0$. Hence $\widehat{fx}(M, \hat{a}) = 0$ for all $(M, \hat{a}) \notin \mathfrak{O}$.

We might add that if $B(G, X)$ is regular, then X is likewise regular. However, this fact will not be used in the following theorem and so we do not enter into its proof.

COROLLARY. *$B(G, X)$ is semi-simple if and only if X is semi-simple.*

THEOREM. *Let X be semi-simple and regular. Suppose that the elements $x \in X$ with $\phi_M(x)$ having compact support in $\mathfrak{M}(X)$ are dense in X. Then every proper closed ideal in $B(G, X)$ is contained in a regular maximal ideal.*

Proof. By the hypothesis and Lemma 3, it follows that $B(G, X)$ is regular and semi-simple. Using the general tauberian theorem (see the introduction), we can prove that any proper closed ideal in B is contained in a regular maximal ideal by showing that if f is any function in B and $\varepsilon > 0$, there exists an $h \in B$ such that $\|f - h\|_B \leqq \varepsilon$ and $\hat{h}(M, \hat{a})$ has compact support in $\mathfrak{M}(X) \times \hat{G}$. Suppose, therefore, that $f \in B$ and $\varepsilon > 0$ are given. We can find $f_i \in L(G)$, $x_i \in X$ $(i = 1, 2, \cdots, n)$, such that

$$\left\| f - \sum_{i=1}^{n} f_i x_i \right\|_B < \varepsilon/3 .$$

We have functions $f'_i \in L(G)$ such that $|f_i - f'_i|_L < \varepsilon/3Kn$ $(i=1, 2, \cdots, n)$, where $K = \sup_{1 \leqq i \leqq n} |x_i|$ and the $\hat{f}'_i$ have compact support $\hat{C}_i \subset \hat{G}$. This follows from the fact that $L(G)$ satisfies the hypotheses of the general tauberian theorem. By the hypotheses on X, we may find x'_i in X such that $|x_i - x'_i| < \varepsilon/3Rn$ $(i=1, 2, \cdots, n)$, where $R = \sup_{1 \leqq i \leqq n} |f'_i|_L$ and the $\phi_M(x_i)$ have compact support $\mathfrak{C}_i \subset \mathfrak{M}(X)$. Now

$$\left\| f - \sum_{i=1}^{n} f'_i x'_i \right\|_B = \left\| f - \sum_{i=1}^{n} f_i x_i + \sum_{i=1}^{n} (f_i - f'_i) x_i + \sum_{i=1}^{n} f'_i (x_i - x'_i) \right\|_B$$
$$\leqq \varepsilon/3 + Kn(\varepsilon/3Kn) + Rn(\varepsilon/3Rn) = \varepsilon .$$

Take (see above) $h \equiv \sum_{i=1}^{n} f'_i x'_i$. We see that $\hat{h}(M, \hat{a})$ has support $\left(\bigcup_{i=1}^{n} \mathfrak{C}_i \right) \times \left(\bigcup_{i=1}^{n} \hat{C}_i \right)$ which is compact in $\mathfrak{M}(X) \times \hat{G}$. The theorem is now proved.

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T-SETS AND ABSTRACT (L)-SPACES

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1. Introduction. The theory of T-sets and of F_T-functionals was developed [4] in reference to abstract (M)-spaces for application to the characterization of Banach spaces which may be represented as Banach spaces of continuous functions. The purpose of this paper is to discuss their use in reference to abstract (L)-spaces [3] for application to the representation of certain Banach spaces as spaces of integrable functions.

A distinction of three types of abstract (L)-spaces is first made and illustrated. Next an extremely simple characterization of the Banach spaces which are susceptible of a semi-ordering under which they become abstract (L)-spaces of the second or third type is established. Then a complete analysis of the role of T-sets and of F_T-functionals in the third and most important type of abstract (L)-space is given. Finally a few remarks are appended relative to T-sets in abstract (L)-spaces of the first type.

2. Preliminary concepts. Let BL be a semi-ordered Banach space which is a linear lattice under its semi-ordering, and in which the collection P of elements $a \geqq 0$ is closed with respect to the norm. Consider, with reference to the subset P of BL, three possible additional requirements:

(I) If $a,\ b \in P$, then $\|a+b\| = \|a\| + \|b\|$.

(II) If $a,\ b \in P$, then $\|a+b\| = \|a\| + \|b\|$, and P is a subset of BL maximal with respect to this property.

(III) If $a,\ b \in P$, then $\|a+b\| = \|a\| + \|b\|$, and if $a \wedge b = 0$, then $\|a-b\| = \|a+b\|$.

A space BL wherein the subset P possesses property III is usually called an abstract (L)-space. If property III obtains in P, then property II also obtains in P with respect to BL. Thus for any $a \in BL$ with $a \notin P$, $a = a^+ - a^-$ with $a^+,\ a^- \in P$, $a^+ \wedge a^- = 0$, while $a^- \neq 0$. Then

$$\begin{aligned}\|a+a^-\| &= \|a^+\| < \|a^+\| + \|a^-\| + \|a^-\| \\ &= \|a^+ + a^-\| + \|a^-\| = \|a^+ - a^-\| + \|a^-\| = \|a\| + \|a^-\| ,\end{aligned}$$

so that P is maximal in BL with respect to the stated property. Thus for the subset P of BL, we have III $\Rightarrow$ II $\Rightarrow$ I. It will presently be seen, however, that I does not imply II and that II does not imply III. Hence let BLI denote the space BL under the additional assumption

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that the subset P possesses property I but not property II, and similarly for BLII while BLIII denotes the space BL under the assumption that the subset P possesses property III. It is known [3] that a space BLI, under an easy change to an equivalent norm, becomes a space BLIII, neither the elements of the subset P nor their norms being disturbed in the process. Hence reference will be made to spaces BLI, BLII and BLIII as abstract (L)-spaces of type I, II and III.

Now let B represent an arbitrary Banach space. Let P be a subset of B maximal with respect to the property: for every finite set of elements $(b_1, \cdots, b_n)$ in P,

$$\left\| \sum_{i=1}^{n} b_i \right\| = \sum_{i=1}^{n} \|b_i\| .$$

Such subsets are called [4] T-sets. Each T-set P of B has the properties [4, Lemma 2.1]: if $a,\ b \in P$, then $a+b \in P$: if $\|a+b\| = \|a\| + \|b\|$ for all $a \in P$, then $b \in P$. In view of these properties the T-sets of B may be described as subsets P of B that are closed under addition and, as subsets of B, are maximal with respect to the property: $a,\ b \in P$ implies $\|a+b\| = \|a\| + \|b\|$.

For each such T-set P of B define an associated F_T-functional F_P with $F_P(a) = \inf_{b \in P} \{\|a+b\| - \|b\|\}$ for each element a of B. Each such F_T-functional F_P has the following pertinent properties [4, Lemma 2.2]: $F_P(b) = \|b\|$ if and only if $b \in P$; the functional F_P is linear over the linear extension of P in B.

The fact and the general form of the role played by T-sets in abstract (L)-spaces is clear from the definition of these spaces and from their representation as spaces of integrable functions. Using this guide, the possibilities when a beginning is made not with a space BL but with an arbitrary Banach space B are not difficult to discern.

Let P be a T-set of Banach space B. Define a relation $\overset{P}{\leqq}$ on $B \times B$ with $a \overset{P}{\leqq} b$ exactly when $(b-a) \in P$. Since every T-set is closed under addition and under scalar multiplication by non-negative real scalars, this relation determines a linear semi-ordering for B. Since every T-set is closed under the norm and contains the zero element, the set of elements $a \overset{P}{\geqq} 0$ of B coincides with P and is closed under the norm. Reference will be made to the relation $\overset{P}{\leqq}$ as the canonical semi-ordering induced on B by P.

Of course, B is not necessarily a linear lattice with respect to this semi-ordering. However, the F_T-functional F_P associated with P provides a simple test of the semi-ordering in this respect. First apply the fact that $F_P(a) = \|a\|$ exactly when a is an element of P. This means that, for any element $a \in B$, an element $a^+ \in P$ with $(a^+ - a) \in P$ serves as the element $a \vee 0$ with respect to $\overset{P}{\leqq}$ exactly when $F_P(b-a^+) = ||b-a^+||$ for each $b \in P$ with $(b-a) \in P$. Next apply the fact that F_P is linear on the linear extension of P in B. This means that with b, $a^+ \in P$, $F_P(b-a^+) = ||b|| - ||a^+||$. Note, lastly that $a=b-c$ with b, $c \in P$ is equivalent to having both b and $(b-a)$ in P. This may be summarized.

LATTICE CRITERION *For any element $a \in B$ and T-set $P \subset B$, an element $a^+ \in P$ with $(a^+ - a) \in P$ serves as the element $a \vee 0$ under the canonical semi-ordering induced on B by P exactly when $a=b-c$, b, $c \in P$, always implies $||b-a^+|| = ||b|| - ||a^+||$.*

If B becomes a linear lattice and thus an abstract (L)-space of at least type II under the canonical semi-ordering induced on B by T-set P, the significance of the functional F_P is easily found. Thus, if $a = a^+ - a^-$ with a^+, $a^- \in P$ and defined as usual, then

$$F_P(a) = F_P(a^+) - F_P(a^-) = ||a^+|| - ||a^-|| ,$$

so that in the representation of B as a space of integrable functions, the value $F_P(a)$ equals the value of the integral over the representing space of the function representing the element a.

Finally, if a particular T-set P and the corresponding F_T-functional F_P may be thus employed in representing the space B, surely the other T-sets and F_T-functionals of B are eligible for similar usages, presumably in respect to the measurable subsets of the representing space.

With this outline of the possibilities completed, attention is turned to specific details.

3. Preliminary examples. Let L_2 be the set $R \times R$ of all ordered pairs of real numbers. Let L_2 be regarded as a linear lattice using the usual definitions of addition and of scalar multiplication, while $(a, b) \geqq (c, d)$ exactly when $a \geqq c$ and $b \geqq d$ as real numbers. Within L_2 distinguish subsets N_1, N_2 and N_3. In geometric terms, let N_3 be the area about the origin bounded by the pairs of lines $x+y=1$, $x+y=-1$ and $x-y=-1$, $x-y=1$. Let N_2 be the area about the origin bounded by the lines $x+y=1$ and $x+y=-1$ in the first and third quadrants, but by the circle $x^2+y^2=1$ in the second and fourth quadrants. Let N_1 be the area about the origin bounded by the lines $x+y=1$, $x+y=-1$ and by the circle $x^2+y^2=5$.

For each element (x, y) of L_2 define

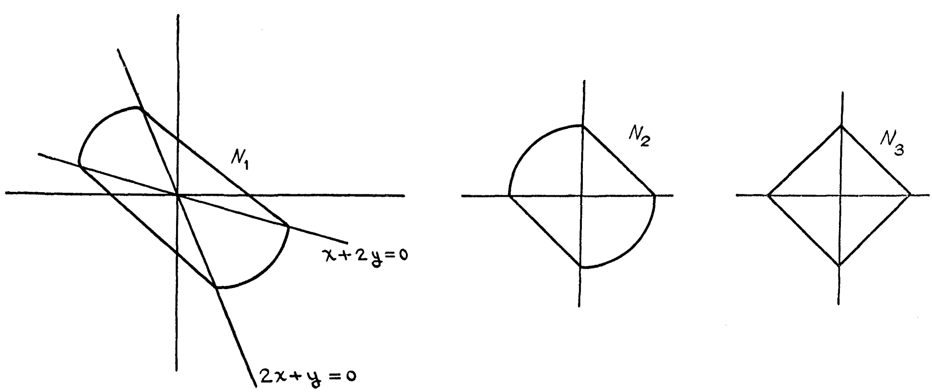
$$||(x, y)||_i = \inf \{a | (x/a, y/a) \in N_i, a > 0\} , \qquad i=1, 2, 3.$$

The third of these norms is familiar: $||(x, y)||_3 = |x| + |y|$ for each element

(x, y) of L_2. The second of these norms was discussed in [3]: $\|(x, y)\|_2 = |x+y| = |x|+|y|$ for elements (x, y) in the first and third quadrants, while $\|(x, y)\|_2 = \sqrt{x^2+y^2}$ for elements (x, y) of L_2 in the second and fourth quadrants. The first of these norms is presumably new: $\|(x, y)\|_1 = (1/\sqrt{5})\cdot\sqrt{x^2+y^2}$ for elements (x, y) on or within the cones formed in the second and fourth quadrants by the intersecting lines $x+2y=0$ and $2x+y=0$, while $\|(x, y)\|_1 = |x+y|$ for all other elements (x, y) of L_2.

Now let BL_2I, BL_2II and BL_2III denote respectively the linear lattice L_2 as under the distinct norms based on the subsets N_1, N_2 and N_3. Then BL_2I is an example of a space BLI wherein the subset P possesses property I but not property II. Specifically, P consists of all points in the first quadrant, while the unique T-set of BL_2I containing P consists of all points on or within the angle determined by the line $x+2y=0$ for $x \geqq 0$ and the line $2x+y=0$ for $y \geqq 0$. Similarly BL_2II is an example of a space BLII while BL_2III is an example of a space BLIII.

The fact that in abstract (L)-spaces of type I the set P is not a T-set complicates the following discussion. The basic relation between T-sets and abstract (L)-spaces of type II and III is treated first. Then, because of its superior importance and because of the perfect application of the T-set theory, the type III situation is discussed in full detail. Last of all, some remarks pertinent to the type I situation will be made.



4. Canonical semi-orderings. This section is devoted to a single theorem.

THEOREM 4.1. *A Banach space B is susceptible of a semi-ordering in respect to which it becomes an abstract (L)-space of type* II *or* III *exactly when it contains a T-set P such that for each $a \in B$ there exist a^+, $a^- \in P$ with the double property that $a = a^+ - a^-$ while $a = b - c$, b, $c \in P$,* *always implies* $\|b-a^+\|=\|b\|-\|a^+\|$, *the semi-ordering then being identical with the canonical semi-ordering induced on* B *by* P. *With this condition satisfied, a space* BLIII *rather than a space* BLII *results exactly when the additional relation* $\|a\|=\|a^+\|+\|a^-\|$ *is satisfied for each* $a \in B$.

Proof. Assume first that B has been endowed with a semi-ordering in respect to which it may be regarded as a space BLII or BLIII. Let P be the subset of B consisting of all elements $a \geqq 0$ under the given semi-ordering. In either case P is a T-set in B: for the case BLII by explicit assumption, and for the case BLIII by assumption and easy conclusion as explained earlier. The canonical semi-ordering induced on B by P obviously duplicates the semi-ordering assumed on B as a space BLII or BLIII.

For any element $a \in B$, let $a^+=a \vee 0$ and $a^-=-(a \wedge 0)$ be as defined under the assumed lattice ordering of B. Then $a=a^+-a^-$ with a^+, $a^- \in P$. Next, if $a=b-c$, b, $c \in P$, then $b \geqq 0$ and $(b-a) \geqq 0$ by definition of P. Hence $b \geqq a$ and $b \geqq a^+$ so that $(b-a^+) \in P$. Then

$$\|b\|=\|(b-a^+)+a^+\|=\|(b-a^+)\|+\|a^+\| \text{ or } \|b-a^+\|=\|b\|-\|a^+\|.$$

Finally, if the assumed ordering is of type III, then for each $a \in B$,

$$\|a\|=\|a^+-a^-\|=\|a^++a^-\|=\|a^+\|+\|a^-\|,$$

since $a^+ \wedge a^-=0$.

Conversely, assume that B contains a T-set P as described in the theorem. Let $\overset{P}{\leqq}$ be the canonical semi-ordering induced on B by P. Then, as explained in the Lattice Criterion, the space B with semi-ordering $\overset{P}{\leqq}$ is a space BLII if not a space BLIII, noting that the existence of $a \vee 0$ in the usual sense is the single additional requirement needed in order that $\overset{P}{\leqq}$ be a linear lattice ordering. Finally, if the condition that $\|a\|=\|a^+\|+\|a^-\|$ for each $a \in B$ is satisfied, then, for $a=b-c$ with $b \wedge c=0$,

$$\begin{aligned}\|b-c\| &= \|(b-c)^+\|+\|(b-c)^-\|=\|(b-c)^++(b-c)^-\| \\ &= \|b \vee c-c+c \vee b-b\|=\|(b-b \wedge c)+(c-c \wedge b)\| \\ &= \|b\|+\|c\|-2\|b \wedge c\|=\|b\|+\|c\|=\|b+c\|.\end{aligned}$$

5. T-sets and F_T-functionals in Type III Spaces. Assume now that Banach space B contains a T-set P_0 such that B is a space BLIII with respect to the canonical semi-ordering $\overset{P_0}{\leqq}$ induced on B by P_0. With P_0 fixed, write $\leqq$ instead of $\overset{P_0}{\leqq}$ and let all lattice notation refer to this fixed lattice ordering of B as BLIII. Certain concepts and results found in [3] will be needed:

(A) For $a, b \in P_0$, $||a-b|| = ||a|| + ||b|| - 2||a \wedge b||$.

(B) An element 1 of P_0, $||1||=1$, is said to be a weak unit in BLIII if $a \wedge 1 > 0$ or, equivalently, $||a-1|| < [||a|| + ||1||]$, for each $a \in P_0$, $a \neq 0$. It is assumed for the present that BLIII contains a weak unit, the adjustments necessary in the contrary case being indicated later.

(C) Associated with each $a \in P_0$ is a projection function P_a. It is defined by the relation $P_a(b) = \lim_n \{[na] \wedge b\}$ for each $b \in P_0$. If $a \wedge b = 0$, then $P_a(c) \wedge P_b(c) = 0$ and $P_c(a) \wedge P_c(b) = 0$ for each $c \in P_0$. If $a \in P_0$ with $P_a(1) = 0$, then $a = 0$.

(D) An element e of P_0 is said to be a characteristic element of BLIII if $e \wedge (1-e) = 0$. For each $a \in P_0$, $P_a(1)$ is a characteristic element of BLIII. For any $a \in P_0$ and any characteristic element e of BLIII, $a = P_1(a) = P_e(a) + P_{1-e}(a)$ with $P_e(a) \wedge P_{1-e}(a) = 0$.

(E) The characteristic elements of BLIII with weak unit form a Boolean algebra, and if $\{e_n\}$ be a sequence of such elements with $e_n \leqq e_{n+1}$, then there is a characteristic element e of BLIII with $e_n \leqq e$ for all n and $\lim_n \{e_n\} = e$ in terms of the norm.

With this information, and with B, P_0, $\leqq$, BLIII, 1 as explained above, two lemmas are in order.

LEMMA 5.1. *For arbitrary T-set P of B the following statements are true*:

(a) *If $a, b \in P$, then $(a^+ + b^+) \wedge (a^- + b^-) = 0$.*

(b) *If $a \in P$, then $a^+ \in P$ and $a^- \in -P$.*

(c) *If $0 \leqq b \leqq a$ with $a \in P$, then $b \in P$.*

(d) *There exists a unique characteristic element e such that $e \in P$ and $(1-e) \in -P$.*

(e) *For this e and for arbitrary $a \in B$, there exist elements $a_e^+ = [P_e(a^+) - P_{1-e}(a^-)]$ and $a_e^- = [P_e(a^-) - P_{1-e}(a^+)]$ in P with the double property that $a = a_e^+ - a_e^-$ with $||a|| = ||a_e^+|| + ||a_e^-||$ while $a = b - c$ with $b, c \in P$ implies $||b - a_e^+|| = ||b|| - ||a_e^+||$.*

(f) *For arbitrary $a \in B$, $\frac{1}{2}\{F_{P_0}(a) + F_P(a)\} = ||P_e(a^+)|| - ||P_e(a^-)||$.*

LEMMA 5.2. *For arbitrary characteristic element e of BLIII, the subset of all elements $P_e(a^+) - P_{1-e}(a^-)$, $a = a^+ - a^- \in BL$III, of B constitute a T-set P of B with $e \in P$ and $(1-e) \in -P$.*

The truth of Lemma 5.2 is easily established in terms of the representation of BLIII as a concrete (L)-space. Because of the routine nature of the proofs for the various parts of Lemma 5.1, attention is restricted to two comments on parts (e) and (f).

First, suppose $a \in P$ with $a \geqq 0$. Then $na \in P$ and thus $[na] \wedge 1 \in P$. Then $e = P_a(1) = \lim_n \{[na] \wedge 1\}$ is in P since every T-set is closed under the norm, and $||e|| \leqq 1$. Let

$$s = \sup \{||e|| \,\big|\, e = P_a(1),\ a \in P,\ a \geqq 0\} .$$

Form $\{a_n\}$, $a_n \in P$, $a_n \geqq 0$, and then $\{e_n\}$ with $e_n = P_{a_n}(1)$ such that $\lim_n \{||e_n||\} = s$. Then let $a_n^* = a_1 + \cdots + a_n$ and $e_n^* = P_{a_n^*}(1)$ so that $a_n^* \in P$, $e_n^* \in P$ with $e_n \leqq e_n^* \leqq e_{n+1}^*$ and $\lim_n \{||e_n^*||\} = s$. Now use (E) to select characteristic element e with $e_n^* \leqq e$ and $\lim_n \{e_n^*\} = e$ under the norm. Since each $e_n^* \in P$, also $e \in P$. Also $||e|| = s$. But if P is a T-set in B, so also is $-P$. Repeating the above process for $-P$, a second characteristic element is obtained which is disjoint from the e obtained above since P and $-P$ have only the zero element in common. It is then but a small matter to show that this second element is $(1-e)$ and that this e and $(1-e)$ are unique with respect to the stated property.

To prove (f), use is again made of the fact that a F_T-functional is linear on the linear extension in B of the T-set used. Thus for arbitrary $a = a^+ - a^- \in B$, with P_0, P and e as above:

$$F_{P_0}(a) = [||P_e(a^+)|| - ||P_{1-e}(a^-)||] - [||P_e(a^-)|| - ||P_{1-e}(a^+)||]$$

$$F_P(a) = [||P_e(a^+)|| + ||-P_{1-e}(a^-)||] - [||P_e(a^-)|| + ||-P_{1-e}(a^+)||] .$$

Finally, consider the case wherein B, P_0, $\leqq$, and BLIII are as before, but in which the existence of a weak unit is not assumed. Then, following [3], it may be shown that there exists a collection 1_α, $\alpha \in \mathscr{A}$. $\mathscr{A}$ an index set, of elements 1_α, $||1_\alpha|| = 1$, of elements of P_0, maximal in BLIII with respect to the property that $1_\alpha \wedge 1_\beta, = 0$ $\alpha \neq \beta$ in $\mathscr{A}$. A characteristic element of BLIII with respect to a particular 1_α, $\alpha \in \mathscr{A}$. is then taken as an element e_α of BLIII such that $e_\alpha \wedge (1_\alpha - e_\alpha) = 0$ in BLIII. Finally, if $\mathscr{E} = \{e_\alpha,\ \alpha \in \mathscr{A}\}$ indicates any definite choice of characteristic elements, one for each 1_α, then Lemmas 5.1 and 5.2 may be restated with each reference to a particular element e replaced by a reference to a particular choice $\mathscr{E}$, and each reference to an element $[P_e(a^\pm) - P_{1-e}(a^\mp)]$ of B replaced by a reference to an element

$$\sum_{\alpha \in \mathscr{A}} [P_{e_\alpha}(a^\pm) - P_{1_\alpha - e_\alpha}(a^\mp)]$$

of B.

These observations are now summarized.

THEOREM 5.3. *Let Banach space B be a space $BL\text{III}_0$ under the canonical semi-ordering induced by a particular T-set P_0 of B. Let* $\{1_\alpha,$

$\alpha \in \mathscr{A}\}$ be a complete set of weak units in $BL\mathrm{III}_0$ and let $\mathscr{E}=\{e_\alpha, \alpha \in \mathscr{A}\}$ denote any chosen family of characteristic elements e_α, one for each 1_α. Then each such family $\mathscr{E}$ determines a unique T-set P of B with $e_\alpha \in P$, $(1_\alpha - e_\alpha) \in -P$. Also every T-set of B is determined in this fashion. Moreover the space B is a space BLIII under the canonical semi-ordering induced by each T-set. Finally, in the concrete representation of $BL\mathrm{III}_0$, for any T-set P of B the function $\frac{1}{2}\{F_{P_0}+F_P\}$ may be interpreted as the result of the associated integration process when restricted to the measurable subsets corresponding to the choice $\{e_\alpha, \alpha \in \mathscr{A}\}$ determining P.

6. Concerning BLI spaces. Let BLI denote an abstract (L)-space of type I and let B denote the same space regarded simply as a Banach space. Let P be the subset of elements $a \in B$ with $a \geqq 0$ as in BLI. By definition of P as in BLI and by Zorn's lemma, there is at least one T-set T of B containing the set P. For elements $a \in P \subset T$, $F_T(a) = \|a\|$. But F_T is linear on the linear extension of T in B. Thus for any element $a \in B$, with $a = a^+ - a^-$ with respect to BLI, $F_T(a) = \|a^+\| - \|a^-\|$. However, $F_{T_1}(a) = F_{T_2}(a)$ for each $a \in B$ implies $T_1 \equiv T_2$. Thus n B the T-set T containing P is uniquely determined.

Next let $a \in B$ be any element of T and let $a = a^+ - a^-$ with respect to BLI. Then a, $a^- \in T$ imply $\|a\| + \|a^-\| = \|a + a^-\| = \|a^+\|$ and so $\|a\| = \|a^+\| - \|a^-\|$. Conversely, let $a \in B$ be such that $\|a\| = \|a^+\| - \|a^-\|$. Then $\|a\| = F_T(a)$, so that $a \in T$. Thus T consists exactly of the elements $a \in B$ for which $\|a\| = \|a^+\| - \|a^-\|$.

It has already been seen that for any space B the type II and type III orderings are mutually exclusive, in the sense that all orderings of either type are canonical semi-orderings based on T-sets, and if one such ordering is of type III so is every other. No success has been had thus far in demonstrating a similar exclusiveness between type I orderings on the one hand, and type II or type III orderings on the other.

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SOME REMARKS ON A PAPER OF ARONSZAJN AND PANITCHPAKDI

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In the paper of the title [1], a number of problems are posed. Negative solutions of two of them (Problems 2 and 3) are derived in a straightforward way from a paper of L. Gillman and the present author [2].

Motivation will not be supplied since it is given amply in [1], but enough definitions are given to keep the presentation reasonably self-contained.

1. A Hausdorff space X is said to satisfy $(Q_{\mathfrak{m}})$, where $\mathfrak{m}$ is an in finite cardinal, if, whenever U and V are disjoint open subsets of X such that each is a union of the closures of *less* than $\mathfrak{m}$ open subsets of X, then U and V have disjoint closures. In particular, a normal (Hausdorff) space X satisfies $(Q_{\aleph_1})$ if and only if disjoint open F_σ-subsets of X have disjoint closures. (For, an open set that is the union of less than $\aleph_1$ closed sets is a fortiori an F_σ. Conversely if U is the union of countably many closed subsets F_n, then since X is normal, for each n there is an open set U_n containing F_n whose closure is contained in U. Thus U is the union of the closures of the open sets U_n.) In Problem 3 of [1], it is asked if every compact (Hausdorff) space satisfying $(Q_{\mathfrak{m}})$ for some $\mathfrak{m}>\aleph_0$ is necessarily totally disconnected, and it is remarked that this is the case if the first axiom of countability is also assumed.

If X is a completely regular space, let $C(X)$ denote the ring of all continuous real-valued functions on X, and let $Z(f)=\{x\in X: f(x)=0\}$, let $P(f)=\{x\in X: f(x)>0\}$, and let $N(f)=P(-f)$. As usual, let βX denote the Stone-Čech compactification of X. If every finitely generated ideal of $C(X)$ is a principal ideal, then X is called an *F-space*. The following are equivalent.

(i) X is an F-space.

(ii) If $f\in C(X)$, then $P(f)$ and $N(f)$ are completely separated [2, Theorem 2.3].

(iii) If $f\in C(X)$, then every bounded $g\in C(X-Z(f))$ has an extension $\bar{g}\in C(X)$ [2, Theorem 2.6].

A good supply of compact F-spaces is provided by the fact that if X is locally compact and σ-compact, then $\beta X-X$ is an F-space [2, Theorem 2.7].

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We remark first that *a normal (Hausdorff) space X satisfies $(Q_{\aleph_1})$ if and only if it is an F-space.*

For, suppose first that X is an F-space, and let U, V be disjoint open F_σ-subsets of X. Since $X-(U\cup V)$ is a closed G_δ in a normal space, there is a bounded $f\in C(X)$ such that $Z(f)=X-(U\cup V)$. Hence by (iii), there is a $\bar{g}\in C(X)$ such that $\bar{g}[U]=0$ and $\bar{g}[V]=1$. In particular, U and V have disjoint closures, so X satisfies $(Q_{\aleph_1})$. Conversely let X satisfy $(Q_{\aleph_1})$, and let $f\in C(X)$. Then $P(f)$ and $N(f)$ are disjoint open F_σ-subsets of X, which by $(Q_{\aleph_1})$ have disjoint closures. So, by Urysohn's lemma, $P(f)$ and $N(f)$ are completely separated. Thus X is an F-space by (ii).

Compact connected F-spaces exist. In particular it is known that if R^+ denotes the space of nonnegative real numbers, then βR^+-R^+ is such a space [2, Example 2.8]. Hence Problem 3 of [1] has a negative solution.

We remark finally that if the first axiom of countability holds at a point of an F-space, then the point is isolated [2, Corollary 2.4]. In particular, every compact F-space satisfying the first axiom of countability is finite.

2. In Problem 2 of [1], it is asked (in different but equivalent language) if for every totally disconnected compact space X satisfying $(Q_{\mathfrak{m}})$ for some $\mathfrak{m}>\aleph_0$, the Boolean algebra $B(X)$ of open and closed subsets of X has the property that every subset of *less* than $\mathfrak{m}$ elements has a least upper bound. A lattice is said to be (conditionally) *σ-complete* if every bounded countable subset has a least upper bound and a greatest lower bound. In view of the above (and since every subset of $B(X)$ is bounded), in case $\mathfrak{m}=\aleph_1$, the problem asks if for every compact totally disconnected F-space X, the Boolean algebra $B(X)$ is σ-complete.

In [3, Theorem 18], it is shown that if X is compact and totally disconnected, then $B(X)$ is σ-complete if and only if $C(X)$ is σ-complete (as a lattice). It is noted in [2, Theorem 8.3, f.f.] that for a completely regular space Y, the lattice $C(Y)$ is σ-complete if and only if $f\in C(Y)$ implies $\overline{P}(f)$ and $\overline{N}(f)$ are disjoint open and closed subsets of Y ($\overline{P}(f)$ denotes the closure of $P(f)$). It is easily seen that Y has this latter property if and only if βY has [2, Lemma 1.6].

In [2, Example 8.10], an example is given of a completely regular space X such that βX is a totally disconnected F-space, and such that $C(X)$ is not σ-complete. By the above, it follows that $B(\beta X)$ yields a negative solution to Problem 2.

We remark also (as was pointed out by J. R. Isbell) that if N denotes the countable discrete space, then $\beta N-N$ is also a totally disconnected compact F-space such that $B(\beta N-N)$ is not σ-complete. The former assertion follows easily from the remarks in § 1, and the latter follows from the fact that $B(\beta N-N)$ is isomorphic to the Boolean algebra of all subsets of N modulo the ideal of finite subsets of N (under the correspondence induced by sending a subset of N to the intersection of its closure in βN with $\beta N-N$). It is easily verified that this latter Boolean algebra is not σ-complete.

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ON THE GENERALIZED RADIATION PROBLEM OF A. WEINSTEIN

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1. Introduction. The generalized radiation problem as formulated and solved by A. Weinstein [8] requires determination of a non-singular solution of the two-dimensional Euler-Poisson-Darboux (abbreviated EPD) equation

$$u_{xx}^{[k]}=u_{yy}^{[k]}+\frac{k}{y}u_{y}^{[k]} \tag{1.1}$$

for $-\infty<k<1$ such that

$$\lim_{y\to 0} u^{[k]}(x,y)=f(x) \quad \text{and} \quad u^{[k]}(x,y)=0 \quad \text{for} \quad y=x \tag{1.2}$$

where $f(x)$ is a function given on some interval $0\leqq x\leqq a$, possessing a specified number of continuous derivatives there and having another specified number of zero derivatives at $x=0$. These conditions on $f(x)$ depend on the parameter k as stated in [8]. The classical radiation problem, requiring an axially symmetric solution of the higher dimensional wave equation with a certain type of singularity, as given in [3], is a special case. If k is an integer and $u^{[k]}$ a solution of the above generalized radiation problem, then

$$u^{(2-k)}(x,y)=\frac{u^{[k]}(x,y)}{y^{1-k}} \tag{1.3}$$

is a solution of the classical radiation problem in an $m=3-k$ dimensional space (not counting time as a dimension). Thus from a regular solution $u^{[k]}$ one generates a solution $u^{[2-k]}$ of the EPD equation with that type of singularity needed to solve the radiation problem.

The first part of this paper will be devoted to uniqueness for the generalized radiation problem. Although a more complete answer to the uniqueness question would be welcome, consideration of solutions which have two continuous derivatives on $y=x$ is natural since such solutions are the ones that correspond closely to radiation phenomena. Let T be a triangle with vertices $(0, 0)$, $(a, 0)$, $(a/2, a/2)$. We define a function to be regular on T if it has two continuous derivatives in some triangle G the interior of which contains T and its sides except for the base line, $y=0$. Only a function satisfying the EPD equation, regular on T, and taking on the given data will be considered a solution of the

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radiation problem. Such considerations cover an important class of the Weinstein solutions.

We are concerned for uniqueness only with the difference of two solutions $u^{[k]}(x, y)$ which take on the given data $f(x)$; that is, we show that $u^{[k]}(x, 0) \equiv u^{[k]}(x, x) \equiv 0$ implies $u^{[k]}(x, y) \equiv 0$. It will be convenient to use several properties of solutions that follow from the general solution of the EPD equation. These general solutions were known to Darboux [4], except for the case $k = -(2n-1)$, $n = 1, 2, \cdots$. We use the E. K. Blum [2] representation of the general solutions.

The recursion

$$u_y^{[k]}(x, y) = y u^{[k+2]}(x, y) \tag{1.4}$$

plays a basic role in our uniqueness considerations. This relation and the relation (1.3) are still valid even where x represents variables x_1, $x_2, \cdots, x_n$ and $u^{[k]}$ is a solution of

$$\Delta_x u = u_{yy} + \frac{k}{y} u_y \ .$$

In their n-dimensional form both recursions are due to A. Weinstein, but in the two-dimensional form used here the recursion (1.3) was known to Darboux. In place of (1.4) Darboux uses a relation which in our notation is

$$k u_y^{[k]}(x, y) = y u^{[k+2]}(x, y)$$

and which therefore does not admit an inversion for $k=0$. Certainly the discovery and emphasis of the very important role of these recursions in the general theory of the EPD equations is the work of A. Weinstein.

Of course, any uniqueness proof which applies to solutions of (1.1) (1.2) also applies when it is required that

$$u^{[k]}(x, x) = g(x) \ , \qquad u^{[k]}(x, 0) = f(x) \tag{1.5}$$

where $f(x)$ and $g(x) \not\equiv 0$ are given functions. A later paper will be devoted to solution of the problem (1.5), and precise conditions on f and g required for existence of solutions regular on T will be given there.

From the Weinstein solutions it can be seen that the region of determination of $f(x)$ defined for $0 \leqq x \leqq a$ is the infinite strip bounded by the lines $y = x$ and $y = x - a$. The uniqueness question, however, can be restricted to consideration of the characteristic triangle T defined above. That is, for uniqueness one considers only the problem $f(x) = 0$. If it follows from this prescription of $f(x)$ that the solution is identically zero in the characteristic triangle, then it is certainly zero on the characteristic $y=-x+a$. But now as the solution has been prescribed to be zero on $y=x$, it can be shown to be zero in the infinite strip by solution of a characteristic problem. The characteristic problem for the two-dimensional EPD equation is classical. It was solved by Riemann [6] in order to obtain the Riemann function for the EPD equation.

2. Some important properties of solutions. In this section we shall be concerned with several properties that are derived from the general solutions of the two-dimensional EPD equation for solutions $u^{[k]}(x, y)$, $k<1$, regular on T, and such that

$$u^{[k]}(x, x)=u^{[k]}(x, 0)=0 .$$

The general solutions which we use are valid on a characteristic triangle in which the solution has two derivatives in a region G containing that characteristic triangle except for the points of its base. Certainly then the general solutions are valid for functions which are regular on T in the sense described above.

The general solutions for k negative are obtained [2] from repeated application of (1.3) and (1.4) (and certain considerations associated with them) to solutions $u^{[s]}(x, y)$, $0\leqq s<2$. Consider coefficients a_{rn} defined by

$$a_{rn}=\left(-\frac{1}{2}\right)^{n-r}\frac{[r+2(n-r)-1]!}{(n-r)!(r-1)!}\,, \qquad a_{nn}=1 . \tag{2.1}$$

The general solutions are:

Case 1. $0<k<1$

$$\begin{aligned} u^{[k]}(x, y)= &-2^{k-1}y^{1-k}\int_1^{-1}\phi[x+\alpha y](1-\alpha^2)^{-k/2}\,d\alpha \\ &-2^{-k+1}\int_1^{-1}\psi[x+\alpha y](1-\alpha^2)^{k/2-1}\,d\alpha . \end{aligned} \tag{2.2}$$

For solutions which are regular on T, the arbitrary functions ϕ and ψ have one continuous derivative on the closed interval $[0, a]$.

Case 2. $k<0$, k non-integral,

$$\begin{aligned} u^{[k]}(x, y) = &-2^{s-1}\sum_{r=1}^{n} a_{rn}r!\sum_{j=0}^{r}\frac{(1-s)(-s)\cdots(-s-r-j)}{j!\,(r-j)!}y^j \\ &\times\int_1^{-1}\phi^{(j)}[x+\alpha y](1-\alpha^2)^{-s/2}\alpha^j\,d\alpha \\ &-2^{-s+1}\sum_{r=1}^{n} a_{rn}y^{s+r-1}\int_1^{-1}\psi^{(r)}[x+\alpha y](1-\alpha^2)^{s/2-1}\alpha^r\,d\alpha \end{aligned} \tag{2.3}$$

where $0<s<2$, $s\neq 1$ and n is an integer given by $2-k=2n+s$. Here if $u^{[k]}$ is regular on T, ϕ and ψ have $(n+1)$ continuous derivatives on $[0, a]$.

Case 3(a). $k=0$, $u^{[0]}(x, y)=F(x+y)+G(x-y)$.

Case 3(b). $k=-2n$, $n=1, 2, \cdots$,

$$u^{[k]}(x, y)=\sum_{r=1}^{n+1} a_{r,n+1}y^{r-1}[F^{(r)}(x+y)+(-1)^r G^{(r)}(x-y)] . \tag{2.4}$$

Here if $u^{[k]}$ is regular on T, F and G have $(n+3)$ continuous derivatives on $[0, a]$.

Case 4. $k=-(2n+1)$, $n=0, 1, 2, \cdots$

$$u^{[k]}(x, y)=\sum_{r=1}^{n-1} a_{r,n+1}y^r \frac{\partial^r u^{[1]}}{\partial y^r} \tag{2.5}$$

where

$$\begin{aligned} u^{[1]}(x, y)&=2\int_1^{-1}\phi[x+\alpha y](1-\alpha^2)^{-1/2}\,d\alpha \\ &+2\int_1^{-1}\psi[x+\alpha y](1-\alpha^2)^{-1/2}\log[y(1-\alpha^2)(1/2)^2]\,d\alpha . \end{aligned} \tag{2.6}$$

Here if $u^{[k]}$ is regular on T, ϕ and ψ have $n+2$ continuous derivatives on $[0, a]$.

LEMMA 1. *If* $u^{[k]}(x, 0)=0$ *and* $u^{[k]}(x, y)$ *is regular on* T, *then*

Case 1. $\psi\equiv 0$

Case 2. $\phi\equiv 0$

Case 3(a). $F\equiv -G$

(b). $F'\equiv -G'$

Case 4. $\psi\equiv 0$.

Proof. There results were known to Blum [2]. The hypothesis is, as stated above, intended in the sense that $\lim_{y\to 0} u^{[k]}(x, y)=0$. In Case 2, for example, let $y\to 0$. As $s+r-1$ is always positive, since $r\geqq 1$, we have

$$u^{[k]}(x, 0) = \lim_{y \to 0} u^{[k]}(x, y)$$
$$= -2^{s-1} \sum_{r=1}^{n} a_{rn}(1-s)(-s)\cdots(-s-r)\phi(x) \int_{1}^{-1} (1-\alpha^2)^{-s/2}\, d\alpha$$

or

$$\phi(x) = \frac{u^{[k]}(x, 0)}{-2^{s-1} \sum_{r=1}^{n} a_{rn}(1-s)(-s)\cdots(-s-r) \int_{1}^{-1} (1-\alpha^2)^{-s/2}\, d\alpha} . \tag{2.7}$$

But the integral cannot be zero since it is a symmetric integral of an even function—in fact, the integral is

$$\frac{-\Gamma(1/2)\Gamma(1-s/2)}{(3/2-s/2)}$$

and it can be shown that $\sum_{r=1}^{n} a_{rn}(1-s)(-s)\cdots(-s-r) \neq 0$. Thus $u^{[k]}(x, 0) = 0$ implies $\phi(x) = 0$ as stated.

Consider now Case 4. Here we have

$$\psi(x) = \frac{u^{[k]}(x, 0)}{\pi\left[a_{1,n+1} + \sum_{r=2}^{n+1} a_{r,n+1}(r-1)(-1)^{r+1}\right]} \tag{2.8}$$

and $\psi \equiv 0$ if $u^{[k]}(x, 0) = 0$. For example, take $k = -1$,

$$u^{[-1]}(x, y) = 2y \int_{1}^{-1} \phi[x+\alpha y](1-\alpha^2)^{1/2}\alpha\, d\alpha$$
$$+ 2\int_{1}^{-1} \psi'[x+\alpha y] y \log\,[y(1-\alpha^2)1/2^2](1-\alpha^2)^{-1/2}\alpha\, d\alpha$$
$$+ 2y \int_{1}^{-1} \psi[x+\alpha y]\left(\frac{1}{y}\right)(1-\alpha^2)^{-1/2}\, d\alpha .$$

On letting $y \to 0$ we notice that $y \log c\ y \to 0$ for any constant c so that

$$0 = u^{(-1)}(x, 0) = 2\int_{1}^{-1} \psi[x](1-\alpha^2)^{-1/2}\, d\alpha$$

and again, since

$$\int_{1}^{-1} (1-\alpha^2)^{-1/2}\, d\alpha \neq 0 , \qquad \psi[x] = 0 .$$

Case 1 and Case 3 are now entirely trivial.

Lemma 2. *For $k < 0$, if $u^{[k]}$ is regular on T and $u^{[k]}(x, 0)$ exists, then*

$$u_y^{[k]}(x, 0)=\lim_{y\to 0} u_y^{[k]}(x, y)=0 .$$

That is, for $k<0$, letting $u^{[k]}(x, 0)=f(x)$, given, the function $u^{[k]}(x, y)$ is a (non-unique) solution of the singular Cauchy problem.[1] This is the main result of Blum [2]; one arbitrary function is determined as seen in Lemma 1 by specification of $f(x)$, the other is left free so that the general solutions then yield the class of all solutions of the Cauchy problem for $k<0$. For $k>0$ solutions of the singular Cauchy problem are unique. One now sees that the solution of the generalized radiation problem for $k<0$ is a solution of the Cauchy problem with one additional condition. It is this condition which must provide uniqueness. The proof of the lemma consists simply in deriving the general solutions with respect to y and examining limits as $y\to 0$. It should be noted that in deriving the general solutions of the EPD equation nothing is said about the behavior of u_y on the line $y=0$. Also, it should be emphasized that one cannot simply look at the term $\frac{k}{y}u_y$ of the EPD equation and conclude the above immediately; for $k\geqq 0$, $u_y^{[k]}(x, 0)$ is not necessarily zero.

Lemma 2 is true for any $u^{[k]}$ regular on T such that $u^{[k]}(x, 0)$ exists, but the problem of uniqueness involves only $u^{[k]}(x, 0)\equiv 0$, and in this case a more general result, valid for $k<-1$ but used here only for $k\leqq -2$, is obtained. In [8] the existence of certain derivatives of $u^{[k]}$ on $y=0$ was (tacitly) assumed. Lemma 3 allows us, for unicity only, to avoid any such assumption.

LEMMA 3. *Let $u^{[k]}(x, y)$, $k<-1$, be any solution of the EPD equation regular on T. Then $u^{[k]}(x, 0)\equiv 0$ implies*

$$\lim_{y\to 0}\frac{u_y^{[k]}(x, y)}{y}\equiv 0 .$$

For $-1<k<0$, a counterexample is $u^{[k]}(x, y)=y^{1-k}$.

Proof. We must again consider separately each of the general solutions. To avoid extensive manipulations a sample case only is presented; k non-integral, $-2<k<-1$.

By Lemma 1 all solutions are of the form

$$u^{[k]}(x, y)=-2^{-s+1}y^s\int_1^{-1}\psi'[x+\alpha y](1-\alpha^2)^{s/2-1}\alpha\, d\alpha$$

with $1<s<2$. We have

[1] For the singular Cauchy problem, specify $f(x)$ and require

$$u^{[k]}(x, 0)=f(x),\quad u_y^{[k]}(x, 0)=0 .$$

$$\lim_{y\to 0}\frac{u_y^{[k]}(x,y)}{y}=\lim_{y\to 0}-2^{-s+1}\left\{sy^{s-2}\int_1^{-1}\psi'[x+\alpha y](1-\alpha^2)^{s/2-1}\alpha\,d\alpha\right.$$
$$\left.+y^{s-1}\int_1^{-1}\psi''[x+\alpha y](1-\alpha^2)^{s/2-1}\alpha^2\,d\alpha\right\}$$
$$=-2^{-s+1}s\lim_{y\to 0}y^{s-2}\int_1^{-1}\psi'[x+\alpha y](1-\alpha^2)^{s/2-1}\alpha\,d\alpha\ .$$

But as $y\to 0$, the integral factor goes to

$$\psi'[x]\int_1^{-1}(1-\alpha^2)^{s/2-1}\alpha\,d\alpha\equiv 0$$

(the integrand is odd), and the L'Hospital rule is applicable. We obtain

$$\lim_{y\to 0}\frac{u_y^{[k]}(x,y)}{y}=-2^{-s+1}\frac{s}{2-s}\lim_{y\to 0}y^{s-1}\int_1^{-1}\psi''[x+\alpha y](1-\alpha^2)^{s/2-1}\alpha^2\,d\alpha\equiv 0\ .$$

LEMMA 4. *If $u^{[k]}$ is regular on T, in the general solution for $u^{[k]}$ we may without loss of generality take*

Case 2. $\psi'(0)=\psi''(0)=\cdots=\psi^{(n)}(0)=0$

Case 3. $F'(0)=F''(0)=\cdots=F^{(n+1)}(0)=0$

or $G'(0)=G''(0)=\cdots=G^{(n+1)}(0)=0$

Case 4. $\phi'(0)=\phi''(0)=\cdots=\phi^{(n+1)}(0)=0$.

The importance of this lemma is that it is essential in the proof of Lemma 5 where these results are used in repeated application of the rule of L'Hospital. Lemma 5 in turn is essential to an important induction used in the uniqueness proof of § 4. For solutions with two derivatives inside T only, the lemma can be extended by replacing the evaluation of ψ, ϕ, F, and G at 0 by evaluation at $c>0$ and considering solutions regular on a triangle T_c contained in T.

Proof. *Case* 2. Let the function $\psi_*^{(1)}(z)$ be defined by

$$\psi_*^{(1)}(z)=\psi^{(1)}(z)-\psi^{(1)}(0)-\psi^{(2)}(0)z-\frac{\psi^{(3)}(0)}{2!}z^2-\cdots-\frac{\psi^{(n)}(0)}{(n-1)!}z^{n-1}\ . \tag{2.14}$$

Of course, $\psi_*^{(1)}(0)=\psi_*^{(2)}(0)=\cdots=\psi_*^{(n)}(0)=0$, and we show that $\psi_*^{(r)}(z)$ can replace $\psi^{(r)}(z)$, $r=1,\cdots,n$, in equation (2.3). Differentiating (2.14) $r-1$ times we obtain

$$\psi^{(r)}(z)=\psi_*^{(r)}(z)+\sum_{m=r}^{n}\frac{\psi^{(m)}(0)}{(m-r)!}z^{m-r}\ ,$$

and using binomial expansion we have

$$(2.15)\qquad \phi^{(r)}(x+\alpha y)=\phi_*^{(r)}(x+\alpha y)+\sum_{m=r}^{n}\sum_{j=0}^{m-r}\frac{\phi^{(m)}(0)}{(m-r)!}\left\{\frac{(m-r)!\,x^{m-r-j}y^j\alpha^j}{j!(m-j-r)!}\right\}.$$

Then using (2.15), we may rewrite equation (2.3) as

$$(2.16)\qquad \begin{aligned} u^{[k]}(x, y) = &-2^{s-1}\sum\cdots\sum\int_1^{-1}\phi\cdots \\ &-2^{s+1}\sum_{r=1}^{n} a_{rn}y^{s+r-1}\int_1^{-1}\phi_*^{(r)}[x+\alpha y](1-\alpha^2)^{s/2-1}\alpha^r\,d\alpha \\ &-2^{s+1}y^s\left\{\sum_{r=1}^{n} a_{rn}\sum_{m=r}^{n}\sum_{j=0}^{m-r}\frac{\phi^{(m)}(0)}{j!(m-j-r)!}x^{m-r-j}y^{j+r-1}\right. \\ &\left.\times\int_1^{-1}(1-\alpha^2)^{s/2-1}\alpha^{r+j}\,d\alpha\right\} \end{aligned}$$

so that our lemma will be proved when we have shown that the last group of terms sum to zero for all x and y. In this group for terms where $r+j$ is odd, the integral factor vanishes. We prove that the indicated brackets is zero for each $r+j$ even. Reordering terms, the brackets in (2.16) becomes

$$(2.17)\qquad \begin{aligned} \sum_{m=r}^{n}\phi^{(m)}(0)&\left\{\sum_{r=1}^{n}\sum_{j=0}^{m-r} a_{rn}\frac{1}{j!(m-j-r)!}x^{m-(r+j)}y^{(r+j)-1}\right. \\ &\left.\times\int_1^{-1}(1-\alpha^2)^{s/2-1}\alpha^{r+j}\,d\alpha\right\} \end{aligned}$$

and it will be possible to show that the new brackets, denoted by $S(n)$ is zero for all n such that $r\leqq m\leqq n$. Letting $2\nu=r+j$, for $0<j<m-r$ we have $r<2\nu<m$ or since the least value of r is 1, $1\leqq\nu\leqq\left[\frac{m}{2}\right]$.[2] Then

$$(2.18)\qquad S(n)=\sum_{\nu=1}^{[\frac{m}{2}]}\frac{1}{(m-2\nu)!}x^{m-2\nu}y^{2\nu-1}\left\{\sum_{r=1}^{2\nu}\frac{a_{rn}}{(2\nu-r)!}\right\}\int_1^{-1}(1-\alpha^2)^{s/2-1}\alpha^{2\nu}\,d\alpha$$

and we only need show that

$$(2.19)\qquad \sigma=\sum_{r=1}^{2\nu}\frac{a_{rn}}{(2\nu-r)!}$$

is zero for all n and ν.

---

[2] $\left[\frac{m}{2}\right]$ is the Legendre symbol—the greatest integer less than or equal to $\frac{m}{2}$.

From (2.19) and (2.1), then

$$\sigma=\sum_{r=1}^{2\nu}(-1/2)^{n-r}\frac{(2n-r-1)!}{(n-r)!\,(r-1)!\,(2\nu-r)!}$$

and this is the quantity which is our present concern.

Consider the polynomial

$$P(z)=\sum_{r=1}^{2\nu}\frac{(-1/2)^{2n-r-1}(1-z)^{2n-r-1}}{(r-1)!\,(2\nu-r)!}=\sum_{l=1}^{2(n-1)}b_l z^l\ . \tag{2.21}$$

Then

$$P^{(n-1)}(z)=\sum_{r=1}^{2\nu}(-1/2)^{2n-r-1}\frac{(2n-r-1)!\,(-1)^{n-1}}{(n-r)!\,(r-1)!\,(2\nu-r)!}(1-z)^{n-r}$$

and

$$P^{(n-1)}(0)=(-1)^{n-1}(-1/2)^{n-1}\sum_{r=1}^{2\nu}(-1/2)^{n-r}\frac{(2n-r-1)!}{(n-r)!\,(r-1)!\,(2\nu-r)!}$$

$$=(-1)^{n-1}(-1/2)^{n-1}\sigma\ .$$

Thus $P(z)$ has been chosen so that it will be sufficient to demonstrate that the coefficient b_{n-1} is zero. Let us rewrite $P(z)$ as follows

$$P(z)=\sum_{r=1}^{2\nu}\frac{(1/2z-1/2)^{2n-r-1}}{(r-1)!\,(2\nu-1)!}=\frac{1}{(2\nu-1)!}\sum_{r=1}^{2\nu}\binom{2\nu-1}{r-1}(1/2z-1/2)^{2n-r-1}$$

$$=\frac{(1/2z-1/2)^{2n-2}}{(2\nu-1)!}\left(1+\frac{1}{1/2z-1/2}\right)^{2\nu-1}$$

$$=\frac{1}{(2\nu-1)!}(1/2z-1/2)^{2(n-\nu)-1}(1/2z+1/2)^{2\nu-1}$$

$$=c(z-1)^{2(n-\nu)-1}(z+1)^{2\nu-1}\,,\qquad c=\frac{1}{(2\nu-1)!}(1/2)^{2(n-1)}$$

$$=cz^{n-\nu-1/2}\left(\sqrt{z}-\frac{1}{\sqrt{z}}\right)^{2(n-\nu)-1}\left(\frac{1}{\sqrt{z}}+\sqrt{z}\right)^{2\nu-1}z^{\nu-1/2}$$

$$=cz^{n-1}Q(z)$$

where

$$Q(z)=\left(\sqrt{z}-\frac{1}{\sqrt{z}}\right)^{2(n-\nu)-1}\left(\frac{1}{\sqrt{z}}+\sqrt{z}\right)^{2\nu-1}\ .$$

We note that $Q(z)=-Q(1/z)$, and that, therefore

$$cQ(z)=\frac{P(z)}{z^{n-1}}=-P\left(\frac{1}{z}\right)z^{n-1}=-cQ\left(\frac{1}{z}\right)$$

or $P(z)\equiv -z^{2(n-1)}P(1/z)$. Thus

$$\sum_{l=0}^{2(n-1)} b_l z^l \equiv -\sum_{l=0}^{2(n-1)} b_l z^{2(n-1)-l} = -\sum_{l=0}^{2(n-1)} b_{2(n-1)-l} z^l$$

and $b_l+b_{2(n-1)-l}=0$. Putting $l=n-1$, the required result $b_{n-1}=0$ is obtained.

It is noted that the coefficients a_{rn} of the general solutions do not arise from consideration of any polynomials.

Case 3(b). It remains only to show that this treatment reduces after a certain point to that of Case 2.

Let

$$F_*^{(1)}(x)=F^{(1)}(x)-G^{(1)}(0)-xG^{(2)}(0)-\frac{x^2}{2!}G^{(3)}(0)-\cdots-\frac{x^n}{n!}G^{(n+1)}(0)$$

and

$$G_*^{(1)}(x)=G^{(1)}(x)-G^{(1)}(0)-xG^{(2)}(0)-\frac{x^2}{2!}G^{(3)}(0)-\cdots-\frac{x^n}{n!}G^{(n+1)}(0)\,.$$

Then $F_*^{(1)}$ and $G_*^{(1)}$ have the required number of continuous derivatives and $G_*^{(0)}(0)=G_*^{(2)}(0)=\cdots=G_*^{(n+1)}(0)=0$. (Of course, if we subtracted the "Taylor part" of $F^{(1)}(x)$ from $F^{(1)}$ and $G^{(1)}$ we would find that $F_*^{(1)}(0)=F_*^{(2)}(0)=\cdots=F_*^{(n+1)}(0)=0$.)

$$F^{(r)}(x)=F_*^{(r)}(x)+\sum_{m=r}^{n+1}\frac{G^{(m)}(0)}{(m-r)!}x^{m-r}$$

and

$$G^{(r)}(x)=G_*^{(r)}(x)+\sum_{m=r}^{n+1}\frac{G^{(m)}(0)}{(m-r)!}x^{m-r}\,.$$

From (2.4)

$$\begin{aligned}u^{[-2n]}=\sum_{r=1}^{n+1}a_{r,n+1}y^{r-1}\Big[&F_*^{(r)}(x+y)+\sum_{m=r}^{n+1}\frac{G^{(m)}(0)}{(m-r)!}(x+y)^{m-r}\\&+(-1)^rG_*^{(r)}(x-y)+(-1)^r\sum_{m=r}^{n+1}\frac{G^{(m)}(0)}{(m-r)!}(x-y)^{m-r}\\=\sum_{r=1}^{n+1}a_{r,n+1}y^{r-1}\,&[F_*^{(r)}(x+y)+(-1)^rG_*^{(r)}(x-y)]-\sum\end{aligned}$$

where

$$\sum = \sum_{r=1}^{n+1} a_{r,n+1} y^{r-1} \left[\sum_{m=1}^{n+1} \frac{G^{(m)}(0)}{(m-r)!} \sum_{j=1}^{m-r} \left(\frac{(m-r)!}{j!\,(m-j-r)!} + \frac{(-1)^r (m-r)!\,(-1)^j}{j!\,(m-j-r)!} \right) \times x^{m-r-j} y^j \right]$$

and, of course, we must show that $\sum$ is zero. We have

$$\sum = \sum_{m=r}^{n+1} G^{(m)}(0) \left\{ \sum_{r=1}^{n+1} \sum_{j=0}^{m-r} a_{r,n+1} \frac{1}{j!\,(m-j-r)!} x^{m-(r+j)} y^{(r+j)-1} \right\} \times (1+(-1)^{r+j}) . \tag{2.22}$$

But the expression in brackets in (2.22) is exactly that of (2.17) with n replaced by $(n+1)$, and the factor $1+(-1)^{r+j}$ plays exactly the role of

$$\int_1^{-1} (1-\alpha^2)^{\frac{s}{2}-1} \alpha^{r+j}\, d\alpha ,$$

each being zero for $r+j$ odd.

LEMMA 5. *If $k<0$ and $u^{[k]}(x, y)$ is a solution of the* EPD *equation regular on T and such that*

$$u^{[k]}(x, x) = 0 ,$$

then

(i) $u_y^{[k]}(x, x) = Bx^{-k/2}$, B *constant*

(ii) $u^{[k]}(x, 0) \equiv 0 \Longrightarrow B = 0$.

That the solution be regular on T implies that all second derivatives exist on the line $y=x$ and that the EPD equation be satisfied there.

Proof.
(i) On $y=x$ the EPD equation may be written, using x as a parameter,

$$\frac{d}{dx}(u_x^{[k]}(x, x) - u_y^{[k]}(x, x)) = \frac{k}{x} u_y^{[k]}(x, x) . \tag{2.23}$$

Differentiating $u^{[k]}(x, y)$ on the line $y=x$, we have

$$0 = u_x^{[k]}(x, x) + u_y^{[k]}(x, x) \tag{2.24}$$

so that (2.23) may be rewritten

$$\frac{d}{dx}(u_y^{[k]}(x, x)) = -\frac{k}{2x} u_y^{[k]}(x, x)$$

and the first part of the lemma follows. This elementary procedure is basic in our problem and similar techniques will be used often.
(ii) To demonstrate the second part of the lemma we note that since $u^{[k]}(x, y)$ has been assumed to be regular on T the general solutions apply on the line $y=x$, and from Lemma 1 the condition $u^{[k]}(x, 0)$ gives the general solutions a simplified form. Thus for Case 2, k non-integral, noting that $k/2=-s/2+1-n$, we have

$$\begin{aligned}(2.25)\qquad B&=x^{k/2}u_y^{[k]}(x, x)\\ &=-2^{-s+1}\sum_{r=1}^{n} a_{rn}\left\{(s+r+1)x^{s/2+r-n-1}\int_1^{-1}\psi^{(r)}[(1+\alpha)x](1-\alpha^2)^{s/2-1}\alpha^r\,d\alpha\right.\\ &\qquad\left.+x^{s/2+r-n}\int_1^{-1}\psi^{(r+1)}[(1+\alpha)x](1-\alpha^2)^{s/2-1}\alpha^{r+1}\,d\alpha\right\}.\end{aligned}$$

We can now conclude that $B=0$ by taking the limit of (2.25) as $x\to 0$. To do this we apply the rule of L'Hospital $(n+1-r)$ times to the r^{th} term in the first set of terms and $(n-r)$ times to the r^{th} term in the second set of terms. The purpose in presenting Lemma 4 was to justify this procedure.

The Cases 1 and 3(a) are irrelevant to this lemma as we require k to be negative. Treatment of Case 4 is precisely analogous to Case 2 except that here, by Lemma 1, $\psi(x)\equiv 0$, and (2.25) appears in terms of integrals of ϕ instead of ψ and with a slightly different kernel.

Consider Case 3(b), $k=-2n$, $n=1, 2, \cdots$. The analogue of (2.25) is

$$\begin{aligned}(2.26)\qquad B=x^{k/2}u_y^{[k]}(x, x)&=\sum_{r=1}^{n+1} a_{r,n+1}x^{r-n-1}[F^{(r+1)}(2x)+(-1)^{r+1}F^{(r)}(0)]\\ &\quad+\sum_{r=1}^{n+1} a_{r,n+1}(r-1)x^{r-n-2}[F^{(r)}(2x)+(-1)^r F^{(r)}(0)].\end{aligned}$$

We again conclude that $B=0$ by taking the limit of (2.26) as $x\to 0$, applying the rule of L'Hospital $(n+1-r)$ times to the r^{th} term of the first set of terms and $(n+2-r)$ times to the r^{th} term of the second set of terms. For this purpose an immediate extension of Lemma 4 is used; that is, without loss of generality, in the expression from the general solutions for $u_y^{[k]}$, we may assume that

$$F^{(1)}(0)=F^{(2)}(0)=\cdots=F^{(n+2)}(0)=0\ ;$$

it is only $u_y^{[k]}$, not $u^{[k]}$ itself, which enters into (2.26).

Since the coefficients of the EPD equation do not depend on x, it is evident that if a solution $u^{[k]}(x, y)$ has three continuous derivatives in a region, then $u_x^{[k]}(x, y)$ is a solution with at least two continuous derivatives in that region. This is the motivation of the following lemma which is essential to the induction of § 4. A solution which has *three* continuous derivatives in a triangle G the interior of which contains the triangle T and its sides except for the base line, will be said to be *regular plus one* on T.

LEMMA 6. *Let $U^{[k]}(x, y)$ be any solution regular on T such that $U^{[k]}(x, 0)=0$. There exists a solution $u^{[k]}(x, y)$ regular plus one on T such that*

$$U^{[k]}(x, y)=u_x^{[k]}(x, y)$$

and such that

$$u^{[k]}(x, 0)=0 \ .$$

Proof. This lemma is obtained in a trivial manner from the general solutions using Lemma 1. For Case 2,

$$\begin{aligned} U^{[k]}(x, y) &= -2^{s+1}\sum_{r=1}^{n} a_{rn}y^{s+r-1}\int_1^{-1}\psi^{(r)}[x+\alpha y](1-\alpha^2)^{s/2-1}\,\alpha^r\,d\alpha \\ &= \frac{\partial}{\partial x}\left[-2^{s+1}\sum_{r=1}^{n} a_{rn}y^{s+r-1}\int_1^{-1}\psi^{(r-1)}[x+\alpha y](1-\alpha^2)^{s/2-1}\alpha^r\,d\alpha\right], \end{aligned}$$

or for Case 3(b)

$$\begin{aligned} U^{[k]}(x, y) &= \sum_{r=1}^{n+1} a_{r,n+1}y^{r-1}\,\{F^{(r)}(x+y)+(-1)^{r+1}F^{(r)}(x-y)\} \\ &= \frac{\partial}{\partial x}\sum_{r=1}^{n+1}\left[a_{r,n+1}y^{r-1}\,\{F^{(r-1)}(x+y)+(-1)^{r+1}F^{(r)}(x-y)\}\right]. \end{aligned}$$

In both cases the quantity in square brackets is a solution of the EPD equation which is regular plus one on T since the arbitrary functions ψ and F have $(n+1)$ and $(n+3)$ continuous derivatives respectively. Of course, $u^{[k]}(x, 0)=0$ as required. Again the treatment of Case 4 is analogous to Case 2.

3. Uniqueness for $-2<k<1$. In this section we show that when $-2<k<1$

$$\left.\begin{array}{l} u^{[k]}(x, y) \text{ regular on } T \\ \lim\limits_{y\to 0} u^{[k]}(x, y)\equiv 0 \\ u^{[k]}(x, x)\equiv 0 \end{array}\right\} \Longrightarrow u^{[k]}(x, y)\equiv 0 \ .$$

The argument is divided into the cases $0<k<1$, $k=0$, and $-2<k<0$. In § 4 it will be shown that uniqueness for all $k\leqq 0$ follows from the uniqueness for $-2<k\leqq 0$.

The $k=0$ case is entirely trivial. We have

$$u^{[0]}(x, y)=F(x+y)+G(x-y) .$$

The boundary conditions yield

$$0=u^{[0]}(x, x)=F(2x)+G(0) \quad \text{or} \quad F(x)=-G(0)$$
$$0=u^{[0]}(x, 0)=F(x)+G(x) \quad \text{or} \quad G(x)=-F(x)=G(0)$$

so that

$$u^{[0]}(x, y)=-G(0)+G(0)\equiv 0 .$$

Consider now $0<k<1$. We have from (2.2) by Lemma 1

$$u^{[k]}(x, y)=-2^{k-1}y^{1-k}\int_1^{-1}\phi[x+\alpha y](1-\alpha^2)^{-k/2}\,d\alpha \tag{3.1}$$

and

$$0=u^{[k]}(x, x)=-2^{k-1}x^{1-k}\int_1^{-1}\phi[x(1+\alpha)](1-\alpha^2)^{-k/2}\,d\alpha .$$

Let $\sigma=x(1+\alpha)$. Then

$$0=2^{k-1}\int_0^{2x}\sigma^{-k/2}\phi[\sigma](\sigma-2x)^{-k/2}\,d\sigma$$

or

$$0=I^{1-k/2}[(2x)^{-k/2}\phi[2x]]$$

where $I^{\alpha}f[x]$ is the Riemann-Liouville integral of f to the order α (see e.g. [8]). Then $(2x)^{-k/2}\phi[2x]=0$ and $\phi[2x]\equiv 0$. Of course, then, from (3.1)

$$u^{[k]}(x, y)\equiv 0 .$$

The case $-2<k<0$ is similar. We treat only the case $k\neq -1$ because using Lemma 1, the treatments of $k=-1$ and k fractional become entirely analogous. We have

$$u^{[k]}(x, y)=-2^{s+1}y^{s}\int_1^{-1}\psi'[x-\alpha y](1-\alpha^2)^{s/2-1}\alpha\,d\alpha \tag{3.2}$$

where $0<s<2$ and ψ has two continuous derivatives on $[0, a]$. Then

$$0=u^{[k]}(x, x)=-2^{s+1}x^{s}\int_1^{-1}\psi'[x(1+\alpha)](1-\alpha^2)^{s/2-1}\alpha\,d\alpha$$

or, integrating once by parts,

$$0=\frac{x^{s+1}}{s}\int_1^{-1}\psi''[x(1+\alpha)](1-\alpha^2)^{s/2}\,d\alpha$$

for all x. As above let $\sigma=x(1+\alpha)$ and obtain

$$0=\frac{1}{s}\int_0^{2x}\sigma^{s/2}\psi''[\sigma](2x-\sigma)^{s/2}\,d\sigma$$

or

$$0=I^{s/2+1}[(2x)^{s/2}\psi''[2x]]\ .$$

Again

$$(2x)^{s/2}\psi''[2x]=0$$

or *for* $x\neq 0$, $\psi''[2x]=0$ and $\psi'[2x]\equiv\text{constant}=K$. But with $\psi'=K$, (3.2) becomes

$$u^{[k]}(x, y)=-2^{s+1}y^sK\int_1^{-1}(1-\alpha^2)^{s/2-1}\alpha\,d\alpha\equiv 0$$

since the integrand is odd.

4. An induction, uniqueness for all $k<1$. Uniqueness for $-2<k\leqq 0$ as proven in the last section together with the lemmas of § 2 are used here to establish uniqueness for all $k\leqq 0$, the case $0<k<1$ having already been considered in § 3.

Define (negative) numbers k_n recursively by the relation $k_{n+1}=k_n-2$, $n=1, 2, \cdots$ where $-2<k_1<0$; that is, such that $-2n<k_n<-2(n-1)$. We apply a complete induction. In § 3 it was shown that for $n=1$ (that is, for any k which is a k_1) $u^{[k]}(x, 0)\equiv u^{[k]}(x, x)\equiv 0$ implies $u^{[k]}(x, y)\equiv 0$ provided $u^{[k]}$ is regular on T. It remains only to show that if this statement is true for $k=k_n$, then it is true for $k=k_{n+1}=k_n-2$.

Induction assumption. $u^{[k_n]}(x, 0)\equiv u^{[k_n]}(x, x)\equiv 0$ implies $u^{[k_n]}(x, y)\equiv 0$ provided $u^{[k_n]}$ is *regular* on T.

(a) Given $u^{[k_{n+1}]}(x, y)$ *regular plus one*[3] on T and such that

$$u^{[k_{n+1}]}(x, 0)=u^{[k_{n+1}]}(x, x)\equiv 0$$

we generate a solution $u^{[k_n]}(x, y)$ of the EPD equation which is *regular* on T by the recursion

$$yu^{[k_n]}(x, y)=u_y^{[k_{n+1}]}(x, y)\ . \tag{4.1}$$

[3] See Lemma 6.

Now by Lemma 5, $u_y^{[k_{n+1}]}(x, x)=0$ so that

$$u^{[k_n]}(x, x)\equiv 0\ . \tag{4.2}$$

Further from (4.1) by Lemma 3

$$u^{[k_n]}(x, 0)=\lim_{y\to 0} u^{[k_n]}(x, y)=\lim_{y\to 0} \frac{u_y^{[k_{n+1}]}(x, y)}{y}\equiv 0\ . \tag{4.3}$$

(b) Now the induction assumption together with (4.2) and (4.3) imply that $u^{[k_n]}(x, y)\equiv 0$. But then by (4.1)

$$u_y^{[k_{n+1}]}(x, y)\equiv 0$$

or

$$u^{[k_{n+1}]}(x, y)=F(x) \qquad \text{for all y.}$$

However, $F(x)$ may be evaluated by setting y equal either to zero or x so that

$$F(x)=u^{[k_{n+1}]}(x, 0)=u^{[k_{n+1}]}(x, x)\equiv 0$$

and

$$u^{[k_{n+1}]}(x, y)\equiv 0\ . \tag{4.4}$$

(c) Consider now $U^{[k_{n+1}]}(x, y)$ *regular* on T and such that

$$U^{[k_{n+1}]}(x, 0)=U^{[k_{n+1}]}(x, x)\equiv 0\ .$$

By Lemma 6 we can write

$$U^{[k_{n+1}]}(x, y)=u_x^{[k_{n+1}]}(x, y) \tag{4.5}$$

where $u^{[k_{n+1}]}$ is regular plus one on T and

$$u^{[k_{n+1}]}(x, 0)\equiv 0\ . \tag{4.6}$$

Let us examine the condition $U^{[k_{n+1}]}(x, x)\equiv 0$ or, equivalently, the condition $u_x^{[k_{n+1}]}(x, x)\equiv 0$. On the line $y=x$, the EPD equation may be written

$$\frac{d}{dx}(u_x^{[k_{n+1}]}(x, x)-u_y^{[k_{n+1}]}(x, x)) = \frac{k_{n+1}}{x} u_y^{[k_{n+1}]}(x, x)$$

and the condition $u_x^{[k_{n+1}]}(x, x)=0$ yields

$$\frac{d}{dx} u_y^{[k_{n+1}]}(x, x) = -\frac{k_{n+1}}{x} u_y^{[k_{n+1}]}(x, x)$$

or

$$u_y^{[k_{n+1}]}(x, x) = Ax^{-k_{n+1}}, \qquad A \text{ arbitrary.} \tag{4.7}$$

Differentiating $u^{[k_{n+1}]}(x, y)$ on the line $y=x$ we have

$$u_x^{[k_{n+1}]}(x, x) + u_y^{[k_{n+1}]}(x, x) = \frac{d}{dx} u^{[k_{n+1}]}(x, x)$$

and, again since $u_x^{[k_{n+1}]}(x, x)=0$, using (4.7) we have

$$Ax^{-k_{n+1}} = u_y^{[k_{n+1}]}(x, x) = \frac{d}{dx} u^{[k_{n+1}]}(x, x)$$

so that

$$u^{[k_{n+1}]}(x, x) = Bx^{1-k_{n+1}} + C\ . \tag{4.8}$$

Here B is arbitrary but C becomes zero since $u^{[k_{n+1}]}(x, 0) \equiv 0$. From parts (a) and (b) above in which the uniqueness of a solution $u^{[k_{n+1}]}$ which is regular plus one on T was established, the unique solution of the boundary value problem (4.6) (4.8) is

$$u^{[k_{n+1}]}(x, y) = By^{1-k_{n+1}}\ .$$

Then by (4.5)

$$U^{[k_{n+1}]}(x, y) \equiv 0$$

and this completes the induction.

The following theorem summarizes the results obtained in §§ 3 and 4.

THEOREM. *For* $-\infty < k < 1$ *there is at most one solution of the* EPD *equation which is regular on* T *and is such that for given functions* $f(x)$ *and* $g(x)$

$$\lim_{y \to 0} u^{[k]}(x, y) = f(x)\ , \qquad u^{[k]}(x, x) = g(x)\ .$$

It should be noted that the uniqueness theorem given in [1] does not apply here for the cases $k<0$ since the EPD equation does not satisfy the relation (A) (5″) of that paper unless $0 \leqq k \leqq 2$

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ON THE INEQUALITY $\Delta u \geqq f(u)$

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We are interested in solutions of the non-linear differential inequality

$$\Delta u \geqq f(u) \tag{1}$$

where $u(x_1, \cdots, x_n)$ is to be defined in some region of Euclidean n-space and $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$ is the Laplacian of u. Wittich [5] considered the corresponding equation

$$\Delta u = f(u) \tag{1a}$$

in two dimensions and found conditions on $f(u)$ which guarantee that (1a) has no solution valid in the whole plane. Haviland [1] found a slightly weaker result in 3 dimensions, and Walter [4] generalized Wittich's theorem to n-dimensions. The method is essentially the same in all three papers, resulting on the one hand in the requirement that the function $f(u)$ be convex, and on the other hand in a rather involved argument for the n-dimensional case. The proofs do extend immediately to the inequality (1).

In the present paper we deal directly with (1), and obtain in particular a simple proof of a stronger theorem (Theorem 1 below) where the convexity of $f(u)$ is no longer required. Our method also yields much more precise information on the behavior of solutions.

Recently Redheffer [3] has obtained in the two-dimensional case an improvement of our Theorem 1, where the monotonicity of $f(u)$ is not needed. Although Redheffers's theorem may very likely be extendable to n dimensions, it does not seem possible by his method to obtain the more precise results mentioned in the remarks following Theorem 1.

The present investigation resulted from an attempt to determine the type of a class of Riemann surfaces. One result, Theorem 2, is given here as an application of Theorem 1.

We should like to mention finally that the method presented here has been developed independently by Keller, who, in a paper to be published, derives further information on the behavior of solutions of (1a), and applies his results to an interesting physical problem described in [2].

Notation. Throughout this paper we shall reserve r for the polar

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distance, $r=\sqrt{x_1^2+\cdots+x_n^2}$, in space of some fixed dimension $n\geqq 1$. We note that if $\varphi(r)$ is considered as a function in this space depending only on r, then

$$\Delta\varphi=\frac{\partial^2\varphi}{\partial r^2}+\frac{n-1}{r}\frac{\partial\varphi}{\partial r}=\frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial\varphi}{\partial r}\right). \tag{2}$$

LEMMA 1. *Let $f(t)$ be a (weakly) monotone increasing continuous function defined for all t. Suppose that there exists a function $\varphi(x)$ satisfying*

$$\varphi''(x)+\frac{n-1}{x}\varphi'(x)=f(\varphi) \tag{3}$$

for $0\leqq x<R$, with $\varphi'(0)=0$ and $\varphi(x)\to+\infty$ as $x\to R$. Then if u is any solution of (1) *for $r\leqq R$, we have $u(x_1,\cdots,x_n)\leqq\varphi(r)$ at each point.*

Proof. By (2) the function $\varphi(r)$ satisfies $\Delta\varphi=f(\varphi)$ for $r<R$. We let $v=u-\varphi$ and wish to show that $v\leqq 0$ for $r<R$. But suppose $v>0$ at some point. Since $v\to-\infty$ as $r\to R$ it would follow that v would take on its maximum at some point P with $r<R$. Then $v>0$ in some neighborhood N of P, that is $u>\varphi$ throughout N. This implies $\Delta v=\Delta u-\Delta\varphi\geqq f(u)-f(\varphi)\geqq 0$, so that Δv would be subharmonic in N, contradicting that v had a maximum at P.

LEMMA 2. *If $f(t)>0$, $f'(t)$ continuous, and $f'(t)\geqq 0$ for all t, then equation* (1) *has a solution u valid for all $(x_1,\cdots,x_n)$ if and only if there is a solution of* (3) *valid for all $x\geqq 0$, with $\varphi'(0)=0$.*

Proof. If such a function φ exists, then $\varphi(r)$ is the desired solution of (1).

Conversely, suppose that no such function $\varphi(x)$ exists. Given an arbitrary real number a, there exists[1] in any case a solution of (3) with initial values $\varphi(0)=a$, $\varphi'(0)=0$, valid in some interval $0\leqq x\leqq x_0$. Then there is a maximal interval $0\leqq x<R$ in which this solution exists. Further, we have by (2) that $\frac{d}{dx}(x^{n-1}\varphi')=x^{n-1}f(\varphi)>0$ for $x>0$, so that $x^{n-1}\varphi'$ is increasing, hence positive for $x>0$ since $\varphi'(0)=0$. Under these conditions we must have $\varphi(x)\to+\infty$ as $x\to R$. Then by Lemma 1 any solution u of (1) would satisfy $u\leqq\varphi$ for $r<R$. In particular we would

[1] The existence does not follow immediately from classical theorems, but may be proved by writing equation (3) in the integral form $\varphi(x)=a+\int_0^x\frac{1}{s^{n-1}}\int_0^s t^{n-1}f(\varphi)\,dt\,ds$ and applying standard iteration procedure.

have $u(0)\leqq\varphi(0)=a$. But since a was arbitrary there could be no solution u valid in $r<R$ for arbitrarily large R.

LEMMA 3. *If $f(t)>0$, $f'(t)$ continuous, and $f'(t)\geqq 0$ for all t, then equation* (3) *has a solution φ with $\varphi'(0)=0$ valid for all $x\geqq 0$ if and only if*

$$\int^{\infty}\left(\int_0^t f(s)\,ds\right)^{-1/2}dt=\infty\ . \tag{4}$$

Proof. Suppose first that there does not exist a solution of (3) valid for all $x\geqq 0$. Then we have seen that if $\varphi(x)$ satisfies (3) in some interval, with $\varphi(0)=0$ and $\varphi'(0)=0$, then for some $R>0$ we will have $\varphi(x)\to+\infty$ as $x\to R$. Further we noted that for $x>0$, $\varphi'(x)>0$, and hence from equation (3), $\varphi''<f(\varphi)$. Thus $\varphi'\varphi''<f(\varphi)\varphi'$ and integrating from $x=0$ to $x=t$ gives

$$[\varphi'(t)]^2<2\int_0^t f(\varphi)\varphi'\,dx=2\int_0^{\varphi(t)} f(\varphi)\,d\varphi\ .$$

Hence

$$\left(\int_0^{\varphi} f(s)\,ds\right)^{-1/2}d\varphi<\sqrt{2}\,dt$$

and integration from $t=0$ to $t=R$ gives

$$\int_0^{\infty}\left(\int_0^{\varphi} f(s)\,ds\right)^{-1/2}d\varphi<\sqrt{2}R\ .$$

Suppose conversely that

$$\int_0^{\infty}\left(\int_0^t f(s)\,ds\right)^{-1/2}dt<\infty\ .$$

Then $t\cdot\left(\int_0^t f(s)\,ds\right)^{-1/2}\to 0$ as $t\to\infty$ since $\left(\int_0^t f(s)\,ds\right)^{-1/2}$ is monotone decreasing. Hence $t^{-2}\cdot\int_0^t f(s)\,ds\to\infty$ and $f(t)/t\to\infty$ since $f(t)$ is monotone increasing. Thus for an arbitrary fixed a, $f(t)>t-a$, for $t>t_0$. Further, if φ is the solution of (3) with $\varphi(0)=a$, $\varphi'(0)=0$, then $\varphi(x)\geqq a$ for $x\geqq 0$, and $f(\varphi)\geqq f(a)$. Hence $(x^{n-1}\varphi')'\geqq f(a)\cdot x^{n-1}$, and integrating twice we find $x^{n-1}\varphi'\geqq\dfrac{f(a)}{n}x^n$, $\varphi\geqq\dfrac{f(a)}{2n}x^2$. Thus $\varphi(x)>t_0$ for $x>x_0$. As above we note that

$$\varphi'^2<2\int_a^{\varphi} f(\varphi)\,d\varphi\leqq 2(\varphi-a)f(\varphi)<2[f(\varphi)]^2 \qquad \text{for } \varphi>t_0\ .$$

Hence $\dfrac{n-1}{x}\varphi'<\dfrac{f(\varphi)}{2}$ for $x>x_1$, and consequently $\varphi''>\dfrac{1}{2}f(\varphi)$ for $x>x_1$.

Thus

$$[\varphi'(x)]^2-[\varphi'(x_1)]^2>\int_{\varphi(x_1)}^{\varphi(x)} f(s)\,ds$$

or

$$[\varphi'(x)]^2>\int_0^{\varphi(x)} f(s)\,ds-C$$

whence

$$\int_0^{\varphi(x)}\left(\int_0^t f(s)\,ds-C\right)^{-1/2} dt>x-x_1 .$$

Since the constant C does not affect the convergence of the integral we have that x must be bounded, which completes the proof of the lemma.

We may note that the proof of Lemma 3 is essentially that of Haviland [1]. The assumption made by Haviland that $f(t)\geqq c>0$ is seen to be unnecessary, but it is interesting to note that the theorem is no longer true in $n\geqq 3$ dimensions if we weaken the requirement to $f(t)\geqq 0$. (If we allow $f(t)=0$ we must speak of *non-constant* solutions of (3) for all x.) The reason for this is that a non-constant subharmonic function in one or two dimensions cannot be bounded above, while in three or more dimensions it can. Thus if we set $f(t)=0$ for $t\leqq 0$ and $f(t)=t^2$ for $t>0$, we see that any negative subharmonic function φ (such as $\varphi(r)=-1/(1+r^2)$ in 4 dimensions) satisfies $\Delta\varphi\geqq f(\varphi)$ throughout space, although the integral in (4) converges.

Combining these three lemmas we obtain the desired result:

THEOREM 1. *Let $f(t)$ be positive, continuous, and monotone increasing for $t\geqq t_0$, and suppose*

$$\int^{\infty}\left(\int_0^t f(s)\,ds\right)^{-1/2} dt<\infty .$$

Then a twice continuously differentiable function u cannot satisfy $\Delta u>0$ throughout space and $\Delta u\geqq f(u)$ outside of some sphere S.

Proof. Suppose such a function u exists. Then it has a maximum t_1 on S, and Δu has a minimum $m>0$ on S. Define $g(t)$ to be continuously differentiable for all t, and such that

a) $g'(t)\geqq 0$ for all t

b) $g(t)\leqq m$ for $t\leqq t_1$

c) $g(t)\leqq f(t)$ for all t

d) $g(t)\geqq f(t)-1$ for $t\geqq t_2$.

Then $\Delta u\geqq g(u)$ throughout space, so that by Lemma 2 there exists a solution of (3) with f replaced by g, and by Lemma 3 we would have

$$\int^{\infty}\left(\int_0^t g(s)\,ds\right)^{-1/2} dt=\infty$$

which, in view of d), contradicts the hypothesis.

Remarks 1. That the integral condition on $f(t)$ is the best possible can be seen most easily, as was pointed out by Walter [4], by noting that for an arbitrary continuous positive function $f(t)$ we can define $u(x_1)$ for $x_1 \geqq 0$ as the inverse of

$$x_1(u)=\frac{1}{\sqrt{2}}\int_0^u\left(\int_0^t f(s)\,ds\right)^{-1/2} dt$$

and for $x_1<0$ by $u(x_1)=u(-x_1)$. Then $\Delta u=\dfrac{\partial^2 u}{\partial x_1^2}=f(u)$ in any number of dimensions, and if the integral diverges this will hold for all x_1, and hence throughout space.

2. We may note that in the proof of Lemma 3 we have obtained somewhat more than the non-existence of a solution for all x. Namely, we have an upper bound on the values of x for which (3) can hold. However, the expression obtained is not a very convenient one, and in any case does not give the best possible bound. The advantage of Lemma 1 is that it allows us to give the best bound whenever we can find the function φ explicitly. For example, if we have the inequality $\Delta u \geqq \varepsilon e^{2u}$, $\varepsilon>0$, in two dimensions, then we can easily verify that

$$\varphi=\log\frac{2R}{\sqrt{\varepsilon}\,(R^2-r^2)}$$

satisfies the hypotheses of Lemma 1, so that $u(0)\leqq\varphi(0)=\log\dfrac{2}{R\sqrt{\varepsilon}}$.

We may therefore state the following result:

If u satisfies $\Delta u \geqq \varepsilon e^{2u}$ for $r\leqq R$ and $u(0)=a$,

$$\textit{then}\quad R\leqq\frac{2}{e^a\sqrt{\varepsilon}}\,.$$

3. We note that in the proof of Lemma 1 we need only assume that φ satisfies the inequality $\varphi''+\dfrac{n-1}{x}\varphi'\leqq f(\varphi)$. In many cases it may be possible to find an explicit solution of this inequality, but not of equation (3). For example, if $f(\varphi)=\varepsilon|\varphi|^{\alpha}$, $\alpha>1$, then the function

$$\varphi=\frac{cR^{2m}}{(R^2-r^2)^m}\,,\qquad c>0$$

satisfies in n dimensions

$$\begin{aligned}\Delta\varphi &= 2mR^{-4}c^{-2/m}(nR^2+(2m+2-n)r^2)\,\varphi^{1+2/m}\\ &\leqq 4m(m+1)R^{-2}c^{-2/m}\varphi^{1+2/m} \qquad \text{if } 2m+2\geqq n,\ r<R\,.\end{aligned}$$

Hence $\Delta\varphi\leqq\varepsilon u^{1+2/m}$ if $R\geqq 2(m+1)\varepsilon^{-1/2}c^{-1/m}$. We can therefore state the following:

If u satisfies $\Delta u\geqq\varepsilon|u|^\alpha$ for $r\leqq R$ in n-dimensions, where $\varepsilon>0$ and $\alpha>1$, and if $u(0)=a>0$, then $R\leqq 2(m+1)\varepsilon^{-1/2}a^{-1/m}$, where

$$m=\max\left\{\frac{n}{2}-1,\ \frac{2}{\alpha-1}\right\}.$$

4. The above remarks may also be viewed from the other direction. That is, if a function u is known to satisfy (1) for $r\leqq R$, then we get a pointwise upper bound on u in terms of the solution of (3). Furthermore, if we know that $u<M$ for $r=R$, then we can improve these bounds. Namely, we have $u\leqq\varphi$, where φ is the solution of (3) with $\varphi'(0)=0$ and $\varphi(R)=M$. Finally, these bounds are again the best possible since $\varphi(r)$ itself satisfies (1).

We turn next to an application of Theorem 1.

THEOREM 2. *If a simply-connected surface S has a Riemannian metric whose Gauss curvature K satisfies $K\leqq-\varepsilon<0$ everywhere, then S is conformally equivalent to the interior of the unit circle.*

Proof. Considering S as a Riemann surface, we know that it can be mapped conformally onto either the interior of the unit circle or else the whole plane. We proceed by contradiction. Suppose we could map S conformally onto the x, y-plane. The Riemannian metric on S could then be expressed as $ds^2=\lambda^2(dx^2+dy^2)$, and we have for the Gauss curvature:

$$K=-\frac{\Delta\log\lambda}{\lambda^2}\,.$$

$K\leqq-\varepsilon$ means that the function $u=\log\lambda$ would have to satisfy $\Delta u\geqq\varepsilon e^{2u}$ throughout the plane, contradicting Theorem 1.

We remark finally that the condition $K\leqq-\varepsilon$ can be weakened slightly to $K<0$, and

$$\frac{\displaystyle\iint_R K\,d\omega}{\displaystyle\iint_R d\omega}\leqq-\varepsilon<0$$

for every region R on the surface including some fixed compact set D, where $d\omega$ is the area element on the surface. The proof of this again involves assuming that the surface may be mapped conformally onto the whole x, y-plane and defining $z(r)$ as the mean value of K over the disk $x^2+y^2 \leqq r^2$. Simple inequalities yield

$$z''+\frac{3z'}{r} \geqq \varepsilon e^{2z} \qquad \text{for } r \text{ sufficiently large,}$$

and

$$z''+\frac{3z'}{r}>0 \qquad \text{everywhere.}$$

Since $z''+\dfrac{3z'}{r}$ is just the Laplacian of z in four dimensions, we again have a contradiction to Theorem 1.

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ON SEMI-NORMAL OPERATORS

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1. A bounded linear operator A in a Hilbert space will be called semi-normal if

$$(1) \qquad H=AA^*-A^*A\geqq 0 \quad (\text{or } \leqq 0)\,.$$

If A is a finite matrix, for instance, then relation (1) implies $H=0$, so that A is even normal ; cf., e.g., [4]. That (1) may hold with $H\neq 0$ is seen if one chooses, for instance, A to to the isometric matrix defined by $A=D=(d_{ij})$ where $d_{i+1,i}=1$ and $d_{ij}=0$ otherwise. The purpose of this note is to investigate the spectrum of the semi-normal operator A and of the associated self-adjoint operators J_θ defined by

$$(2) \qquad J_\theta=\frac{A_\theta+A_\theta^*}{2}\,, \quad A_\theta=Ae^{-i\theta}\ (\theta \text{ real})\,.$$

It is seen that, in particular, J_θ becomes the real or the imaginary part of A according as $\theta=0$ or $\theta=\pi/2$.

A number λ belonging to the spectrum of A (sp (A)) will be called accessible if there exists a sequence of numbers λ_n not belonging to sp (A) for which $\lambda_n\to\lambda$ as $n\to\infty$. If M is any self-adjoint operator, $\max M$ and $\min M$ will denote the greatest and the least points respectively of the set sp (M).

The following theorems will be proved:

THEOREM 1. *Let A be semi-normal with $H\geqq 0$ and let $\lambda=re^{i\theta}$ (r real, $\geqq 0$) be an accessible point of the spectrum of A. Then*

$$(3) \qquad (\max J_\theta)^2\geqq \min AA^*$$

and

$$(4) \qquad |r-\max J_\theta|\leqq((\max J_\theta)^2-\min AA^*)^{1/2}\,,$$

where J_θ is defined by (2).

THEOREM 2. *Let A be semi-normal and let $J=J_\theta$ have the spectral resolution $J=\int\lambda dE$. Then, if $S=S_\theta$ is any measurable set for which*

$$(5) \qquad \int_S dE=I\,,$$

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there holds the inequality

$$\|H\| \leqq 4\|A\| \text{ meas } S\,. \tag{6}$$

The proof of Theorem 1 will be given in § 2 below. The assertion of Theorem 2 can be considered as a supplement to Corollary 3 of [5]. The proof follows readily from the Lemma, *loc. cit.*, p. 1027 if one notes that $H/2 = J_\theta A_\theta^* - A_\theta^* J_\theta$ and that $\|A_\theta\| = \|A\|$.

Various corollaries can be obtained from the two theorems. For instance, as a consequence of Theorem 1, one has the

COROLLARY 1. *If V is isometric and not unitary, then its spectrum is the disk $|\lambda| \leqq 1$ of the complex plane.*

Actually it is possible to deduce this result from a normal form for such operators; cf., e.g., [8, p. 351 ff]. It should be noted that the spectrum of the isometric matrix D defined earlier in this paper, and which occurs in the normal form, is the disk $|\lambda| \leqq 1$; cf. [9, p. 279].

The proof of the corollary as a consequence of Theorem 1 however is as follows. Put $A^* = V$ so that $AA^* = I$; clearly V is semi-normal and $H \geqq 0$. Let $\lambda = re^{i\theta}$ $(r \geqq 0)$ be an accessible point in the spectrum of A (that is, of A^* or V). Then, by (3), $|\max J_\theta| \geqq 1$. On the other hand, $\|A\| = 1$, and hence $|\max J_\theta| \leqq 1$. Thus $|\max J_\theta| = 1$ and (4) implies $r = 1$; consequently, the only possible accessible points of the spectrum of an isometric operator lie on the circle $|\lambda| = 1$. However, if the operator is not unitary, then $\lambda = 0$ lies in its spectrum. Hence, the entire disk $|\lambda| \leqq 1$ is in the spectrum and the proof is complete.

Another consequence of Theorem 1 is

COROLLARY 2. *If A is semi-normal, if* 0 *lies in the spectrum of A, and if* $\min AA^* > 0$, *then for any θ the circular disk*

$$|\lambda| \leqq \max J_\theta - ((\max J_\theta)^2 - \min AA^*)^{1/2}$$

lies in the spectrum of A (where, of course, $\max J_\theta > 0$).

The proof follows from the observation that $\lambda = 0$ is in $\operatorname{sp}(A)$ but no accessible points of the spectrum can lie in the disk in question.

It can be remarked that if A is an arbitrary bounded linear operator (not necessarily semi-normal), and if the conditions that 0 be in $\operatorname{sp}(A)$ and $\min AA^* > 0$ are fulfilled, then there surely exists *some* circular disk $|\lambda| \leqq \text{const.}$ in the spectrum of A; sec, e.g., [7, pp. 76–78]. If however A is semi-normal, the radius of the corollary can even be specified.

An immediate consequence of Theorem 2 is the

COROLLARY 3. *If A is semi-normal but not normal, then the spectrum* *of* J_θ (*in particular, of the real or imaginary part of* A) *has a positive measure not less than* $\|H\|/4\|A\|$.

It should be noted that (5) surely holds if S is the spectrum of J although it may hold for a set of measure less than that of the spectrum (but whose closure would, of course, contain the spectrum).

It seems natural to conjecture that the spectrum of (say) the real part, $J=(A+A^*)/2$ of any semi-normal, but not normal, operator A must be an interval. Evidence to support the conjecture is furnished by the isometric, but not unitary, operators V, in which case the spectrum of $(V+V^*)/2$ is the interval $-1\leqq\lambda\leqq 1$. This fact also follows from the normal form for isometric operators referred to above and from the fact that the spectrum of $(D+D^*)/2$ is the interval $-1\leqq\lambda\leqq 1$ (cf., e.g., [3, p. 155]). Further evidence is furnished by the (bounded) matrices $A=(c_{j-i})$, where $c_n=0$ if $n<0$, for which the spectra of the associated Toeplitz materices $J=(A+A^*)/2$ are intervals, provided J is not a multiple of the unit matrix (in which case A is also); see [1, p. 361] and [2, p. 868]. It was shown in [6] that the matrices A are semi-normal.

The conjecture will remain unsettled. In fact, it will remain undecided whether or not the spectrum of the real part J of a semi-normal, but not normal, operator must even contain some interval. The assertion of Corollary 3 does not seem to preclude the possibility of, for instance, a nowhere dense spectrum (of positive measure).

2. Proof of Theorem 1. Let $\lambda_n=r_ne^{i\theta_n}$ be chosen so that λ_n is not in $\operatorname{sp}(A)$ and $\lambda_n\to\lambda$ as $n\to\infty$. Put $A_n=A-\lambda_nI$. Then $A_nA_n^*=A_nA_n^*A_nA_n^{-1}$, so that the spectra of $A_nA_n^*$ and $A_n^*A_n$, hence the spectra of $AA^*-2r_nJ_{\theta_n}$, and $A^*A-2r_nJ_{\theta_n}$, are (respectively) identical. Since $\lambda=re^{i\theta}$ is in the spectrum of A, then either $(A-\lambda)x_m\to 0$ or $(A-\lambda)^*x_m\to 0$ for some sequence of unit vectors x_m. In either case, it follows from (1) that $\limsup(x_m, A^*Ax_m)\leqq r^2$ as $m\to\infty$ and that $(x_m, J_{\theta_n}x_m)\to r$ as $m, n\to\infty$. Consequently, $\min(AA^*-2rJ_\theta)\leqq -r^2$ and hence $\min AA^*-2r\max J_\theta+r^2\leqq 0$. The desired relations (3) and (4) follow and the proof of Theorem 1 is complete.

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BOUNDS FOR THE PRINCIPAL FREQUENCY OF THE NONHOMOGENEOUS MEMBRANE AND FOR THE GENERALIZED DIRICHLET INTEGRAL

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Introduction. In § § 1 and 2 of this paper we consider an arbitrarily shaped membrane of variable density and uniform tension. We assume that this nonhomogeneous membrane is stretched in a given frame and obtain bounds for its principal frequency (fundamental tone). Before describing our results we quote the analogous result for the nonhomogeneous string proved in a paper by P. R. Beesack and the author [1, Theorem 2].

Let $p(x)$ *be continuous and not identically zero for* $-x_0 \leqq x \leqq x_0$, $0 < x_0 < \infty$, *and let* $p^+(x)$ *and* $p^-(x)$ *be the rearrangement of* $p(x)$ *in symmetrically increasing respectively decreasing order. Consider the three differential systems*

$$y''(x) + \lambda p(x) y(x) = 0 \ , \qquad y(\pm x_0) = 0 \ ;$$

$$u''(x) + \lambda^+ p^+(x) u(x) = 0 \ , \qquad u(\pm x_0) = 0 \ ;$$

$$v''(x) + \lambda^- p^-(x) v(x) = 0 \ , \qquad v(\pm x_0) = 0 :$$

denote their least positive eigenvalues also by λ, λ^+ *and* λ^- *respectively. Then* $\lambda^- \leqq \lambda$ *even if* $p(x)$ *changes sign finitely often while* $\lambda \leqq \lambda^+$ *holds if* $p(x) \geqq 0$.

For the nonhomogeneous membrane we consider a domain D bounded by a Jordan curve C. The differential system (for the original density) is given by

$$\Delta u(x, y) + \lambda p(x, y) u(x, y) = 0$$

for (x, y) in D and $u(C) = 0$. We base the existence of the first eigenfunction and its minimum property on the classical treatment of Courant-Hilbert [3, vol. 2, Chapter VII]. We assume therefore that $p(x, y)$ is positive and continuous in $\bar{D}$ and has continuous first derivatives in D. Together with $p(x, y)$ we consider its rearrangements in symmetrically increasing respectively decreasing order. The symmetrization is with respect to a point: $p^+(x, y) = p^+(r)$ and $p^-(x, y) = p^-(r)$ are defined in a closed disk $\bar{D}^*$ of the same area as D. The properties of $p(x, y)$ imply that $p^+(x, y)$ and $p^-(x, y)$ are positive and continuous in $\bar{D}^*$. However,

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their first derivatives may be discontinuous along infinitely many concentric circles which accumulate to circles lying in the open disk D^*. λ^+ and λ^- can thus not be defined as the classical first eigenvalues of a circular membrane with the density function p^+ or p^-, but are easily defined as a generalization of this notion (see formulas (8^+) and (8^-) below). The actual statement of Theorems 1 and 2 uses only density functions with continuous first derivatives, so that all eigenvalues are in the classical sense. Here we summarize these results as follows: In § 1 it is shown that if the original domain D is a disk, then $\lambda \leqq \lambda^+$ (Theorem 1). In § 2 we prove that for any domain D (bounded by a Jordan curve) $\lambda^- \leqq \lambda$. This Theorem 2 is a generalization of the theorem of Rayleigh, Faber and Krahn and it implies (essentially) a result of Szegö on the principal frequency of nonhomogeneous membranes [10, § V]. In Theorem 2′ we formulate these results in complete analogy to [1, Theorem 2], using generalized first eigenvalues.

Following Szegö([10] and [9, Note D]), we consider in § 3 a ring-shaped domain D and the class of the admissible functions $\varphi(x,y)$ in D. These admissible functions satisfy a smoothness condition, vanish on the inner boundary of D and are equal to 1 on its outer boundary. $p(x, y)$ is defined in $\overline{D}$ and satisfies the same conditions as in §§ 1 and 2; p^+ and p^- are now defined in a closed annulus $\overline{D}^*$. We denote the minimum of the generalized Dirichlet integral

$$\iint_D \{|\operatorname{grad} \varphi|^2 + p\varphi^2\}\, d\sigma$$

in the above class by $4\pi\gamma$ and define γ^+ and γ^- in a similar way. Theorem 3 states that for any ring-shaped domain D (bounded by two Jordan curves) $\gamma^- \leqq \gamma$. After restating this theorem in terms of Szegö's—slightly different—definition of the generalized Dirichlet integral, we show that it implies (essentially) Szegö's result on this integral. Theorem 4 states that if the original domain is an annulus, then $\gamma \leqq \gamma^+$. We conclude with two theorems which are one-dimensional analogues of the results on the generalized Dirichlet integral.

Throughout this paper, symmetrization which respect to a point is the main tool. We rely in § 2 on Krahn's paper [7] and in § 3 on Szegö's paper [10], and we use their results with regard to the behavior of the (ordinary) Dirichlet integral under this symmetrization (see (11^-) and (11^+) below). In addition, we use a well known theorem of Hardy, Littlewood and Pólya on the rearrangements of functions ([5, Theorem 378] and [9, p. 153]).

1. The nonhomogeneous membrane I. We start with the definition of the symmetrical rearrangements of a function $p(x,y)$ (cf. [5], [6] and [9]). Let D be a simply connected bounded domain in the x, y-plane and let $p(x, y)$ be defined and continuous in the closure $\bar{D}$ of D and be positive in D. We denote by D^* the open disk with the same area as D. R is the radius of D^* and $r=(x^2+y^2)^{1/2}$ the distance from its center. (By using x, y-coordinates for the planes containing D and D^* we do not imply that these planes have to coincide). The rearrangements of $p(x, y)$ in symmetrically increasing and decreasing order will be denoted by $p^+(x, y)$ and $p^-(x, y)$ respectively. They are uniquely defined in the closure $\bar{D}^*$ of D^* by the following three requirements: First, both functions have circular symmetry, $p^+(x, y)=p^+(r)$, $p^-(x, y)=p^-(r)$, $0 \leqq r \leqq R$, and $p^+(r)$ is a nondecreasing, $p^-(r)$ a nonincreasing function of r. Secondly, both functions are equimeasurable to $p(x, y)$; that is denoting by $A(z)$ the area of the open set in D for which $p(x, y) > z$ and similarly by $A^+(z)$ and $A^-(z)$ the area of the set in D for which $p^+(x, y) > z$ and $p^-(x, y) > z$ respectively, then we require that for each $z \geqq 0$ $A(z)=A^+(z) =A^-(z)$. Finally, at the center $r=0$ of D^* we let $p^+(p^-)$ be equal to the minimum (maximum) of p in $\bar{D}$ and we complete $p^+(p^-)$ to the closure $\bar{D}^*$ of D^* by assuming that its value on the boundary circle C^* is equal to the maximum (minimum) of p in $\bar{D}$.

The two rearrangements are connected by the formula $p^-(r)= p^+((R^2-r^2)^{1/2})$, $0 \leqq r \leqq R$. If p is positive in $\bar{D}$ then, clearly, the same holds for p^+ and p^- in $\bar{D}^*$. Moreover, the continuity of p in $\bar{D}$ implies the continuity of its rearrangements in $\bar{D}^*$ (cf. [6, Theorem 5]). Indeed, the continuity of $p(x, y)$ implies that $A(z)$ is a strictly decreasing function of z (for the z-interval bounded by the minimum and maximum of $p(x, y)$ in $\bar{D}$). As $p^+(r)$ and $p^-(r)$ are monotonic functions their only possible discontinuities would be jumps. Such a jump would imply that $A^+(z)$ or $A^-(z)$ had to be constant for the corresponding z-interval. But, as $A^+(z) \equiv A^-(z) \equiv A(z)$, this possibility is excluded.

Though not necessary for the following proofs, we wish to justify our above statement concerning the discontinuities of the first derivatives of $p^+(x, y)$ and $p^-(x, y)$. We assume therefore that $p(x, y)$ has continuous partial derivatives of first order—or, indeed, of any desired order—and we consider the surface $z=p(x, y)$ lying above D. Let us perform the transition from $p(x, y)$ to $p^-(x, y)=p^-(r)$ in the direction of decreasing z-values. The absolute maximum of $p(x, y)$ in $\bar{D}$ becomes $p^-(0)$ and every z-value, smaller than this absolute maximum, for which $p(x, y)$ has a local extremum induces a jump of dp^-/dr at the corresponding value $p^-(r)=z$. Clearly, the values of the local extrema of $p(x, y)$ may accumulate to one or more values lying in the open interval bounded by the absolute extrema of $p(x, y)$ in $\bar{D}$. This case generates the situation mentioned in the introduction with respect to the discontinuities of the first derivatives of $p^-(x, y)$ and $p^+(x, y)$. We shall return to this question in a special case (for the function $u^-(x, y)$ appearing in the proof of Theorem 2).

We state now the following.

THEOREM 1. *Let D be the disk $0 \leqq x^2+y^2 < R^2$, $0 < R < \infty$, and denote its boundary by C. Let the function $p(x, y)$ be positive and continuous in $\bar{D}$ $(=D \cup C)$ and have continuous first derivatives in D. Let $p^+(x, y) = p^+(r)$ $(r^2 = x^2+y^2,\ 0 \leqq r \leqq R)$ be the rearrangement of $p(x, y)$ in symmetrically increasing order defined in $\bar{D}$ $(=\bar{D}^*)$. Further let $m(x, y)$ be a function which is positive and continuous in $\bar{D}$, has continuous first derivatives in D and satisfies for each $(x, y) \in \bar{D}$*

$$(1^+) \qquad m(x, y) \leqq p^+(x, y) .$$

Consider the differential systems

$$(2) \qquad \Delta u(x, y) + \lambda p(x, y) u(x, y) = 0 \quad \text{for} \quad (x, y) \in D , \quad u(C) = 0 ,$$

and

$$(3^+) \qquad \Delta v(x, y) + \mu m(x, y) v(x, y) = 0 \quad \text{for} \quad (x, y) \in D , \quad v(C) \equiv 0 ,$$

and denote their first eigenvalues by λ and $\mu = \mu(m)$ respectively. Then

$$(4^+) \qquad \lambda \leqq \mu(m) .$$

For the proof we need the properties of the first eigenfunction. As mentioned, we rely on the last chapter of Courant-Hilbert [3, Vol. 2, Chapter VII]. In our § § 1 and 2 we deal with the eigenvalue problem for vanishing boundary values. (See their § 3 ; and put in their notation $p \equiv 1$, $a \equiv b \equiv q \equiv 0$, and replace their k—in case of our system (2)—by p). Throughout this paper we use the result of their § 4 ; this implies that if the domain D is bounded by a Jordan curve C, then a function belonging to their classes $\mathring{\boldsymbol{D}}$ and $\boldsymbol{F}$ is continuous in the closure $\bar{D}$ of D and vanishes on the boundary C. We state now all the needed properties, e.g. for system (2).

A first eigenfunction $u(x, y)$ of the system (2) is defined as a (non-trivial) solution of this system corresponding to the first eigenvalue $\lambda (\lambda > 0)$. $u(x, y)$ is continuous in $\bar{D}$, vanishes on C, has continuous derivatives of first and second order in D and the integral

$$\iint_D |\operatorname{grad} u|^2 d\sigma = \iint_D (u_x^2 + u_y^2) d\sigma$$

exists. $d\sigma$ denotes the area element of D and throughout this paper all area integrals are improper Riemann integrals [3, Vol. 2, p. 478]. Moreover, $u(x, y) \neq 0$ in D [3, Vol. 1, Chapter VI, § 6] and the first eigenfunction is therefore essentially unique (i.e. except for a multiplicative constant). The Rayleigh ratio

$$\iint_D |\operatorname{grad} \varphi|^2 d\sigma \Big/ \iint_D p\varphi^2 \, d\sigma$$

attains its minimum λ in the class of all admissible functions $\varphi(x, y)$ for $\varphi \equiv u$. Here a function $\varphi(x, y)$ is called admissible in D if it is continuous in $\overline{D}$, vanishes on C, has piecewise continuous[1] first derivatives in D and if the integral

$$\iint_D |\operatorname{grad} \varphi|^2 d\sigma$$

exists.

To prove Theorem 1 assume first that $m(x, y)$ has circular symmetry in $D, m(x, y) = m(r)$. Let $v(x, y)$ be a fixed first eigenfunction of (3^+). As the first eigenfunction is essentially unique, it follows from the circular symmetry of $m(r)$ that $v(x, y)$ too has circular symmetry, $v(x, y) = v(r)$. (3^+) becomes therefore

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}v(r)\right\} + \mu m(r)v(r) = 0 \quad \text{for} \quad 0 < r < R\,, \quad v(R) = 0\,.$$

As $v \neq 0$ in D, we may assume that $v(r) > 0$, $0 \leqq r < R$, and it follows that

$$\frac{d}{dr}\left\{r\frac{d}{dr}v(r)\right\} < 0 \quad \text{for} \quad 0 < r < R\,.$$

This inequality and

$$\lim_{r=0}\left\{r\frac{d}{dr}v(r)\right\} = 0$$

imply

$$\frac{d}{dr}v(r) < 0 \quad \text{for} \quad 0 < r < R\,.$$

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[1] A function is called piecewise continuous in a domain D if it is continuous there except for arbitrary discontinuities at isolated points and discontinuities of the first kind (jumps) along smooth arcs; and it is required that each closed subdomain of D has a nonempty intersection with only a finite number of these arcs [**3**, Vol. 2, p. 473].

$v(x, y)=v(r)$ is therefore symmetrically decreasing in D and, as $v>0$, the same holds for v^2. We have now

$$(5)\qquad \mu(m)=\frac{\iint |\operatorname{grad} v|^2 d\sigma}{\iint m v^2 d\sigma} \geqq \frac{\iint |\operatorname{grad} v|^2 d\sigma}{\iint p^+ v^2 d\sigma}$$

$$\geqq \frac{\iint |\operatorname{grad} v|^2 d\sigma}{\iint p v^2 d\sigma} \geqq \min \frac{\iint |\operatorname{grad} \varphi|^2 d\sigma}{\iint p \varphi^2 d\sigma} = \lambda .$$

All the integrals are taken over the disk D. The first inequality sign follows from (1^+). The second inequality sign is justified by the above mentioned theorem on the rearrangements of functions [9, p. 153]. To apply this theorem, we note that p and p^+ are equimeasurable and that p^+ and v^2 are oppositely ordered. The minimum in (5) is taken over the class of the admissible functions φ, and v clearly belongs to this class. We proved thus (4^+) under the additional assumption that $m(x, y)$ has circular symmetry.

We define now

$$(6^+)\qquad \lambda^+ = \text{g.l.b. } \mu(m) ;$$

here the g.l.b. is taken over all functions $m(x, y)$ fulfilling the requirements stated in the theorem and having, in addition, circular symmetry. Hence, we have until now established that

$$(7^+)\qquad \lambda \leqq \lambda^+ .$$

λ^+ is connected with the function p^+ in a more direct way ; that is, we show that

$$(8^+)\qquad \lambda^+ = \text{g.l.b.} \frac{\iint |\operatorname{grad} \varphi|^2 d\sigma}{\iint p^+ \varphi^2 d\sigma} ,$$

where the g.l.b. is taken over all admissible functions $\varphi(x, y)$. To prove (8^+) let us denote its right hand side by λ_+. (6^+) implies that for every $\varepsilon>0$ there exists a circular symmetric function $m(x, y)=m(r)$, fulfilling all our above requirements, for which $\mu(m) \leqq \lambda^+ + \varepsilon$. Denoting the corresponding first eigenfunction by v and using (1^+) we obtain

$$\lambda^+ + \varepsilon \geqq \mu(m) = \frac{\iint |\operatorname{grad} v|^2 d\sigma}{\iint m v^2 d\sigma} \geqq \frac{\iint |\operatorname{grad} v|^2 d\sigma}{\iint p^+ v^2 d\sigma} \geqq \lambda_+ .$$

It follows that

$$(9) \qquad \lambda^+ \geqq \lambda_+ .$$

On the other hand, given any ε, $0 < \varepsilon < 1$, there exists an admissible function $\varphi(x, y)$ such that

$$\lambda_+ + \varepsilon \geqq \frac{\iint |\operatorname{grad} \varphi|^2 d\sigma}{\iint p^+ \varphi^2 d\sigma} .$$

Furthermore, by using the Weierstrass approximation theorem with respect to $p^+(r)$, we can find a function $m(x, y) = m(r)$ which, in addition to all our former requirements, fulfills also $p^+(r)(1-\varepsilon) \leqq m(r)$ for $0 \leqq r \leqq R$. Hence,

$$\lambda_+ + \varepsilon \geqq \frac{\iint |\operatorname{grad} \varphi|^2 d\sigma}{\iint p^+ \varphi^2 d\sigma} \geqq (1-\varepsilon) \frac{\iint |\operatorname{grad} \varphi|^2 d\sigma}{\iint m\varphi^2 d\sigma}$$

$$\geqq (1-\varepsilon)\mu(m) \geqq (1-\varepsilon)\lambda^+ .$$

This implies

$$(10) \qquad \lambda_+ \geqq \lambda^+$$

(9) and (10) give (8^+).

Let us interpret the g.l.b. $\mu(m)$ in a less restrictive way than in (6^+) ; that is, we take now this g.l.b. over all functions $m(x, y)$ fulfilling the requirements stated in the theorem (and drop the additional requirement of circular symmetry). By a proof entirely analogous to the one given just now, it follows that also this g.l.b. $\mu(m)$ (for the wider class) is equal to the right hand side of (8^+). This and (8^+) imply that λ^+, that is, the g.l.b. $\mu(m)$ for the restricted class (of circular symmetric functions), is equal to the g.l.b. $\mu(m)$ for the wider class of functions $m(x, y)$ (not necessarily having circular symmetry). (7^+) establshes therefore (4^+) for any function $m(x, y)$ fulfilling the requirements stated in the theorem. This concludes the proof of Theorem 1.

In the special case of $p^+(x, y)$ having continuous first derivatives in D, λ^+ is the first eigenvalue (in the classical sense) of the differential system

$$\Delta v(x, y) + \lambda^+ p^+(x, y)\, v(x, y) = 0 \quad \text{for} \quad (x, y) \varepsilon D,\ v(C) = 0 .$$

In any case we shall call λ^+ the generalized first eigenvalue of this system.

2. The nonhomogeneous membrane II.

THEOREM 2. *Let D be a domain in the x, y-plane bounded by a Jordan curve C. Let the function $p(x, y)$ be positive and continuous in $\bar{D}\,(=D\cup C)$ and have continuous first derivatives in D. Let $p^-(x, y)=p^-(r)$ $(r^2=x^2+y^2,$ $0\leqq r\leqq R)$ be the rearrangement of $p(x, y)$ in symmetrically decreasing order defined in the closed disk $\bar{D}^*$ (whose boundary we denote by C^*). Further let $k(x, y)$ be a function which is positive and continuous in $\bar{D}^*$, has continuous first derivatives in the open disk D^* and satisfies for each $(x, y)\in\bar{D}^*$*

$$k(x, y)\geqq p^-(x, y)\ . \tag{1$^-$}$$

Consider the differential systems

$$\Delta u(x, y)+\lambda p(x,y)\,u(x, y)=0 \quad \textit{for} \quad (x, y)\in D,\ u(C)=0\ , \tag{2}$$

and

$$\Delta w(x, y)+\kappa k(x, y)\,w(x, y)=0 \quad \textit{for} \quad (x, y)\in D^*,\ w(C^*)=0\ , \tag{3$^-$}$$

and denote their first eigenvalues by λ and $\kappa=\kappa(k)$ respectively. Then

$$\lambda\geqq\kappa(k)\ . \tag{4$^-$}$$

For the proof set

$$\lambda^-=\text{l.u.b.}\ \kappa(k)\ , \tag{6$^-$}$$

where the l.u.b. is taken over all functions $k(x, y)$ satisfying the just stated conditions. The theorem will be proved if we show that

$$\lambda\geqq\lambda^-\ . \tag{7$^-$}$$

Similar to (8^+), it follows that

$$\lambda^-=\text{g.l.b.}\ \frac{\displaystyle\iint_{D^*}|\operatorname{grad}\varphi|^2\,d\sigma}{\displaystyle\iint_{D^*}p^-\varphi^2\,d\sigma}\ ; \tag{8$^-$}$$

here the g.l.b. is taken over all admissible functions $\varphi(x, y)$ in D^*. We shall use (8^-) for the proof of (7^-).

In the proof we make use of the first eigenfunction $u(x,y)$ of (2) and of its rearrangement in symmetrically decreasing order $u^-(x, y)=u^-(r)$. In particular, we have to show that u^- is an admissible function in D^* (see (12) below). u^- is continuous in $\bar{D}^*$ and vanishes on C^*; it is, however, doubtful whether in the case of a general $p(x, y)$, satisfying the conditions of the theorem, the first derivatives of $u^-(x, y)$ are piecewise continuous in D^*. But this is true, as we shall see presently, in the case in which the function $p(x, y)$ is analytic.

We therefore prove (7^-) first under the assumption that $p(x, y)$ is positive and continuous in $\bar{D}$ and analytic in D. The first eigenfunction $u(x,y)$ of (12) is then also analytic in D [8, p. 162]. We assume $u(x, y)$ fixed so that $u(x, y) > 0$ for $(x, y) \in D$. Following Krahn [7], we consider the planes z=constant which touch the surface $z=u(x, y)$, $(x, y) \in D$, and we claim that this (finite or infinite) set of horizontal planes can be enumerated $z=z_i$, $i=1, 2, \cdots$, in such a way that $z_1 > z_2 > \cdots, z_i > 0$, and that (in case of infinitely many such planes) $\lim z_i = 0$. Indeed, as $u(x, y)$ is continuous in $\bar{D}$, positive in D and vanishes on C, if this were not so then we could find a sequence $(x_n, y_n) \in D$, $n=1, 2, \cdots$, with the following properties:

(a) $\lim\limits_{n=\infty} (x_n, y_n) = (x_0, y_0) \in D$;

(b) $\operatorname{grad} u(x_n, y_n) = 0$, $n=1, 2, \cdots$;

(c) $u(x_m, y_m) \neq u(x_n, y_n)$ for $m \neq n$, $m, n=1, 2, \cdots$. We show now that the existence of such a sequence (x_n, y_n) is impossible. Let us consider the two sets of points (x, y) in D given by $u_x(x, y)=0$ and $u_y(x, y)=0$ respectively. u_x and u_y are together with u analytic functions in D. As u is a solution of (2) the identically vanishing of u_x or u_y is excluded. Hence, both these sets consist of analytic curves (or arcs) and we consider these curves near (x_0, y_0). Using $\Delta u < 0$ and, if necessary, rotating the coordinate system of the plane, we may assume that both $u_{xx} \neq 0$ and $u_{yy} \neq 0$ at (x_0, y_0). The curve $u_x(x, y)=0$ is thus near (x_0, y_0) represented by a power series of the form $x-x_0=P_1(y-y_0)$. Similary, $u_y(x, y)=0$ is there represented by $y-y_0=P_2(x-x_0)$. The expansion for $u_x(x, y)=0$ may be solved by $y-y_0=P_3((x-x_0)^{1/k})$, where $k \geq 1$ is given by the index of the first nonvanishing coefficient of P_1. By the above properties (a) and (b) of the sequence (x_n, y_n) it follows that $P_2(x_n-x_0)=P_3((x_n-x_0)^{1/k})$, $n=1, 2, \cdots$. As infinitely many of these last equalities hold for a fixed branch of $(x-x_0)^{1/k}$, it follows that $P_2(x-x_0) \equiv P_3((x-x_0)^{1/k})$ and that $k=1$. u_x and u_y vanish along this analytic curve which contains all the points (x_n, y_n). This gives the desired contradiction to property (c) and we have justified the enumeration of the horizontal tangential planes $z=z_i$.

Using $\Delta u < 0$—which excludes the existence of minima of $u(x, y)$—it follows that there are no closed curves along which $\operatorname{grad} u=0$. Arcs, ending at the boundary C of D, along which $\operatorname{grad} u=0$ are clearly excluded. This implies that no sequence (x_n, y_n) having the above properties (a) and (b) exists. Hence, each critical plane $z=z_i$ touches the surface $z=u(x, y)$ only in a finite number of points (and, for $i=2, 3, \cdots$, cuts the surface along certain analytic curves).

For any z, $0<z<z_1$, denote by $C(z)$ the level set $u(x, y)=z$ and let $A(z)$ be the area of the open set in D for which $u(x, y)>z$. $C(z)$ consists of the boundary of this open set and contains for $z=z_i$ $(i=2, 3, \cdots)$ perhaps an additional finite number of points. For $z\neq z_i$ $C(z)$ separates into a finite number of simple closed analytic curves and it follows that for each z, $0<z<z_1$, $C(z)$ is of finite positive length. We consider now the open intervals $z_i>z>z_{i+1}$ $(i=1, 2, \cdots)$, where in the case of only a finite number n of critical values z_i the last interval is $z_n>z>0$. For each z in one of these open intervals we have ([7, formula (10)] and [10, § II])

$$\frac{dA}{dz}=-\int_{C(z)}\frac{ds}{|\operatorname{grad} u|},$$

where ds denotes the length element of $C(z)$. Clearly $dA/dz<0$ $(z\neq z_i)$.

Let k_i denote the number of simple closed analytic curves into which $C(z)$ separates for z in the open interval (z_{i+1}, z_i), $i=1, 2, \cdots$. k_i is a function of i only, and it follows from the last formula that dA/dz is continuous for (z_{i+1}, z_i). The same consideration implies the existence of the one-sided limits of dA/dz as z tends to $z_{i+1}+0$ and z_i-0. These limits may conceivably be equal to $-\infty$[2], but are different from zero. Indeed, as $C(z_{i+1})$ is of positive length $(i=1, 2, \cdots)$, it follows that for $z\to z_{i+1}+0$ at least one of the k_i families of simple closed curves, into which the level sets $C(z)$ separate for z in (z_{i+1}, z_i), converges to a part of positive length of $C(z_{i+1})$. The same argument holds for $z\to z_i-0$ $(i=2, 3, \cdots)$.

As remarked in § 1, $A(z)$ is a strictly decreasing function of z, $0\leqq z\leqq z_1$. In the present case $A(z)$ is also continuous in this interval. This follows from the fact that $u(x, y)$ achieves a fixed z-value only for finitely many curves and (perhaps) points in D and not for a set of positive area. The definition of $u^-(x, y)=u^-(r)$ and the continuity of $A(z)$ imply that $A(z)=\pi r^2$ for $u^-(r)=z$ $(0\leqq z\leqq z_1,\ 0\leqq r\leqq R)$. Hence, $u^-(r)=A^{-1}(\pi r^2)$ and $u^-(r)$ is not only continuous (see § 1) but also strictly decreasing. The critical z-values $z_1, z_2, \cdots$ correspond to the critical r-values $r_1, r_2, \cdots$ with $r_1=0<r_2<r_3\cdots, r_i<R$ and (in case of infinitely many critical values) $\lim_{i=\infty} r_i=R$. As

$$\frac{du^-(r)}{dr}=\frac{dA^{-1}(\pi r^2)}{d(\pi r^2)}2\pi r=2\pi r\bigg/\left(\frac{dA}{dz}\right),$$

[2] This, indeed, cannot occur. We do not have to bring the argument which excludes this case, as we may allow that the one-sided limits of du^-/dr at $r_i (i=2, 3, \cdots)$ are equal to 0. Similary, it can be shown that dA/dz tends to a (finite) negative value as $z\to z_1-0$ so that $du^-/dr\to 0$ as $r\to 0$. This again will not be needed as an arbitrary singularity of u_x^- and u_y^- at (0, 0) does not invalidate their being piecewise continuous in D^*. (See below).

it follows that du^-/dr is continuous for each open interval $r_i < r < r_{i+1}$ $(i=1, 2, \cdots)$ and that its discontinuities at the values r_i $(i=2, 3, \cdots)$ are of the first kind (jumps). Every interval $0 \leqq r \leqq \rho$, $0 < \rho < R$, contains only a finite number of critical values r_i and every closed subdomain of D^* intersects therefore with only a finite number of critical circles $x^2+y^2=r_i^2$. The continuity of du^-/dr at $r \neq r_i$ implies the continuity of u_x^- and u_y^- at all points of D^* different from the center and not lying on these critical circles. At the critical circles $x^2+y^2=r_i^2$ $(i=2, 3, \cdots)$ u_x^- and u_y^- have (at most) jumps and it follows that these first derivatives of $u^-(x, y)$ are piecewise continuous in D^*. Moreover, as was shown by Faber [4] and Krahn [7] in their proofs of Rayleigh's conjecture, the Dirichlet integral

$$\iint_{D^*} |\operatorname{grad}(u^-)|^2 d\sigma$$

exists and fulfills the inequality.

$$\iint_{D} |\operatorname{grad} u|^2 d\sigma \geqq \iint_{D^*} |\operatorname{grad}(u^-)|^2 d\sigma, \tag{11$^-$}$$

which we shall use presently. All this, together with the previously established continuity of $u^-(x, y)$ in $\bar{D}^*$ and its vanishing on C^*, prove finally that the function $u^-(x, y)$ is admissible in D^*.

We have now

$$\lambda = \frac{\iint_D |\operatorname{grad} u|^2 d\sigma}{\iint_D p u^2 d\sigma} \geqq \frac{\iint_{D^*} |\operatorname{grad}(u^-)|^2 d\sigma}{\iint_{D^*} p^-(u^-)^2 d\sigma} \tag{12}$$

$$\geqq \text{g.l.b.} \frac{\iint_{D^*} |\operatorname{grad} \varphi|^2 d\sigma}{\iint_{D^*} p^- \varphi^2 d\sigma} = \lambda^- .$$

To justify the first inequality sign in (12) we use (11$^-$) for the numerators and for the denominators we apply again the theorem on the rearrangements of functions. (As $u > 0$, $(u^-)^2$ is together with u^- symmetrically decreasing, and p^- and $(u^-)^2$ are therefore similarly ordered.)[3]

[3] The integrals in the theorem on the rearrangements of functions [**9**, p. 153] are taken over the same bounded region. Our case, integrating once over D and the other time over D^*, can easily be reduced to that case of the same region of integration. We embed D and D^* into the same plane and take all integrals over a bounded region G containing both D and D^*, after having completed p, p^-, u and u^- into G by steting $p \equiv u \equiv 0$ in $G-\bar{D}$ and $p^- \equiv u^- \equiv 0$ in $G-\bar{D}^*$.

The g.l.b. appearing in (12) is taken over all admissible functions φ in D^* and is thus by (8^-) equal to λ^-. We proved (7^-), and hence the theorem, under the additional assumption of $p(x, y)$ being analytic in D.

This special case implies now (7^-) for any function $p(x, y)$ satisfying the conditions stated in the theorem. Indeed, as $p(x, y)$ is positive and continuous in $\bar{D}$, the Weierstrass approximation theorem assures that for every $\delta > 0$ there exists a polynomial $p_\delta(x, y) = p_\delta$, so that

$$(13) \qquad 0 < p(x, y) \leqq p_\delta(x, y) \leqq p(x, y)(1+\delta)$$

holds for all points (x, y) of $\bar{D}$. Denoting by $\lambda(\delta)$ the (classical) first eigenvalue of the differential system with the density function p_δ, the minimum property of the first eigenvalue implies

$$(14) \qquad \lambda(\delta) \leqq \lambda \leqq \lambda(\delta)(1+\delta) .$$

Let $p_\delta^-(x, y) = p_\delta^-(r)$ be the rearrangement of p_δ in symmetrically decreasing order defined in $\bar{D}^*$. (13) gives

$$(13^-) \qquad 0 < p^-(r) \leqq p_\delta^-(r) \leqq p^-(r)(1+\delta)$$

for $0 \leqq r \leqq R$. For the corresponding generalized first eigenvalues it follows by (8^-) and the analogous definition of $\lambda^-(\delta)$ that

$$(14^-) \qquad \lambda^-(\delta) \leqq \lambda^- \leqq \lambda^-(\delta)(1+\delta) .$$

For each polynomial $p_\delta(x, y)$ we proved

$$\lambda(\delta) \geqq \lambda^-(\delta) .$$

As δ tends to 0, we obtain from (14), (14^-) and the last inequality

$$(7^-) \qquad \lambda \geqq \lambda^- .$$

Theorem 2 is therefore established.

Inequality (11^-), i.e. the fact that the Dirichlet integral of the first eigenfunction decreases under symmetrization, was an essential step in our proof. On the other hand, this inequality constitutes Faber's and Krahn's proof of Rayleigh's conjecture. It is thus by no means surprising that Theorem 2 includes the theorem of Rayleigh, Faber and Krahn as the special case $p(x, y) \equiv 1$. However, Theorem 2 implies only a weakened from of their theorem, since with regard to inequality (11^-) Faber and Krahn proved more than we used. They showed that equality in (11^-) can occur only if D is a circle. Their theorem thus states that *for all homogeneous membranes with constant area the minimum of the principal frequency is achieved for the disk and only for the disk.* As for any homogeneous membrane $\lambda^+ = \lambda^-$, it follows that if $p \equiv 1$ and D

is not a disk then $\lambda > \lambda^+$. Hence, *Theorem* 1 *can not be extended to any noncircular domain.* For any such domain there exist functions $p(x, y)$, for example, all the positive constants, so that $\lambda > \lambda^+$, and at least for nearly circular domains there exist functions so that $\lambda < \lambda^+$. This last fact follows from the continuity of the first eigenvalue as a function of the domain [3, Vol. 1, Chapter VI, Theorem 11] (and we assume that for some functions $p(x, y)$ in the disk the proper inequality sign holds in (7^+)).

A lower bound for the principal frequency of nonhomogeneous membranes was obtained by Szegö in his paper on the generalized Dirichlet integral [10]. In this case the density function $p(x, y)$ is given in the whole x, y-plane (except at the origin) and satisfies there the following conditions:

(a) $p(x, y)$ is positive in the whole x, y-plane (with the exception of the origin);

(b) $p(x, y)$ has circular symmetry, $p(x, y) = p(r)$, and $p(r)$ is a non-increasing function of r, $r > 0$;

(c) $rp(r)$ is integrable in a neighborhood of $r = 0$. Considering membranes lying in this plane, Szegö's result is that *for all membranes with given area the minimum of the principal frequency is achieved for the disk whose center coincides with the origin of the plane.* [10, § V]. While keeping Szegö's condition (b), we replace his conditions (a) and (c) by the following more restrictive assumptions: (a′) $p(x, y)$ is positive and continuous in the whole x, y-plane; (c′) $p(x, y)$ has continuous first derivatives in the whole x, y-plane. Under these more restrictive conditions (a′), (b) and (c′), Szegö's result follows from Theorem 2. Indeed, let D be a domain in the x, y-plane with the given density function $p(x, y)$. Let D^* and $p^-(x, y)$ be defined as in Theorem 2, but put the center of D^* into the origin of the given x, y-plane. As $p(r)$ is a non-increasing function of r, $r \geqq 0$, it follows that for each $(x, y) \in D^*$

$$p(x, y) \geqq p^-(x, y) . \tag{15}$$

(a′), (c′) and (15) imply that $p(x, y)$ in D^* satisfies all the conditions which were in Theorem 2 required of $k(x, y)$. (4^-) is thus the desired conclusion. (For a one-dimensional analogue of Szegö's theorem see [1, Lemma 3].)

We state now our results on the nonhomogeneous membrane in a form involving only generalized first eigenvalues. We drop therefore the requirement that the original density function $p(x, y)$ has continuous first derivatives.

THEOREM 2′. *Let D be a domain in the x, y-plane bounded by a Jordan curve C and let $p(x, y)$ be positive and continuous in $\overline{D}$. Let*

$p^+(x, y)=p^+(r)$ and $p^-(x, y)=p^-(r)$ be the rearrangements of $p(x, y)$ in symmetrically increasing respectively decreasing order defined in the closed disk $\bar{D}^*$. Consider the three differential systems

$$\Delta u(x, y)+\lambda p(x, y)\, u(x, y)=0 \quad \textit{for} \quad (x, y) \in D, \quad u(C)=0\,;$$

$$\Delta v(x, y)+\lambda^+ p^+(x, y)\, v(x, y)=0 \quad \textit{for} \quad (x, y) \in D^*, \quad v(C^*)=0\,;$$

$$\Delta w(x, y)+\lambda^- p^-(x, y)\, w(x, y)=0 \quad \textit{for} \quad (x, y) \in D^*, \quad w(C^*)=0\,;$$

and denote their generalized first eigenvalues by λ, λ^+ and λ^- respectively. λ is defined by

$$\lambda=\text{g.l.b.} \frac{\iint_D |\operatorname{grad} \varphi|^2 d\sigma}{\iint_D p\varphi^2 d\sigma}, \tag{8}$$

where the g.l.b. *is taken over all admissible functions $\varphi(x, y)$ in D, and λ^+ and λ^- are analogously defined by* (8^+) *and* (8^-). *Then $\lambda^- \leqq \lambda$. In the special case of D being a disk ($D=D^*$) we have in addition $\lambda \leqq \lambda^+$.*

To prove this let us again approximate $p(x, y)$ by polynomials $p_\delta(x, y)$ satisfying (13). This implies (14), with λ now being defined by (8); (14) and (14^-) give as before (7^-), that is, $\lambda^- \leqq \lambda$. The additional result for the disk follows, by the same approximation, from Theorem 1.

We conclude the treatment of the nonhomogeneous membrane with the following remarks. It is known that the second proper frequency of a homogeneous membrane of given area does not attain its minimum for the disk [9, p. 168]. This implies that Theorem 2 cannot be extended to the second proper frequency; i.e. under its assumptions the relation $\lambda_2^- \leqq \lambda_2$ cannot be proved. Even for the circular nonhomogeneous membrane we are not able to establish any inequality—or equality—between λ_2, λ_2^+ and λ_2^-. It is thus of some interest to note that for the one-dimensional case (see [1, Theorem 2]) $\lambda_2^- = \lambda_2^+$. This follows easily from the relation $p^-(x)=p^+(x_0-x)$, $0 \leqq x \leqq x_0$.

Finally, an intuitive proof gives the following analogue of Theorem 2. *The principal frequency of a nonhomogeneous membrane of arbitrary shape decreases* (*i.e. does not increase*) *under Steiner symmetrization or under Pólya* (*circular*) *symmetrization.* (cf. [9, Note A] and [6, Chapter I]). Indeed, formula (12) holds also for these symmetrizations. The Dirichlet integral of the first eigenfunction decreases and we apply the one-dimensional case of the theorem on the rearrangements of functions for each member of an (obvious) one parameter family of straight or circular segments respectively. (Note that if D is not convex with respect to this family, then p^- is in general not continuous in D^*.

On the other hand, u^- is always continuous in D^*.) It is easily seen that Steiner and Pólya symmetrizations are weaker than Schwarz symmetrization used in Theorem 2; the lower bounds obtained by the first two kinds of symmetrization are not smaller than λ^- of Theorem 2.

3. The generalized Dirichlet integral. In this section we follow closely Szegö's treatment of the generalized Dirichlet integral ([10] and [9, Note D]); however, our definition of this integral will be somewhat simpler than Szegö's. We consider a ring-shaped domain D in the x, y-plane, that is, D is bounded by two Jordan curves C_0 and C_1 such that C_0 is completely in the interior of C_1. We call C_0 and C_1 the inner and outer boundary of D respectively and we denote the interior of C_1 by G. Let D^* be the open annulus which has the circle C_0^* of radius R_0 as inner boundary and the (concentric) circle C_1^* of radius R_1 as outer boundary $(0<R_0<R_1<\infty)$. The radii are so chosen that the disk bounded by C_0^* has the same area as the interior of C_0 and that the disk G^* bounded by C_1^* has the same area as G. Hence D^* has the same area as D and we assume that the center of D^* is the origin of a (new) x, y-plane and use again $r=(x^2+y^2)^{1/2}$.

Let $p(x, y)$ be nonnegative and continuous in the closure $\bar{D}$ of D. Its rearrangements in symmetrically increasing and decreasing order are defined in complete analogy to the case of a simply connected domain: $p^+(x, y)$ and $p^-(x, y)$ are defined in $\bar{D}^*$; both functions have circular symmetry $p^+(x, y)=p^+(r)$, $p^-(x, y)=p^-(r)$ and $p^+(r)$ is a nondecreasing, $p^-(r)$ a nonincreasing function of r, $R_0 \leqq r \leqq R_1$; p, p^+ and p^- are equimeasurable; finally, $p^+(R_0)$ $(p^-(R_0))$ is equal to the minimum (maximum) of p in $\bar{D}$ and $p^+(R_1)$ $(p^-(R_1))$ is equal to the maximum (minimum) of p in $\bar{D}$. Both rearrangements are nonnegative and continuous in $\bar{D}^*$.

The admissible functions are now defined as follows. A function $\varphi(x, y)$ is called admissible in D if it is continuous in $\bar{D}$, vanishes on C_0, is equal to 1 on C_1, has piecewise continuous first derivatives in D and if the integral

$$\iint_D |\operatorname{grad} \varphi|^2 d\sigma$$

exists. The admissible functions in D^* are defined analogously and will be denoted by $\psi(x, y)$. Using these definitions, we state.

THEOREM 3. *Let D be a ring-shaped domain in the x, y-plane and let the Jordan curves C_0 and C_1 be the inner and outer boundary of D respectively. Let the function $p(x, y)$ be positive and continuous in $\bar{D}$ and have continuous first derivatives in D. Let* $p^-(x, y)=p^-(r)$ $(R_0 \leqq r \leqq R_1)$

be the rearrangement of $p(x, y)$ in symmetrically decreasing order defined in the closed annulus $\overline{D}$. Denote by $4\pi\gamma$ the minimum of the generalized Dirichlet integral*

$$E(\varphi)=\iint_D \{|\operatorname{grad} \varphi|^2+p\varphi^2\}\, d\sigma \tag{16}$$

in the class of all admissible functions $\varphi(x, y)$ in D. Similarly, denote by $4\pi\gamma^-$ the g.l.b. *of the generalized Dirichlet integral*

$$E^-(\psi)=\iint_{D^*} \{|\operatorname{grad} \psi|^2+p^-\psi^2\}\, dr \tag{16$^-$}$$

in the class of all admissible functions $\psi(x, y)$ in D^* which satisfy $|\psi|\leqq 1$. Then[4]

$$\gamma\geqq\gamma^- . \tag{17$^-$}$$

We rely again on Courant-Hilbert [3, Vol. 2, Chapter VII]. To minimize $E(\varphi)$ in the class of all admissible functions $\varphi(x, y)$ in D is a special case of their Variational Problem I corresponding to the first boundary value problem. (See their § 2; and put in their notation $p=k=1$, $a=b=f=0$, and replace their q by our p. To assure that all their assumptions are satisfied, we have to show that there exists a function g which is continuous in $\overline{D}$, vanishes on C_0, is equal to 1 on C_1 and has piecewise continuous first derivatives in D which are such that

$$\iint_D |\operatorname{grad} g|^2\, d\sigma$$

exists. The existence of such a function g follows by conformal mapping. Set $z=x+iy$ and let $\zeta=\varphi(z)$ be the function which maps D onto the annulus $\rho<|\zeta|<1$. The harmonic function $g(x, y)=g(z)$,

$$g(z)=\log \frac{|\varphi(z)|}{\rho} \Big/ \log \frac{1}{\rho}$$

has all the required properties.)

We again use the result of their § 4 with an implication similar to the one stated in our § 1. With regard to the same problem for $E^-(\psi)$, the conditions of Courant-Hilbert are satisfied only if $p^-(x, y)$ has continuous first derivatives in D^*. As this is in general not true, $4\pi\gamma^-$ has to be defined as the g.l.b. $E^-(\psi)$.

[4] The words "which satisfy $|\psi|\leqq 1$" may of course be deleted. But we shall need the above given formulation of Theorem 3 to obtain Theorem 3′.

The variational problem to minimize $E(\varphi)$ in the class of all admissible functions $\varphi(x, y)$ in D has a unique solution $u(x, y)$. This admissible function $u(x, y)$ has continuous derivatives of first and second order in D and is also the unique solution of the corresponding boundary value problem; that is, $u(x, y)$ solves the system

(18) $\Delta u(x, y)-p(x, y)u(x, y)=0$ for $(x, y)\in D$, $u(C_0)=0$, $u(C_1)=1$,

and is the only admissible function having continuous first and second derivatives which solves this system. (18) and $p(x, y)>0$ imply $0\leqq u(x, y)\leqq 1$ for $(x, y)\in \bar{D}$.

For the same reason as in § 2, we prove Theorem 3 first under the assumption that $p(x, y)$ is not only positive and continuous in $\bar{D}$ but is also analytic in D. (18) implies the analyticity of $u(x, y)$ in D and in complete analogy to § 2—using (11^+) below— it follows that $u^+(x, y)=u^+(r)$ is an admissible function $\psi(x, y)$ in D^* which, by the above, satisfies $|\psi|\leqq 1$. We have now

$$(19)\qquad 4\pi\gamma=\iint_D\{|\operatorname{grad} u|^2+pu^2\}\,d\sigma\geqq\iint_{D^*}\{|\operatorname{grad}(u^+)|^2+p^-(u^+)^2\}\,d\sigma$$

$$\geqq \text{g.l.b.}\iint_{D^*}\{|\operatorname{grad}\psi|^2+p^-\psi^2\}\,d\sigma=4\pi\gamma^- .$$

To establish (19) it remains only to justify its first inequality sign. For this purpose we use

$$(11^+)\qquad \iint_D|\operatorname{grad} u|^2 d\sigma\geqq\iint_{D^*}|\operatorname{grad}(u^+)|^2 d\sigma\ ;$$

that is, the fact, proved by Szegö [10], that also in this case the Dirichlet integral decreases under symmetrization. The remaining inequality

$$\iint_D pu^2\,d\sigma\geqq\iint_{D^*}p^-(u^+)^2\,d\sigma$$

is again a consequence of the theorem on the rearrangements of functions. (See footnote 3) and complete u, u^+, p and p^- in an obvious way into a bounded region containing D and D^*.) This establishes (19) and thus proves Theorem 3 for analytic functions $p(x, y)$.

This special case implies (17^-) for any function $p(x, y)$ satisfying the conditions stated in the theorem. We use the same approximation as in the analogue step in § 2. $p_\delta(x, y)=p_\delta$ is again a polynomial satisfying (13) in $\bar{D}$ and (13^-) holds therefore for $\bar{D}^*$. Replacing in (16) p by p_δ and in (16^-) p^- by p_δ^-, we denote the corresponding minimum and g.l.b.

by $4\pi\gamma(\delta)$ and $4\pi\gamma^-(\delta)$ respectively. By (13) and the definitions of γ and $\gamma(\delta)$ we obtain (using that $0 \leqq u \leqq 1$)

$$4\pi\gamma = \iint_D \{|\operatorname{grad} u|^2 + pu^2\}\, d\sigma \geqq \iint_D |\operatorname{grad} u|^2 d\sigma + \left(1 - \frac{\delta}{1+\delta}\right)\iint_D p_\delta u^2 d\sigma$$
$$\geqq 4\pi\gamma(\delta) - \delta P d \ ,$$

where P is the maximum of $p(x, y)$ in $\overline{D}$ and d denotes the area of D. Setting $\alpha = Pd/4\pi$ we have

(20) $$\gamma \geqq \gamma(\delta) - \delta\alpha \ .$$

By (13^-) and the definitions of γ^- and $\gamma^-(\delta)$, there exists for each $\varepsilon > 0$ an admissible function $\psi(x, y)$ in D^*, satisfying $|\psi| \leqq 1$, so that

$$4\pi\gamma^-(\delta) + \varepsilon \geqq \iint_{D^*} \{|\operatorname{grad} \psi|^2 + p_\delta^- \psi^2\} d\sigma \geqq \iint_{D^*} \{|\operatorname{grad} \psi|^2 + p^- \psi^2\} d\sigma \geqq 4\pi\gamma^- \ ;$$

hence,

(21) $$\gamma^-(\delta) \geqq \gamma^- \ .$$

For each polynomial $p_\delta(x, y)$ we proved

$$\gamma(\delta) \geqq \gamma^-(\delta) \ .$$

As δ tends to 0, we obtain from (20), (21) and the last inequality the desired conclusion (17^-) and we thus completed the proof of Theorem 3.

The assumptions of this theorem can be weakened ; that is, as in Theorem 2′, there is no need to assume the existence (and continuity) of the first derivatives of $p(x, y)$. *Theorem* 3 *remains correct if we assume with respect to* $p(x, y)$ *only its being positive and continuous in* $\overline{D}$ *and if we accordingly define* $4\pi\gamma$ *as the* g.l.b. $E(\varphi)$ *in the class of all admissible functions* $\varphi(x, y)$ *in* D *which satisfy* $|\varphi| \leqq 1$. Indeed, the just given proof remains unchanged except for a slight modification in the derivation of (20).

We mentioned that definition (16) differs from Szegö's definition of the generalized Dirichlet integral. In order to obtain his result on this integral it will be convenient to restate Theorem 3 using his definition.

THEOREM 3′ *Let* D, C_0, C_1, D^*, C_0^* *and* C_1^* *have the same meaning as in Theorem* 3 *and denote the interior of* C_1 *by* G *and the interior of* C_1^* *by* G^*. *Let* $p(x, y)$ *be positive and continuous in* $\overline{G}$ *and have continuous first derivatives in* G (*or at least in* D). *Let* $p^-(x, y) = p^-(r)$ $(0 \leqq r \leqq R_1)$ *be the rearrangement of* p *in symmetrically decreasing order* (*in the sense of* § 1) *defined in* $\overline{G}^*$. *Further let* $k(x, y)$ *be positive and* *continuous in* $\overline{G}^*$, *have continuous first derivatives in* G^* (*or at least in* D^*) *and satisfy for each* $(x, y) \in \overline{G}^*$

$$k(x, y) \geqq p^-(x, y) . \tag{1$^-$}$$

Denote by $4\pi c$ *the minimum of the generalized Dirichlet integral*

$$D(\varphi) = \iint_D \{|\operatorname{grad} \varphi|^2 + p\varphi^2\} \, d\sigma - \iint_G p d\sigma \tag{22}$$

in the class of all admissible functions $\varphi(x, y)$ *in* D. *Similarly, denote by* $4\pi c(k)$ *the minimum of the generalized Dirichlet integral*

$$D_k(\psi) = \iint_{D^*} \{|\operatorname{grad} \psi|^2 + k\psi^2\} \, d\sigma - \iint_{G^*} k \, d\sigma \tag{23}$$

in the class of all admissible functions $\psi(x, y)$ *in* D^*. *Then*

$$c \geqq c(k) . \tag{24}$$

For the proof let $4\pi c^-$ be the g.l.b. of the generalized Dirichlet integral

$$D^-(\psi) = \iint_{D^*} \{|\operatorname{grad} \psi|^2 + p^- \psi^2\} d\sigma - \iint_{G^*} p^- d\sigma \tag{22$^-$}$$

in the class of all admissible functions $\psi(x, y)$ in D^* which satisfy $|\psi| \leqq 1$. We show first that

$$c \geqq c^- . \tag{25}$$

As

$$\iint_G p d\sigma = \iint_{G^*} p^- d\sigma$$

and as these two integrals are independent of φ and ψ respectively, (25) is equivalent to

$$\min \iint_D \{|\operatorname{grad} \varphi|^2 + p\varphi^2\} \, d\sigma \geqq \text{g.l.b.} \iint_{D^*} \{|\operatorname{grad} \psi|^2 + p^- \psi^2\} d\sigma \; ; \tag{26}$$

here the minimum is taken over all admissible functions φ in D, the g.l.b. only over those admissible functions ψ in D^* which satisfy $|\psi| \leqq 1$. p^- in (26) is obtained by rearranging — in the sense of § 1 — the in $\overline{G}$ defined function p and then considering this rearrangement only in $\overline{D}^*$. p^- in (16$^-$) is the rearrangement—in the sense of the beginning of this section—of the restriction of the function p to $\overline{D}$. It is easily seen that, at each point $(x, y) \in \overline{D}^*$, $p^-(x, y)$ in the sense of (26) is not larger than $p^-(x, y)$ in the sense of (16^-). Theorem 3 implies thus (26) and hence the proof of (25).

Let now $k(x, y)$ be any function satisfying the conditions stated in Theorem 3′. By the definition of c^-, there exists for each $\varepsilon > 0$ an admissible function ψ in D^* satisfying $|\psi| \leqq 1$, so that $4\pi c^- + \varepsilon \geqq D^-(\psi)$. Using ($1^-$), ($22^-$), (23) and $|\psi| \leqq 1$ we obtain

$$4\pi c^- + \varepsilon \geqq D^-(\psi) = \iint_{D^*} |\operatorname{grad} \psi|^2 d\sigma - \iint_{D^*} p^-(1-\psi^2) d\sigma - \iint_{G^*-D^*} p^- d\sigma$$

$$\geqq \iint_{D^*} |\operatorname{grad} \psi|^2 d\sigma - \iint_{D^*} k(1-\psi^2)\, d\sigma - \iint_{G^*-D^*} k d\sigma = D_k(\psi) \geqq 4\pi c(k) \,.$$

We thus obtain $c^- \geqq c(k)$ which together with (25) gives (24). Theorem 3′ is therefore established.

We state now Szegö's theorem on the generalized Dirichlet integral ([10], [9, Note D]) in the following restricted form: *Let the function $p(x, y)$ be given in the whole x, y-plane and satisfy there conditions* (a′), (b) *and* (c′) *stated in* § 2. *Let D be a ring-shaped domain in this plane bounded by the inner Jordan curve C_0 and the outer Jordan curve C_1. Denote by $4\pi c$ the minimum of the generalized Dirichlet integral*

$$D(\varphi) = \iint_D \{|\operatorname{grad} \varphi|^2 + p\varphi^2\}\, d\sigma - \iint_G p d\sigma \tag{22}$$

in the class of all admissible functions $\varphi(x, y)$ in D. Of all ring-shaped domains D with given area and with given area of the containing simply connected domain G, the annulus whose center coincides with the origin of the given plane has the minimum generalized capacity c.

This theorem follows from Theorem 3′ in the same way as our restricted form of Szegö's theorem on the membrane followed from Theorem 2. ((15) holds now in $\overline{G}^*$.) Szegö proves this theorem on the generalized Dirichlet integral assuming only conditions (b) and (c) stated in § 2[5] instead of our more restrictive conditions (a′), (b) and (c′).

Similarly to the final remark of § 2, it follows intuitively that *Theorem* 3 *and Theorem* 3′ *remain correct if we use Steiner or Pólya symmetrization instead of Schwarz symmetrization.* For the analogues of Theorem 3, Steiner and Pólya symmetrizations of functions given in a ring-shaped domain have to be defined in an obvious way.

Theorem 3 corresponds to Theorem 2 on the membrane. We state now a theorem on the generalized Dirichlet integral which corresponds to Theorem 1.

[5] We are unable to follow Szegö's argument allowing to drop the condition $p > 0$ (that is, condition (a) of § 2).

THEOREM 4. *Let D be the annulus $R_0^2 < x^2+y^2 < R_1^2$, $0 < R_0 < R_1 < \infty$ and denote its inner boundary by C_0 and its outer boundary by C_1. Let $p(x, y)$ be positive and continuous in $\overline{D}$ and have continuous first derivatives in D. Let $p^+(x, y) = p^+(r)$ $(R_0 \leqq r \leqq R_1)$ be the rearrangement of $p(x, y)$ in symmetrically increasing order defined in $\overline{D}$ $(=\overline{D}^*)$. Let γ have the same meaning as in Theorem 3 and denote by $4\pi\gamma^+$ the* g.l.b. *of the generalized Dirichlet integral*

$$E^+(\varphi) = \iint_D \{|\operatorname{grad} \varphi|^2 + p^+ \varphi^2\}\, d\sigma \tag{16$^+$}$$

in the class of all admissible functions $\varphi(x, y)$ in D which satisfy $|\varphi| \leqq 1$. Then

$$\gamma \leqq \gamma^+ . \tag{17$^+$}$$

For the proof let $m(x. y) = m(r)$ be a function having circular symmetry in $\overline{D}$ and assume that m is continuous in $\overline{D}$ and has continuous first derivatives in D. Moreover, for each $(x, y) \in \overline{D}$ let $m(x, y) \geqq p^+(x, y)$. Denote by $4\pi\gamma(m)$ the minimum of

$$E_m(\varphi) = \iint_D \{|\operatorname{grad} \varphi|^2 + m\varphi^2\}\, d\sigma$$

in the class of all admissible functions $\varphi(x, y)$ in D. Then it is easily proved that

$$\gamma^+ = \text{g.l.b. } \gamma(m) ,$$

where the g.l.b. is taken over all functions $m(x, y) = m(r)$ satisfying the above conditions. Let now m be such a function and let $v(x, y)$ be the uniquely given admissible function for which $E_m(v) = 4\pi\gamma(m)$. The uniqueness of v and the circular symmetry of m imply that v too has circular symmetry, $v(x, y) = v(r)$. As $v(r)$ solves the differential system

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}v(r)\right\} - m(r)\,v(r) = 0 \quad \text{for } R_0 < r < R_1 , \quad v(R_0) = 0 , \quad v(R_1) = 1 ,$$

and as $m(r) > 0$ and $v(r) \geqq 0$ for $R_0 \leqq r \leqq R_1$, it follows that $v(r)$ is a non-decreasing function of r in this interval. We thus obtain

$$\begin{aligned} 4\pi\gamma(m) &= \iint_D \{|\operatorname{grad} v|^2 + mv^2\}\, d\sigma \geqq \iint_D \{|\operatorname{grad} v|^2 + p^+v^2\}\, d\sigma \\ &\geqq \iint_D \{|\operatorname{grad} v|^2 + pv^2\}\, d\sigma \geqq 4\pi\gamma . \end{aligned}$$

This proves Theorem 4. The last step of this proof shows that *the* *(italicized) statement following the proof of Theorem 3 holds also true with respect to Theorem 4.*

We started this paper with quoting the one-dimensional analogue of the results on the nonhomogeneous membrane. With regard to the generalized Dirichlet integral we state now the one-dimensional analogue of Theorem 3. It will be convenient to exchange the boundary conditions. We thus require the vanishing of the admissible functions at the outer endpoints of the two disjoint segments, so that p^+ (instead of p^- of Theorem 3) appears in our statement. Moreover, we let the inner endpoints of the two segments coincide and thus obtain

THEOREM 5. *Let $p(x)$ be positive and continuous for $-x_0 \leqq x \leqq x_0$, $0 < x_0 < \infty$, and let $p^+(x)$ be the rearrangement of $p(x)$ in symmetrically increasing order. Let $u(x)$ be the unique solution of the differential system*

$$u''(x) - p^+(x)u(x) = 0 \quad \textit{for} \quad -x_0 \leqq x \leqq 0\,, \quad u(-x_0) = 0\,, \quad u(0) = 1\,,$$

and set $\alpha = 2u'(0)$. Let $\varphi(x)$ be any function of class D' in $-x_0 \leqq x \leqq x_0$[6] *which satisfies $\varphi(-x_0) = \varphi(x_0) = 0$ and denote the maximum of $|\varphi(x)|$ in this interval by ϕ. Then*

$$\int_{-x_0}^{x_0} (\varphi'^2 + p\varphi^2)dx \geqq \alpha\phi^2\,.$$

Equality is obtained in the case $p(x) = p^+(x)$ and $\varphi(x) = Cu(x)$ for $-x_0 \leqq x \leqq 0$, $\varphi(x) = \varphi(-x)$ for $0 \leqq x \leqq x_0$.

For the proof let x_1 be a point in $\langle -x_0, x_0 \rangle$ such that $|\varphi(x_1)| = \phi$ and assume that $\varphi(x_1) = \phi$. Let us minimize the integral

$$\int_{-x_0}^{x_1} (y'^2 + py^2)dx$$

under the boundary conditions $y(-x_0) = 0$ and $y(x_1) = \phi$. The Euler equation $y'' - py = 0$ has (by $p > 0$) a unique solution satisfying the boundary conditions and it follows by standard criteria of the calculus of variations[7] that this unique extremal satisfying the boundary conditions gives the absolute (strong) minimum of the variational problem. Considering also the analogue problem for $\langle x_1, x_0 \rangle$ with the boundary conditions $y(x_1) = \phi$ and $y(x_0) = 0$ we finally obtain

$$\int_{-x_0}^{x_0} (\varphi'^2 + p\varphi^2)\,dx \geqq \int_{-x_0}^{x_0} (y'^2 + py^2)dx\,,$$

[6] See [**2**, p. 7].

[7] See Bolza [**2**; pp. 101, 102] and use his conditions (I), (IIb′) and (III′).

where $y(x)$ is the unique solution of $y''-py=0$ for $-x_0\leqq x\leqq x_1$ and $x_1\leqq x\leqq x_0$ which satisfies $y(-x_0)=y(x_0)=0$, $y(x_1)=\phi$. (Note that $0\leqq y\leqq\phi$ follows). We have now

$$\int_{-x_0}^{x_0}(y'^2+py^2)dx\geqq\int_{-x_0}^{x_0}\{(y^-)'^2+p^+(y^-)^2\}\,dx=2\int_{-x_0}^{0}\{(y^-)'^2+p^+(y^-)^2\}\,dx$$

$$\geqq 2\phi^2\int_{-x_0}^{0}(u'^2+p^+u^2)dx=\alpha\phi^2\ .$$

Here we used again the calculus of variations to justify the last inequality sign and we obtained the last equality by partial integration of u'^2. This completes the proof of Theorem 5.

In case of $p(x)$ being a monotonic function we obtain

THEOREM 6. *Let $p(x)$ be positive, continuous and non-decreasing for $a\leqq x\leqq b$ $(-\infty<a<b<\infty)$. Let $y_1(x)$ and $y_2(x)$ be any (nontrivial) solutions of*

$$y''(x)-p(x)y(x)=0\ ,\qquad a\leqq x\leqq b\ ,$$

which satisfy $y_1(a)=y_2(b)=0$. Then

$$\frac{y_1'(b)}{y_1'(a)}\geqq\frac{y_2'(a)}{y_2'(b)}(>0)\ .$$

For the proof we may assume $y_1(b)=y_2(a)=1$. As the Wronskian of the two solutions $y_1(x)$ and $y_2(x)$ is constant, (and using $p>0$) we obtain

$$y_1'(a)=-y_2'(b)>0\ .$$

Setting $p^*(x)=p(a+b-x)$, $a\leqq x\leqq b$, we have

$$y_1'(b)=\int_a^b(y_1'^2+py_1^2)dx\geqq\int_a^b(y_1'^2+p^*y_1^2)\,dx$$

$$\geqq\int_a^b(y_2'^2+py_2^2)dx=-y_2'(a)(>0)\ .$$

Dividing $y_1'(b)\geqq -y_2'(a)$ by $y_1'(a)=-y_2'(b)$ we obtain the assertion of the theorem.

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TECHNION—ISRAEL INSTITUTE OF TECHNOLOGY, HAIFA

MOREY'S REPRESENTATION THEOREM FOR SURFACES IN METRIC SPACES

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1. Introduction. In 1935 Morrey showed that a non-degenerate surface of finite Lebesgue area has a quasi-conformal representation on the unit circle. He made use of Schwarz' result for polyhedral surfaces and was able to use a limiting process after he had shown that the representations of the surfaces involved were sufficiently well behaved for the area to be given by the usual integral. The limiting process depended upon Tonelli's result concerning the lower semi-continuity of the Dirichlet integral.

Several years later Cesari reduced the dependence upon complex variable theory by the use of a variational technique to obtain a slightly weaker version of Schwarz' result, but he showed that for the remainder of Morrey's argument his form was adequate.

The purpose of this paper is to remove the restriction that the surfaces be in Euclidean space; the method is that of Cesari.

Morrey's theorem has proved useful in the study of certain two-dimensional problems in the calculus of variations. It is hoped that the extension of his theorem will permit corresponding extensions of that theory [3, 6, 12].

A desirable feature of quasi-conformal mappings is that the area of the surface is given by one half the Dirichlet integral. To retain this property for surfaces which are not in Euclidean space requires the definition of an appropriate integral to complement the definition of area. The definition of (Lebesgue) area used in this paper is that given in [13] which agrees with the usual definition in case the surface is in Euclidean space.

We shall make use of the ideas of [13] in two other respects. First, we need only solve our problem for surfaces in m, the space of bounded sequences [1], since the definitions are chosen so as to be invariant under an isometry and we can map other surfaces isometrically into m. Second, we shall make use of the fact that the area of a function in m depends only upon its distinct components. The last remark results from the definition of the area of a triangle. Let $r=\{r^i\}$, $s=\{s^i\}$, and $t=\{t^i\}$ be three points in m. Then the area of the triangle with these points as vertices is, by definition,

$$\frac{1}{2}\sup_{i,k}\begin{vmatrix} r^i & r^k & 1 \\ s^i & s^k & 1 \\ t^i & t^k & 1 \end{vmatrix}.$$

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2. A closure theorem for A.C.T. functions. Certain definitions applying to real-valued functions must be modified to apply to functions which range in a metric space.

DEFINITION 1. Let φ be defined on the interval $[a, b]$ with range in a metric space D. Let Φ be the interval function defined by

$$\Phi([c, d]) = \delta(\varphi(c), \varphi(d)) \qquad a \leqq c \leqq d \leqq b ,$$

where $\delta(r, s)$ is the distance between r and s in D. Then φ is B.V. or A.C. according as Φ is B.V. or A.C. Define $D\varphi = D\Phi$ wherever the right hand side exists.

With this definition of bounded variation and absolute continuity of a function of one real variable in a metric space D, we extend verbatim the definitions of bounded variation and absolute continuity in the sense of Tonelli, B.V.T. and A.C.T., to apply to functions of two variables with range in D [10].

If x is continuous on an open set G contained in E_2 into D, define, where the right hand sides exist,

$$D_u x(u, v) = D\varphi(u) \qquad \text{where } \varphi(t) = x(t, v) ,$$

$$D_v x(u, v) = D\psi(v) \qquad \text{where } \psi(t) = x(u, t)$$

If x is B.V.T. then $D_u x$ and $D_v x$ exist a.e. [8].

If φ is defined on $[a, b]$ into m and is A.C. it is still possible that $\lim_{w \to t} \dfrac{\varphi(w) - \varphi(t)}{w - t}$ may not exist anywhere [5]. Hence we define a *component-wise* derivative φ' by $\varphi' = \{\varphi^{i\prime}\}$. Since φ is A.C. it follows that all of the φ^i are also and that $\varphi^{i\prime}$ and $D\varphi$ exist almost everywhere. That $D\varphi \geqq |\varphi^{i\prime}|$ for each i is evident, hence φ' is defined, and in m, almost everywhere in $[a, b]$.

THEOREM 1. *If φ is* A.C. *then $D\varphi$ exists and is equal to $\|\varphi'\|$ wherever φ' exists.*

Proof. Suppose that the theorem is true whenever φ has only a finite number of non-zero components. Let φ_n be that function whose only non-zero components are the first n, and these are the first n components of φ. Then (see the proof of Theorem 10) length $\varphi = \lim_{n\to\infty}$ length φ_n. Hence

$$\int D\varphi = \text{length } \varphi = \lim_{n\to\infty} \text{ length } \varphi_n = \lim \int D\varphi_n = \lim \int \|\varphi_n'\| = \int \|\varphi'\| .$$

Thus we may as well suppose φ has only a finite number of non-zero components. Let t be a point where φ' is defined. It suffices to show that

$$D^+\varphi(t) = \limsup_{w \to t} \frac{\|\varphi(w)-\varphi(t)\|}{|w-t|} \leqq \|\varphi'\| .$$

For some i there exists a sequence of numbers $w_n \to t$ such that

$$\lim_{n\to\infty} \frac{\|\varphi(w_n)-\varphi(t)\|}{|w_n-t|} = D^+\varphi(t)$$

and

$$|\varphi^i(w_n)-\varphi^i(t)| = \|\varphi(w_n)-\varphi(t)\| .$$

The existence of this sequence implies that $D^+\varphi(t) = |\varphi^{i\prime}(t)| \leqq \|\varphi'\|$.

DEFINITION 2. If x is continuous on an open set G into m, define, where the right hand sides exist,

$$x_u(u, v) = \varphi'(u) \qquad \text{where } \varphi(t) = x(t, v) ,$$
$$x_v(u, v) = \psi'(v) \qquad \text{where } \psi(t) = x(u, t) .$$

THEOREM 2. *If x is* A.C.T. *on G into m then*

$$\|x_u\| = D_u x \quad \textit{and} \quad \|x_v\| = D_v x$$

wherever the left hand sides exist.

DEFINITION 3. If x is A.C.T. on G into D and if $D_u x$ and $D_v x$ are in L^2, then x is a *D-mapping* [4]. Let

$$D(x) = \iint_G [(D_u x)^2 + (D_v x)^2] .$$

It was shown in [13] that if x is a D-mapping on a Jordan region into a metric space, then the Lebesgue area of x, $L(x)$, is given by what corresponds to the usual integral (see § 6).

Let Π^N be the projection of m defined by

$$\Pi^N(\{a^i\}) = \begin{cases} a^i & i \leqq N , \\ 0 & i > N . \end{cases}$$

Put $\Pi^N x = {}_N x$.

THEOREM 3. *If x_m is a sequence of* A.C.T. *functions on a bounded open set G into m, if $x_m \to x$ uniformly in each closed set H contained in G, if the norms of the partial* (*component-wise*) *derivatives $p_m = \|x_{mu}\|$,*

$q_m=\|x_{mv}\|$ are in L^α, $\alpha>1$, and $\iint_G[p_m^\alpha+q_m^\alpha]<M$ for all m, then x is A.C.T. *in G, the norms of its partials $p=\|x_u\|$ and $q=\|x_v\|$ are in L^α and*

$$\iint_G p^\alpha=\liminf_{m\to\infty}\iint_G p_m^\alpha\,, \qquad \iint_G q^\alpha=\liminf_{m\to\infty}\iint_G q_m^\alpha\,.$$

Proof. Let us first suppose that $x_m={}_Nx_m$ for each m and fixed N. The hypothesis, together with the closure theorem for A.C.T. real-valued functions, assures us that x^i is A.C.T. for each i. Hence ${}_Nx$ is A.C.T.

The remainder of the proof, in case $x_m={}_Nx_m$, deviates slightly from that given in [4] for real-valued functions.

Let K be a closed set contained in G whose distance from the boundary of G is $2\rho>0$. Let K_ρ be the closed set of all points whose distance from K does not exceed ρ. Let $n>2/\rho$. Define $(n;x)$ by

$$(n;x)=\{(n;x,i)\} \text{ and}$$
$$(n;x,i)(u,v)=n^2\int_u^{u+1/n}\int_v^{v+1/n}x^i(r,s)\,dr\,ds \qquad \text{for } (u,v)\in K\,.$$

Then $(n;x)$ has continuous first partial derivatives, $(n;x)_u=(n;x_u)$, $(n;x)_v=(n;x_v)$, and $(n;x_m)_u\to(n;x)_u$, $(n;x_m)_v\to(n;x)_v$. Furthermore, if $\|y\|$ is in L^α, where $y=\{y^i\}$ is defined on G into m, each y^i being measurable, then

$$\iint_K\|(n;y)\|^\alpha\leqq\iint_G\|y\|^\alpha\,.$$

Thus

$$\iint_K\|(n;x)_u\|^\alpha=\lim_{m\to\infty}\iint_K\|(n;x_m)_u\|^\alpha\leqq\liminf_{m\to\infty}\iint_G\|(x_m)_u\|^\alpha\leqq M\,.$$

Since x is A.C.T. and x_u^i is integrable for each i, $(n;x_u^i)\to x_u^i$ a.e. in K and $\|(n;x_u)\|\to\|x_u\|$ a.e. in K. Thus $\|x_u\|$ is in L^α and

$$\iint_K\|x_u\|^\alpha\leqq\liminf_{n\to\infty}\iint_K\|(n;x_u)\|^\alpha\leqq\liminf_{m\to\infty}\iint_G\|(x_m)_u\|^\alpha\leqq M\,.$$

Finally, $p^\alpha=\lim_{N\to\infty}\|{}_Nx_u\|^\alpha$ and

$$\iint_K p^\alpha=\lim_{N\to\infty}\iint_K\|{}_Nx_u\|^\alpha\leqq\lim_{N\to\infty}\liminf_{m\to\infty}\iint_G\|({}_Nx_m)_u\|^\alpha\leqq\liminf_{m\to\infty}\iint_G\|(x_m)_u\|\leqq M\,.$$

Similarly

$$\iint_K q^\alpha=\liminf_{m\to\infty}\iint_G q_m^\alpha\leqq M\,.$$

3. Equicontinuity theorems. The theorems listed in this section are taken from [4], except that now the surfaces need not be in Euclidean space. The proofs carry over almost without change.

Let Q be the square $[0 \leqq u, v \leqq 1]$, let Q^* be its boundary, and x be defined on Q into a metric space D.

THEOREM 4. [L. C. Young]. *Given two positive numbers N and ε there exists a positive number η depending only upon N and ε such that for any D-mapping x with $D(x) < N$ there exists a δ, $\eta < \delta < \varepsilon$, and a finite subdivision of Q into rectangles whose side-lengths lie between δ and 2δ and such that image of each side of such rectangles not on Q^* is a rectifiable curve whose length is less than ε. A subdivision may be obtained by means of straight lines parallel to the sides of Q.*

THEOREM 5. *Let S be a base (or open non-degenerate) surface, let S_n, $n=1, 2, \cdots$, be a sequence of surfaces such that $\|S_n, S\| \to 0$, each S_n having a D-representation x_n on Q with $D(x_n) < M$ ($\|S_n, S\|$ is the Fréchet distance between the surfaces S_n and S). Then the mappings x_n are equicontinuous in each closed set $K \subset Q^\circ$ (the interior of Q).*

THEOREM 6. *Let S be an open non-degenerate surface and S_n be a sequence of surfaces with $\|S_n, S\| \to 0$ such that each S_n has a D-representation x_n on Q with $D(x_n) < M$ and such that there exist points $w_{im} \in Q^*$, $i=1, 2, 3$, and a positive number m with $\|w_{in} - w_{jn}\| > m$, $(x_n(w_{in}), x_n(w_{jn})) > m$ for $i \neq j$. Then the mappings are equicontinuous in an open set containing Q^*. That is, for each $\varepsilon > 0$ there is a $\delta > 0$ such that if w, $w' \in Q$, $\|w - w'\| < \delta$,* $\operatorname{dist}(w, Q^*) < \delta$, *and* $\operatorname{dist}(w', Q^*) < \delta$, *then $\delta(x_n(w), x_n(w')) < \varepsilon$.*

4. Lower semi-continuity theorems. The results in this section follow from [10].

If y is a D-mapping on G into m, let

$$E_n(y) = \iint_G \sup_{i < k \leqq n} \{i, k, y\}$$

$$E(y) = \iint_G \sup_{i < k} \{i, k, y\}$$

where

$$\{i, k, y\} = \{i, y\} + \{k, y\}$$

and

$$\{p, y\} = (y_u^p)^2 + (y_v^p)^2 .$$

Let

$$F(y)=\iint\limits_{G} \sup_{i} \{i, y\} .$$

THEOREM 7. *If* x_k *and* x *are continuous on* $\overline{G}$ *(the closure of* G*) into* m *and are D-mappings on* G *with* $x_k \to x$ *uniformly on* $\overline{G}$, G *being of finite measure, then*

$$D(x) \leqq \liminf_{k\to\infty} D(x_k) , \qquad E(x) \leqq \liminf_{k\to\infty} E(x_k) ,$$
$$F(x) \leqq \liminf_{k\to\infty} F(x_k) .$$

Proof. We shall prove that E is lower semi-continuous. The other two parts are proved in a similar manner.

The hypothesis and Theorem 2.1 [10, p. 26] show that E_n is lower semi-continuous. The theorem follows since $E_n \leqq E_{n+1}$ and $E=\lim_{n\to\infty} E_n$.

5. Quasi-conformal representations for surfaces in m**.** Much of this section is lifted bodily from [2]. The principal problem is to obtain a desirable representation for certain polyhedra. After this representation has been obtained, Morrey's technique yields a similar representation for other surfaces in m.

LEMMA 1. *Let* $a_n \geqq 0$, b_n, *and* c_n *be constants,* $n=1, 2, \cdots, N$. *If* $f(t)=\max_n [a_n t^2+b_n t+c_n]$ *then for some* m, $f^+(0)=f'_m(0)$ *where* $f^+(0)=\lim_{t\to 0^+} (f(t)-f(0))/t$ *and* $f_m(t)=a_m t^2+b_m t+c_m$.

Proof. That $f^+(0)$ exists is a result of the convexity of f. Now let $w_k>0$, $w_k \to 0$. Then for some m we have $f(w_k)=f_m(w_k)$ for an infinite set of k's, and in addition, $f(0)=f_m(0)$. Therefore

$$f^+(0)=\lim_{k\to\infty} \frac{f(w_k)-f(0)}{w_k}=\lim_{k\to\infty} \frac{f_m(w_k)-f_m(0)}{w_k}=f'_m(0) .$$

LEMMA 2. *Let* a_n, b_n, *and* c_n *be measurable functions on a bounded measurable set* E *with* $a_n(x) \geqq 0$, $n=1, 2, \cdots, N$. *Let* a, b, *and* c *be summable functions on* E *such that* $a_n(x) \leqq a(x)$, $|b_n(x)| \leqq b(x)$, *and* $|c_n(x)| \leqq c(x)$. *In addition let* M *be a positive constant and* A *and* B *be measurable functions on* E *such that* $|A(x)|<2M$ *and* $|B(x)|<2M^2$ *on* E. *Let*

$$f_n(x, t)=(1+A(x)t+B(x)t^2)^{-1}(a_n(x)t^2+b_n(x)t+c_n(x))$$
$$f(x, t)=\max_n f_n(x, t)$$

and

$$\varphi(t)=\int_E f(x, t)\,dx .$$

Then, for each x, there is an $r=n(x)$ such that

$$\varphi^+(0)=\int_E f_{rt}(x, 0)\,dx .$$

Proof. If we examine the proof of the theorem permitting differentiation under the integral sign [7] we see that it is sufficient to show the existence of a summable function g such that, for some $\eta>0$,

$$\left|\frac{f(x, t)-f(x, 0)}{t}\right|\leqq g(x) , \qquad 0<t<\eta .$$

If we take $\eta<(5M)^{-1}$ we may take $g(x)=2[\eta a(x)+b(x)]$.

If y is a D-mapping into m, let

$$[i, k, y]=y_u^i y_v^k-y_v^i y_u^k .$$

Then

$$L(y)=\iint \sup_{i,k} [i, k, y] .$$

THEOREM 8. *An open non-degenerate polyhedron $\mathscr{P}$ contained in range Π^N for some N has a representation x^* on the unit circle $\mathscr{C}$ such that x^* is a D-mapping and*

$$\max_i \{i, x^*\}=\max_{i,k} [i, k, x^*] \qquad \text{a.e. } in\ \mathscr{C}.$$

Proof. Let X be a representation of $\mathscr{P}$ on Q and let C=range $X|Q^*$. Consider the class K of all representations x of $\mathscr{P}$ which are D-mappings on $\mathscr{C}$. Since $\mathscr{P}$ is a polyhedron, K is not empty. Let $I=\inf E(x)$ for all $x\in K$. We shall see that the infimum is attained for $x=x^*$.

Let $\bar{x}_n$ be a minimizing sequence with $E(\bar{x}_n)<I+1/n$. Fix three distinct points $\overline{P_i}$ on Q^* with $Q_i=X(\overline{P_i})$ also distinct. For each n, choose P_{in} on $\mathscr{C}^*$ so that $\bar{x}_n(P_{in})=Q_i$. Let P_i^* be three distinct points of $\mathscr{C}^*$. By means of a conformal transformation taking $\mathscr{C}$ into itself and P_{in} into P_i^*, the functions $\bar{x}_n$ are transformed into x_n where $x_n(P_i^*)=Q_i$. It is easy to verify that $E(x_n)=E(\bar{x}_n)$.

Theorems 5 and 6 assure us that the sequence $\{x_n\}$ is equicontinuous and hence a subsequence of the x_n converges uniformly to x^*. The closure theorem for D-mappings enables us to conclude that $x^*\in K$. By Theorem 7, $E(x^*)=I$.

Now let φ and ψ be Lipschitzian with constant M in $\overline{\mathscr{C}}$ and vanish on $\mathscr{C}^*$. Then [2] the transformations T and T^{-1},

$$T:\quad \alpha=u+\varepsilon\varphi(u, v)\,, \qquad \beta=v+\varepsilon\psi(u, v)\,,$$

are both Lipschitzian if $|\varepsilon|<1/(3M)$. Put

$$x^*[u(\alpha, \beta, \varepsilon), v(\alpha, \beta, \varepsilon)]=x(\alpha, \beta, \varepsilon)\,,$$

Then $x\in K$ [10].

Now put

$$J(\varepsilon)=E(x)=\iint_{\mathscr{C}} \max_{i\neq k}\ \{i, k, x\}(\alpha, \beta)\, d\alpha\, d\beta\ .$$

A straightforward computation shows that

$$J(\varepsilon)=\iint_{\mathscr{C}} D^{-1} \max_{i\neq k}\ [E^*_{ik}(\alpha_v^2+\beta_v^2)-2F^*_{ik}(\alpha_u\alpha_v+\beta_u\beta_v)+G^*_{ik}(\alpha_u^2+\beta_u^2)]\, du\, dv\ ,$$

where

$$E^*_{ik}=(x_u^{*i})^2+(x_u^{*k})^2\,, \qquad G^*_{ik}=(x_v^{*i})^2+(x_v^{*k})^2\,, \qquad F^*_{ik}=x_u^{*i}x_v^{*i}+x_u^{*k}x_v^{*k}\,,$$
$$D=\frac{\partial(\alpha, \beta)}{\partial(u, v)}\ .$$

We apply Lemma 2 to compute

$$J^+(0)=\iint_{\mathscr{C}} \{[-(E^*_{rs}-G^*_{rs})\varphi_u-2F^*_{rs}\varphi_v]+[(E^*_{rs}-G^*_{rs})\psi_v-2F^*_{rs}\psi_u]\}\, du\, dv\ ,$$

where r and s depend upon (u, v). That $J^+(0)\geqq 0$ is evident since J assumes its minimum at $\varepsilon=0$.

From the arbitrariness and independence of φ and ψ we obtain, first of all, that

$$\iint_{\mathscr{C}} [(E^*_{rs}-G^*_{rs})\varphi_u-2F^*_{rs}\varphi_v]\, du\, dv\leqq 0$$

and

$$\iint_{\mathscr{C}} [(E^*_{rs}-G^*_{rs})\psi_v-2F^*_{rs}\psi_u]\, du\, dv\geqq 0\ .$$

Next we see that if we replace φ by $-\varphi$ and ψ by $-\psi$ then the equality must hold in each case.

The remainder of Cesari's proof now goes through without change, and we conclude that $E^*_{rs}=G^*_{rs}$, $F^*_{rs}=0$ almost everywhere. It is clear that

$$[i,\, k,\, x^*] \leqq \tfrac{1}{2}\{i,\, k,\, x^*\}$$

for all i, k. Also, where the equalities above hold, if we order r and s properly we see that

$$x_u^{*r} = x_v^{*s},\; x_v^{*r} = -x_u^{*s}\,, \qquad [r,\, s,\, x^*] = \tfrac{1}{2}\{r,\, s,\, x^*\} = \{r,\, x^*\} = \{s,\, x^*\}\ .$$

Also, from the maximizing property

$$\begin{aligned}\max_{i \neq k}\, \{i,\, k,\, x^*\} &= \{r,\, s,\, x^*\} = \{r,\, x^*\} + \{s,\, x^*\} = 2\{r,\, x^*\} \\ &= 2\{s,\, x^*\} = 2[r,\, s,\, x^*] = 2\max_{i,\,k}\,[i,\, k,\, x^*]\ .\end{aligned}$$

Finally $\{r,\, x^*\} = \max_i\, \{i,\, x^*\}$ for otherwise $\{i,\, s,\, x^*\} > \{r,\, s,\, x^*\}$.

LEMMA 3. *Let $\mathscr{P}$ be a non-degenerate polyhedron in m. Then for some N, the projected polyhedron $\Pi^n \mathscr{P}$ is non-degenerate for all $n > N$.*

Proof. The hypothesis implies that the vertices of each triangle of $\mathscr{P}$ are distinct and not on a line. It is clear that N may be taken large enough for $\Pi^n \mathscr{P}$ to have this property for all $n > N$.

LEMMA 4. *If $\mathscr{P}$ is a non-degenerate polyhedron with representation x, and if in the countable set of functions x^i there are only a finite number of distinct functions, then $\mathscr{P}$ has a representation x^* on the unit circle $\mathscr{C}$ such that*

$$\max_i\, \{i,\, x^*\} = \max_{i,\,k}\,[i,\, k,\, x^*] \qquad \text{a.e. } in\ \mathscr{C}.$$

DEFINITION 4. A D-mapping x is *quasi-conformal* in a Jordan region R if

$$\sup_i\, [(x_u^i)^2 + (x_v^i)^2] = \sup_{i,\,k}\, [x_u^i x_v^k - x_v^i x_u^k] \qquad \text{a.e. in } R\ .$$

THEOREM 9. *If x_n and x are quasi-conformal mappings on R with x_n converging uniformly to x and $L(x_n) \to L(x)$, then x is quasi-conformal.*

Proof. From $\|y_u\|^2 + \|y_v\|^2 \leqq 2 \sup_i\, \{i,\, y\}$ it follows that $D(x_n) \leqq 2L(x_n)$ and hence that $D(x_n) < M$ for some M. The closure theorem for A.C.T. functions assures us that x is a D-mapping and $D(x) \leqq M$. More exactly, we have

$$L(x) \leqq \iint_R \sup_i\, \{i,\, x\} \leqq \liminf_{n \to \infty} \iint_R \sup_i\, \{i,\, x_n\} = \liminf_{n \to \infty}\, L(x_n) = L(x)\ .$$

Hence $\sup_{i,k}[i, k, x]=\sup_i \{i, x\}$ a.e.

THEOREM 10. *An open non-degenerate surface $\mathscr{S}$ of finite Lebesgue area has a quasi-conformal representation on $\mathscr{C}$.*

Proof. There exists a sequence of polyhedral surfaces $\mathscr{P}_n^*$ approaching $\mathscr{S}$ with $L(\mathscr{P}_n^*)\to L(\mathscr{S})$, and we may suppose that each $\mathscr{P}_n^*$ is open non-degenerate.

Using the idea of [13, § 8] we can, for each n, determine a polyhedron $\mathscr{P}_n$ with the properties

(a) The Fréchet distance between $\mathscr{P}_n$ and $\mathscr{P}_n^*$ is less than $1/n$.

(b) $L(\mathscr{P}_n^*)\geqq L(\mathscr{P}_n)>L(\mathscr{P}_n^*)-1/n$.

(c) If x_n is a representation of $\mathscr{P}_n$ then there are only a finite number of distinct functions in the collection x_n^i.

(d) The $\mathscr{P}_n$ are open non-degenerate.

Hence the sequence $\mathscr{P}_n$ approaches $\mathscr{S}$ and $L(\mathscr{S})=\lim_{n\to\infty}(\mathscr{P}_n)$.

The remainder of the proof is the same as that for a surface in Euclidean space [4].

The idea referred to is the following. If y is a representation of a polyhedron $\mathscr{P}$ then the sequence y^i is uniformly bounded and equicontinuous, thus totally bounded. Hence for each $\varepsilon>0$ there exists a finite subset y^{i_j} of the y^i with the property that $\sup|y^i-y^{i_j}|<\varepsilon$ for each i and some i_j. If $\mathscr{P}$ is open non-degenerate and $\Pi^n\mathscr{P}$ is also, then adjoin y^k, $k=1, 2, \cdots, n$ to the y^{i_j}. Now replace those components of y which are not in the subset by one which is and is within ε of it. The resulting function represents an open non-degenerate polyhedron whose Fréchet distance from $\mathscr{P}$ does not exceed ε and whose area does not exceed that of $\mathscr{P}$.

6. Isometric surfaces in m. For later applications it is convenient to know that if x is quasi-conformal and y is isometric with x, then y is also quasi-conformal.

Let a, b, A and B be points of m.

LEMMA 5. *If $\|a\cos\theta+b\sin\theta\|=\|A\cos\theta+B\sin\theta\|$ for all θ then* $\sup_i [(a^i)^2+(b^i)^2]=\sup_i [(A^i)^2+(B^i)^2]$.

Proof. Suppose that for some p we have $(A^p)^2+(B^p)^2>0$. Then there exist real numbers $\lambda>0$ and θ such that $A^p=\lambda\cos\theta$ and $B^p=\lambda\sin\theta$. Thus

$$\begin{aligned}(A^p)^2+(B^p)^2&=\lambda^{-2}[(A^p)^2+(B^p)^2]^2\leqq\lambda^{-2}\sup[A^pA^i+B^pB^i]^2\\&=\sup[A^i\cos\theta+B^i\sin\theta]^2=\|A\cos\theta+B\sin\theta\|^2=\|a\cos\theta+b\sin\theta\|^2\\&=\sup|a^i\cos\theta+b^i\sin\theta|^2\leqq\sup[(a^i)^2+(b^i)^2]\ .\end{aligned}$$

Similarly

$$\sup[(a^i)^2+(b^i)^2]\leqq\sup[(A^i)^2+(B^i)^2]\ .$$

COROLLARY 1. *If* $\{\theta_j;\ j=1,2,\cdots\}$ *is dense in* $[0,2\pi]$ *and if* $\|a\cos\theta_j+b\sin\theta_j\|=\|A\cos\theta_j+B\sin\theta_j\|$ *for all* j, *then* $\sup_i[(a^i)^2+(b^i)^2]=\sup_i[(A^i)^2+(B^i)^2]$.

Fix θ and let $u=r\cos\theta-s\sin\theta$, $v=r\sin\theta+s\cos\theta$. Suppose that x is A.C.T. on G into m and define y by $y(r,s)=x(u,v)$. Since x^i is A.C.T. for each i, so is y^i. Furthermore, except for a set Z of measure 0, $y^i_r=x^i_u\cos\theta+x^i_v\sin\theta$ for all i. Thus for $s_0\notin Z$ we have

$$\begin{aligned}\text{length } y(r,s_0)&=\lim_{N\to\infty}\text{length } \Pi^N y(r,s_0)=\lim_{N\to\infty}\int D_r(\Pi^N y)\\&=\lim_{N\to\infty}\int\sup_{i\leqq N}|x^i_u\cos\theta+x^i_v\sin\theta|\leqq\int\|x_u\cos\theta+x_v\sin\theta\|\leqq\int\|x_u\|+\int\|x_v\|\end{aligned}$$

where the first integral is taken over the intersection of dom y with the line $s=s_0$ and the other integrals are taken over the intersection of G with the line $[-u\sin\theta+v\cos\theta]=s_0$. Thus

$$\int_{s_0}\text{length } y(r,s_0)\leqq\iint_G\|x_u\|+\iint_G\|x_v\|$$

and since r and s may be interchanged in this argument, we see that y is A.C.T.

The partials of y are, of course, directional derivatives of x. We can now apply Theorem 1 to obtain, almost everywhere in G,

$$x_\theta=\{x^i_u\cos\theta+x^i_v\sin\theta\}\ \text{ and }\ D_\theta x=\|x_u\cos\theta+x_v\sin\theta\|$$

where, if $\varphi(s)=x(u+s\cos\theta,\ v+s\sin\theta)$, then $x_\theta=\varphi'(0)$ and $D_\theta x=D\varphi(0)$ (see Definition 1).

Now let θ_j, $j=1,2,\cdots$, be dense in $[0,2\pi]$. Let W be that set of measure 0 in the complement of which $x_{\theta_j}=\{x^i_u\cos\theta_j+x^i_v\sin\theta_j\}$ and $D_{\theta_j}x=\|x_u\cos\theta_j+x_v\sin\theta_j\|$.

Observe that if x and y are isometric (dom $x=$ dom y and $\|x(p)-x(q)\|=\|y(p)-y(q)\|$ for all $p,q\in\operatorname{dom}x$) then $D_{\theta_j}x=D_{\theta_j}y$ wherever either side exists.

THEOREM 11. *If* x *is quasi-conformal and* y *is isometric with* x, *then* y *is quasi-conformal.*

Proof. That y is a D-mapping follows directly from the definitions. By the preceding remarks and Corollary 1 we have $\sup\{i, x\} = \sup\{i, y\}$ almost everywhere. In [13] it was shown that $L(x) = L(y)$. Hence

$$L(y) = \iint \sup[i, k, y] \leqq \iint \sup\{i, y\} = \iint \sup\{i, x\} = \iint \sup[i, k, x] = L(x)$$

from which we can conclude that $\sup[i, k, y] = \sup\{i, y\}$ almost everywhere.

7. Almost conformal representations for surfaces in a metric space.

If a surface is in a metric space, then there exists an isometric surface in m. The definition of 'almost-conformal' is phrased so as to be invariant under isometries. Hence the result of the last section can be applied to surfaces in metric spaces.

DEFINITION 5. Let X be continuous on a Jordan region R into a metric space D. Then X is *almost-conformal* if there exists a quasi-conformal map x on R into m which is isometric with X.

We can now repeat some familiar reasoning of [13] to obtain the following.

THEOREM 12. *An open non-degenerate surface in a metric space has an almost-conformal representation upon the unit circle.*

Proof. Let X be a representation on Q of an open non-degenerate surface $\mathscr{T}$. If p_i, $i = 1, 2, \cdots$, is dense in range X then X is isometric with $x = \{X^i\}$, where $X^i(q) = \delta(p_i, X(q))$ for all $q \in Q$. By Theorem 10 there is a quasi-conformal map y on the unit circle $\mathscr{C}$ which is Fréchet equivalent to x. Define Y on $\mathscr{C}$ into D by $Y(s) = X(r)$ where $x(r) = y(s)$. If $x(r) = y(s)$ and $x(r') = y(s)$ then $X(r) = X(r')$, so Y is well defined. The map Y is a representation of $\mathscr{T}$ upon $\mathscr{C}$ which is isometric to a quasi-conformal map y. Hence Y is almost-conformal.

Let $\mathscr{S}$ be a surface in D and suppose $\mathscr{S}$ has an almost-conformal representation X on a Jordan region R. Then X is a D-mapping and $L(\mathscr{S}) = \iint_R \sup_{i,k} [i, k, X]$ where X^j is defined as in the proof of Theorem 12.

Finally we observe that if X is a D-mapping then X is almost-conformal if $\sup\{i, X\} = \sup[i, k, X]$, and conversely. The direct statement is an immediate consequence of the definition. For the converse note that if $x = \{X^i\}$ then x is isometric with X and is quasi-conformal.

8. Surfaces in a Banach space. If a D-mapping has range in a Banach space B then it is possible to give a definition of quasi-conformality which is analogous to that for the case $B = m$. Then we shall see that the notions of quasi-conformal and almost-conformal are equivalent and, in case $B=E_n$, they are both equivalent to the original definition of Morrey.

Let X be defined on a Jordan region R into B. There exists a smallest (separable) subspace $B(X)\subset B$ which contains range X. A sequence $\{f_n\}$ of linear functionals of norm one over B is *admissible* with respect to X if $\sup f_i(r)=\|r\|$ for each $r\in B(X)$. The transformation T: $B(X)\to m$ defined by $T(r)=\{f_i(r)\}$ is an isometry. It was shown in [13] that such an admissible sequence always exists.

Let $\{f_i, X\}=\{i, TX\}$ and $[f_i, f_k, X]=[i, k, TX]$.

DEFINITION 6. In the notation of the preceding paragraphs, X is *quasi-conformal* if X is a D-mapping and if $\sup\{f_i, X\}=\sup[f_i, f_k, X]$ almost everywhere in R.

Theorem 11 assures us that this definition is equivalent to that given earlier for the case $B=m$.

THEOREM 13. *A necessary and sufficient condition that X be quasi-conformal is that X be almost-conformal.*

Proof. The function TX is isometric with X. If X is quasi-conformal then

$$\sup\{i, TX\}=\sup\{f_i, X\}=\sup[f_i, f_k, X]=\sup[i, k, TX]\ .$$

Thus TX is quasi-conformal in m and X is almost-conformal. If X is almost-conformal there exists a quasi-conformal function y which is isometric with X and, therefore, with TX. (The function y has the same domain as X and has range in m.) Thus TX is also quasi-conformal and

$$\sup\{f_i, X\}=\sup\{i, TX\}=\sup[i, k, TX]=\sup[f_i, f_k, X]\ .$$

Hence X is quasi-conformal.

Now suppose that B is E_n. If f is a linear functional of norm one then there exists a point p with $\|p\|=1$ such that $f(r)=p\cdot r$ for each $r\in E_n$. Since $\{f_i\}$ is admissible, $\sup p_i\cdot r=\|r\|$. If r and s are two points in E_n with $\|r\|=\|s\|$ and $r\cdot s=0$, then $(r\cdot p)^2+(s\cdot p)^2\leqq r\cdot r$ for any p with $\|p\|=1$.

If X is quasi-conformal in the sense of Morrey (almost-conformal [4]) then X is a D-mapping and $E=G$, $F=0$ almost everywhere ($E=X_u\cdot X_u$, $F=X_u\cdot X_v$, $G=X_v\cdot X_v$). Where these equations hold, $(X_u\cdot p)^2+(X_v\cdot p)^2\leqq E$ for any p on the unit sphere. Hence $\sup\{f_i, X\}\leqq E=$area of the square determined by X_u and $X_v=\sup[f_i, f_k, X]\leqq\sup\{f_i, X\}$. Thus X is quasi-conformal in the sense of this paper.

Now let X be quasi-conformal in the sense of this paper. Since E_n has the property that an absolutely continuous function on an interval into E_n does have a derivative almost everywhere, we can conclude that X_u and X_v exist almost everywhere (not only component-wise derivatives). If $\sup \{f_i, X\}=0$, then $E=F=G=0$. If X_u and X_v both exist and $\sup \{f_i, X\}>0$, it is easy to see that

$$\sup [f_i, f_k, X] = \max_{|a|=|b|=1} [(a\cdot X_u)(b\cdot X_v)-(a\cdot X_v)(b\cdot X_u)] = \sqrt{EG-F^2},$$

$$\sup \{f_i, X\} = \max_{|a|=1} [(a\cdot X_u)^2+(a\cdot X_v)^2]$$

$$=\left(\frac{E+G}{2}\right)+\sqrt{\left(\frac{E+G}{2}\right)^2-(EG-F^2)};$$

clearly these are equal only if $E=G$, $F=0$. We conclude that the definitions of almost-conformal and quasi-conformal as given in this paper are equivalent to the original definition of Morrey.

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CERTAIN GENERALIZED HYPERGEOMETRIC IDENTITIES OF THE ROGERS-RAMANUJAN TYPE (II)

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1. Introduction. Nearly two years ago, Alder [1] established the following generalizations of the well-known Rogers-Ramanujan identities:

$$\prod_{n=1}^{\infty}\frac{(1-x^{(2M+1)n-M})(1-x^{(2M+1)n-(M+1)})(1-x^{(2M+1)n})}{(1-x^n)}=\sum_{t=0}^{\infty}\frac{G_{M,t}(x)}{(x)_t}, \tag{1}$$

$$\prod_{n=1}^{\infty}\frac{(1-x^{(2M+1)n-1})(1-x^{(2M+1)n-2M})(1-x^{(2M+1)n})}{(1-x^n)}=\sum_{t=0}^{\infty}x^t\frac{G_{M,t}(x)}{(x)_t}, \tag{2}$$

where $G_{M,t}(x)$ are polynomials which reduce to x^{t^2} for $M=2$ and

$$(x)_t=(1-x)(1-x^2)\cdots(1-x^t)\,, \qquad (x)_0=1\,.$$

In a recent paper [6] I gave a simple alternative proof of (1) and (2). We used the result

$$\begin{aligned}1+\sum_{s=1}^{\infty}(-1)^s k^{Ms}x^{\frac{1}{2}s\{(2M+1)s-1\}}(1-kx^{2s})\frac{(kx)_{s-1}}{(x)_s}\\ =\prod_{n=1}^{\infty}(1-kx^n)\sum_{t=0}^{\infty}\frac{k^tG_{M,t}(x)}{(x)_t}\,, \qquad M=2,3,\cdots\end{aligned} \tag{3}$$

Alder in his paper states that identities involving the generating function for the number of partitions into parts not congruent to 0, $\pm(M-r)(\bmod 2M+1)$, where $0\leqq r\leqq M-1$, can be obtained by his method and indicates the result for $r=1$.

In the present paper I give a simple method of obtaining the M identities for each modulus $(2M+1)$. In § 4 identities for which $r\geqq\frac{1}{2}M$ have been deduced and in § 5 those for which $r\leqq\frac{1}{2}M$ have been obtained for any r such that $0\leqq r\leqq M-1$. The identities given in § 5 have not been mentioned by Alder. As a corollary, an interesting identity between two infinite series is given.

2. Notations. Assuming $|x|<1$, let

$$(\alpha)_n\equiv(\alpha)_{x,n}=(1-\alpha)(1-\alpha x)\cdots(1-\alpha x^{n-1})\,, \qquad (\alpha)_0=1\,,$$

$$(\alpha)_{-n}=(-1)^n x^{\frac{1}{2}n(n+1)}/\alpha^n(x/\alpha)_n\,,$$

$$x_n=1+x+x^2+\cdots+x^{n-1}\,.$$

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$$P_{m,t}(x) \equiv x^{\frac{1}{2}(t+1-m)t} \frac{(x)_m}{(x)_t (x)_{m-t+1}} (1 - x^{m-2t+1}) , \tag{4}$$

and let

$$\phi(M, x^r) = 1 + \sum_{s=1}^{\infty} (-1)^s x^{Mrs} x^{\frac{1}{2}s\{(2M+1)s-1\}} (1 - x^{2s+r})(x^{s+1})_{r-1} , \tag{5}$$

so that (3) can be written as

$$\phi(M, x^r) = \prod_{n=1}^{\infty} (1 - x^n) \sum_{t=0}^{\infty} \frac{x^{rt} G_{M,t}(x)}{(x)_t} . \tag{6}$$

3. The polynominals $u_n(x)$. Before proceeding to deduce the generalized identities, we first give a few properties of a sequence of polynomials with the help of an operator. These we will need in later sections. Let us define a sequence $\{u_n(x)\}$ of polynomials by the relations

(i) $\qquad u_0(x) = 0$

(ii) $\qquad u_n(x) = u_{n-1}(x) + x^{n-1} x_n , \qquad n \geqq 1 .$

Let $\mathscr{R}$ be an operator which replaces x_m by $u_m(x)$ in any $u_n(x)$, that is,

$$\mathscr{R} u_n(x) = \mathscr{R} u_{n-1}(x) + x^{n-1} u_n(x) .$$

Also

$$\mathscr{R}^n u_m(x) = \mathscr{R}^{n-1} \{\mathscr{R} u_m(x)\} .$$

Then we have

$$(\alpha)_n = 1 - \alpha x_n + \sum_{s=2}^{n} (-\alpha)^s x^{\frac{1}{2}s(s-1)} \mathscr{R}^{s-2} u_{n-s+1}(x) . \tag{7}$$

As can be easily shown

$$\mathscr{R}^{s-2} u_{n-s+1}(x) = \frac{(x)_n}{(x)_s (x)_{n-s}} . \tag{8}$$

The above polynomials (8) have also recently occurred in a paper by Carlitz [3].

Comparing the coefficients of α^{s-1} in

$$(\alpha)_n = (-1)^n \alpha^n x^{\frac{1}{2}n(n-1)} (1/\alpha x^{n-1})_n , \tag{9}$$

we get the relation

$$\mathscr{R}^{s-3} u_{n-s+2}(x) = \mathscr{R}^{n-s-1} u_s(x) , \qquad s = 1, \cdots, (n+1) . \tag{10}$$

We can thus write (7) as

$$(\alpha)_n=\sum_{s=0}^{n}(-\alpha)^s x^{\frac{1}{2}s(s-1)}\mathscr{R}^{s-2}u_{n-s+1}(x)\ , \tag{11}$$

where negative indices of $\mathscr{R}$ are defined by (10). Again comparing the coefficients of α^s in

$$(\alpha/x^{n-1})_{x,n}=(\alpha)_{1/x,n}\ ,$$

we get with the help of (11),

$$\mathscr{R}^{s-2}u_{n-s+1}(x)=x^{s(n-s)}\mathscr{R}^{s-2}u_{n-s+1}(x^{-1})\ . \tag{12}$$

In particular

$$u_n(x)=x^{2n-2}u_n(x^{-1})\ . \tag{13}$$

The following values of $\mathscr{R}^m\ u_n(x)$ will also be required:

$$\mathscr{R}^{n-3}u_2(x)=x_n\ , \qquad \text{from (10)}$$

$$u_1(x)=1$$
$$u_3(x)=1+x+2x^2+x^3+x^4$$
$$u_4(x)=1+x+2x^2+2x^3+2x^4+x^5+x^6=\mathscr{R}\,u_3(x)\ .$$

4. Now we proceed to deduce identities involving the generating function for the number of partitions into parts not congruent to 0, $\pm(M-r)(\operatorname{mod} 2M+1)$. From (11), we have

$$\begin{aligned}(x^{n-r+1})_{2r-1}&(1-x^{2n})\\ &=\left[1+\sum_{s=1}^{2r-2}\{(-x^{n-r+1})^s x^{\frac{1}{2}s(s-1)}\mathscr{R}^{s-2}u_{2r-s}(x)\}-x^{(2r-1)n}\right](1-x^{2n})\ ,\end{aligned}$$

whence

$$\begin{aligned}1+x^{(2r+1)n}&\\ &=\left[x^{2n}+x^{(2r-1)n}-\left\{\sum_{s=1}^{2r-2}x^{ns}(-1)^s x^{\frac{1}{2}s(s+1)-rs}\mathscr{R}^{s-2}u_{2r-s}(x)\right\}(1-x^{2n})\right]\\ &\qquad\qquad+(1-x^{2n})(x^{n-r+1})_{2r-1}\ .\end{aligned} \tag{14}$$

And since, because of (8) or (10), the terms equidistant from the two ends in the sum on the right of (14) have equal coefficients of powers of x^n, the expression in square brackets can be written as

$$\sum_{t=1}^{r}(-1)^{t-1}x^{tn}\{1+x^{(2r-2t+1)n}\}U_{r,t}(x) \tag{15}$$

where

$$\begin{aligned}U_{r,t}(x)&=x^{\frac{1}{2}t(t+1)-tr}\mathscr{R}^{t-2}u_{2r-t}(x)-x^{\frac{1}{2}(t-2)(t-1)-(t-2)r}\mathscr{R}^{t-4}u_{2r-t+2}(x)\\ &=P_{2r,t}(x)\ , \qquad \text{using (4) and (8).}\end{aligned} \tag{16}$$

The polynomials $U_{r,t}(x)$ may be called "reciprocal" since they are such that the terms equidistant from the two ends have equal coefficients. Taking $n=0$ in (14) we see that

$$\sum_{t=1}^{r}(-1)^{t-1}U_{r,t}(x)=1 . \tag{17}$$

Also, with the help of (12), we have

$$U_{r,t}(x)=U_{r,t}(x^{-1}) . \tag{18}$$

Now from (15)

$$\begin{aligned}\sum_{n=-\infty}^{\infty} &(-1)^n x^{\frac{1}{2}n\{(2M+1)n+2r+1\}} \\ &=1+\sum_{n=1}^{\infty}\frac{(-1)^n x^{\frac{1}{2}n\{(2M+1)n-1\}}}{x^{rn}}\{1+x^{(2r+1)n}\} \\ &=1+\sum_{t=1}^{r}(-1)^{t-1}U_{r,t}(x)\sum_{n=1}^{\infty}\frac{(-1)^n x^{\frac{1}{2}n\{(2M+1)n-1\}}}{x^{(r-t)n}}\{1+x^{(2r-2t+1)n}\} \\ &\quad+\sum_{n=1}^{\infty}\frac{(-1)^n x^{\frac{1}{2}n\{(2M+1)n-1\}}}{x^{rn}}(1-x^{2n})(x^{n-r+1})_{2r-1} .\end{aligned} \tag{19}$$

For $n=s+r$, the last series on the right-hand side of (19) becomes

$$(-1)^r x^{Mr^2-\frac{1}{2}r(r+1)}\phi(M, x^{2r})$$

since the first $(r-1)$ terms of the series vanish because of the factor $(x^{n-r+1})_{2r-1}$. Then using (17) and writing

$$F(M, r)=\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{1}{2}n\{(2M+1)n+2r+1\}}$$

we obtain for (19) the form

$$F(M, r)=\sum_{t=1}^{r}(-1)^{t-1}U_{r,t}(x)F(M, r-t)+(-1)^r x^{Mr^2-\frac{1}{2}r(r+1)}\phi(M, x^{2r}) . \tag{20}$$

Thus, using Jacobi's classical identity

$$\sum_{n=-\infty}^{\infty}(-1)^n x^{n^2}z^n=\prod_{n=1}^{\infty}(1-x^{2n-1}z)(1-x^{2n-1}/z)(1-x^{2n}) \tag{21}$$

to express the infinite series in (20) as infinite products, we could find for any given r, such that $0\leqq r\leqq M-1$, an expression for the generating function for the number of partitions into parts not congruent to $0, \pm(M-r)(\bmod 2M+1)$ in terms of generating functions for the number of partitions into parts not congruent to $0, \pm(M-s)(\bmod 2M+1)$,

$(s=0, 1, 2, \cdots, r-1)$. Since $F(M, 0)=\phi(M, 1)$, the F-series can be successively expressed in terms of ϕ-series and, with the help of (6), we get

THEOREM 1.

$$\prod_{n=1}^{\infty}\frac{(1-x^{(2M+1)n-(M-r)})(1-x^{(2M+1)n-(M+r+1)})(1-x^{(2M+1)n})}{(1-x^n)}=\sum_{t=0}^{\infty}\frac{A_r(x,t)G_{M,t}(x)}{(x)_t} \tag{22}$$

where

$$A_r(x, t)=\sum_{s=0}^{r}(-1)^s x^{Ms^2-\frac{1}{2}s(s+1)+2st}U'_{r,s+1}(x)\ .$$

The polynomials $U'(x)$ are of the "reciprocal" kind, with

$$U'_{r,r+1}(x)=1$$

$$U'_{r,s+1}(x)=\sum_{m=1}^{r-s}(-1)^{m-1}U_{r,m}(x)U'_{r-m,s+1}(x)\ , \qquad s\neq r\ .$$

so that

$$U'_{r,1}(x)=1\ , \qquad \textit{because of } (17)$$

and

$$U'_{r,m}(x)=U'_{r,m}(x^{-1})\ , \qquad \textit{because of } (18).$$

As an example of Theorem 1, taking the case $r=1$, we have

$$1+x^{3n}=x^n(1+x^n)+(1-x^n)(1-x^{2n})\ .$$

Therefore

$$\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{1}{2}n\{(2M+1)n+3\}}=\phi(M, 1)-x^{M-1}\phi(M, x^2) \tag{23}$$

which is equivalent to equation (23) of Alder [1].

From (23), using (21) and (6), we get the identity

$$\prod_{n=1}^{\infty}\frac{(1-x^{(2M+1)n-(M-1)})(1-x^{(2M+1)n-(M+2)})(1-x^{(2M+1)n})}{(1-x^n)} =\sum_{t=0}^{\infty}\frac{(1-x^{M+2t-1})}{(x)_t}G_{M,t}(x)\ . \tag{24}$$

For $r=2$,

$$\prod_{n=1}^{\infty}\frac{(1-x^{(2M+1)n-(M-2)})(1-x^{(2M+1)n-(M+3)})(1-x^{(2M+1)n})}{(1-x^n)} =\sum_{t=1}^{\infty}\frac{\{1-U'_{2,2}(x)x^{M+2t-1}+x^{4M+4t-3}\}}{(x)_t}G_{M,t}(x)\ , \tag{25}$$

where

$$U'_{2,2}(x)=x^{-1}+1+x\ .$$

Similarly for $r=3$ we get

$$U'_{3,2}(x)=x^{-2}+x^{-1}+2+x+x^2$$
$$U'_{3,3}(x)=x^{-2}+x^{-1}+1+x+x^2\ ,$$

and so on for any r such that $0\leqq r\leqq M-1$.

5. In this section identities involving the generating function for the number of partitions into parts not congruent to 0, $\pm r\ (\mathrm{mod}\ 2M+1)$ are obtained.

From (11) we have

$$(x^{n-r+2})_{2r-2}(1-x^{2n+1})$$
$$=\left[1+\left\{\sum_{s=1}^{2r-3}(-x^{n-r+2})^s x^{\frac{1}{2}s(s-1)}\mathscr{R}^{s-2}u_{2r-s-1}(x)\right\}+x^{(2n+1)(r-1)}\right](1-x^{2n+1})\ ,$$

whence

$$1-x^{(2n+1)r}$$
$$=\left[x^{2n+1}-x^{(2n+1)(r-1)}-\left\{\sum_{s=1}^{2r-3}x^{ns}(-1)^s x^{\frac{1}{2}s(s+3)-rs}\mathscr{R}^{s-2}u_{2r-s-1}(x)\right\}(1-x^{2n+1})\right]$$
$$+(1-x^{2n+1})(x^{n-r+2})_{2r-2}\ . \tag{26}$$

In the expression in square brackets in (26), the terms containing x^{nr} cancel and the other terms can again be grouped in pairs to give

$$1-x^{(2n+1)r}=\sum_{t=1}^{r-1}(-1)^{t-1}V_{r,t}(x)x^{tn}\{1-x^{(2n+1)(r-t)}\}+(1-x^{2n+1})(x^{n-r+2})_{2r-2}\ , \tag{27}$$

where

$$V_{r,t}(x)=x^{\frac{1}{2}t(t+3)-rt}\mathscr{R}^{t-2}u_{2r-t-1}(x)-x^{\frac{1}{2}t(t-1)-r(t-2)}\mathscr{R}^{t-4}u_{2r-t+1}(x)$$
$$=x^{\frac{1}{2}t}P_{2r-1,t}(x)\ . \tag{28}$$

The polynomials $V(x)$ are less symmetric than $U(x)$. In particular, corresponding to (17) and (18), they satisfy the relations

$$\sum_{t=1}^{r-1}(-1)^{t-1}V_{r,t}(x)x_{r-t}=x_r\ , \tag{29}$$

and

$$V_{r,t}(x)=x^t V_{r,t}(x^{-1})\ . \tag{30}$$

Now

$$\sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n\{(2M+1)(n+1)\}-rn}$$
$$=\sum_{n=0}^{\infty} \frac{(-1)^n x^{\frac{1}{2}n^2(2M+1)+(M-\frac{1}{2})n}}{x^{(r-1)n}} \{1-x^{(2n+1)r}\} .$$

Denoting the left-hand side of the last equation by $\psi(M, r)$ and using (27) and (5), we get, after slight simplification,

$$\psi(M, r)=\sum_{t=1}^{r-1}(-1)^{t-1}\psi(M, r-t)V_{r,t}(x)+(-1)^{r-1}x^{\frac{1}{2}(2M-1)r(r-1)}\phi(M, x^{2r-1}) . \tag{31}$$

Using (21), the generating function for the number of partitions into parts not congruent to 0, $\pm r$ (mod $2M+1$) can now be expressed in terms of the generating function for the number of partitions into parts not congruent to 0, $\pm s$ (mod $2M+1$), $(s=1, 2, \cdots, r-1)$. Thus, we finally have

THEOREM 2.

$$\prod_{n=1}^{\infty} \frac{(1-x^{(2M+1)n-r})(1-x^{(2M+1)n-(2M+1-r)})(1-x^{(2M+1)n})}{(1-x^n)} = \sum_{t=0}^{\infty} \frac{B_r(x, t)G_{M,t}(x)}{(x)_t} , \tag{32}$$

where

$$B_r(x, t)=\sum_{s=1}^{r} (-1)^{s-1}x^{\frac{1}{2}(2M-1)s(s-1)+(2s-1)t} V'_{r,s}(x) ,$$

and $V'_{r,s}(x)$ *are polynomials with*

$$V'_{r,r}(x)=1$$
$$V'_{r,s}(x)=\sum_{m=1}^{r-s}(-1)^{m-1}V_{r,m}(x)V'_{r-m,s}(x) , \qquad s\neq r ,$$

so that

$$V'_{r,1}(x)=x_r$$

and

$$V'_{r,t}(x)=x^{r-t}V'_{r,t}(x^{-1}) .$$

As an illustration, for $r=2$ in Theorem 2, we have

$$1-x^{4n+2}=x^n(1+x)(1-x^{2n+1})+(1-x^{2n+1})(1-x^n)(1-x^{n+1}) .$$

Therefore

$$\sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n^2(2M+1)+(M-\frac{3}{2})n}=(1+x)\phi(M, x)-x^{2M-1}\phi(M, x^3) ,$$

which gives us the identity

$$\prod_{n=1}^{\infty} \frac{(1-x^{(2M+1)n-2})(1-x^{(2M+1)n-(2M-1)})(1-x^{(2M+1)n})}{(1-x^n)} \tag{33}$$
$$= \sum_{t=0}^{\infty} \frac{\{(1+x)x^t - x^{2M+3t-1}\}}{(x)_t} G_{M,t}(x) \ .$$

COROLLARY. *If r is replaced by $M-r$ in Theorem 2 then the left-hand sides of* (22) *and* (32) *become the same and we have*

$$\sum_{t=0}^{\infty} \frac{A_r(x, t) G_{M,t}(x)}{(x)_t} = \sum_{t=0}^{\infty} \frac{B_{M-r}(x, t) G_{M,t}(x)}{(x)_t} \ , \tag{34}$$
$$r=0, 1, \cdots, M-1 \ , \ M=2, 3, \cdots .$$

For $M=2$ and $r=0$ and 1 we get respectively the relations

$$\sum_{t=0}^{\infty} \frac{x^{t^2}}{(x)_t} = \sum_{t=0}^{\infty} \frac{\{(1+x)x^t - x^{3(t+1)}\}}{(x)_t} x^{t^2}$$

$$\sum_{t=0}^{\infty} \frac{(1-x^{2t+1})}{(x)_t} x^{t^2} = \sum_{t=0}^{\infty} \frac{x^{t+t^2}}{(x)_t}$$

the truth of which can easily be verified.

Some time ago, Slater ([4] and [5]) gave a very large number of identities of the Rogers-Ramanujan type using Bailey's summation theorem [2] for a well-poised ${}_6\Psi_6$. It is interesting to note that, as special cases of our identities, we get some of those given by Slater, differing only in form as can be easily verified. To mention an example, let us take equation (90) of Slater [5]:

$$\prod_{n=1}^{\infty} \frac{(1-x^{27n-3})(1-x^{27n-24})(1-x^{27n})}{(1-x^n)} = \sum_{t=0}^{\infty} \frac{(x^3)_{x^3, t} x^{t(t+3)}}{(x)_t (x)_{2t+2}} \ , \tag{35}$$

If we put $M=13$, $r=3$ in Theorem 2, we obtain another series for the product on the left of (35). I propose to study the equivalence of identities (22) and (32) above and those of Slater in a subsequent paper, as also identities involving products in which the powers increase by $2M$.

I would like to express my gratitude to Dr. R. P. Agarwal for suggesting the present work and for his kind help in the preparation of this paper.

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Corrigenda. In [6] the following corrections may be noted:

p. 1011. The series for $T_{n,m}$ runs up to

$$t_n = \left[\frac{M-n-1}{M-n} t_{n-1}\right].$$

p. 1012. In the line immediately preceding (3.3), a_{2Mn-1} should be a_{2M+1}.

In the right hand side of (3.4) a factor $(kn;t)$ should be inserted in the denominator of the outer series.

p. 1014. In the right hand side of the last identity of the paper, we should have Π instead of π.

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A DETERMINANT IN CONTINUOUS RINGS

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1. Introduction. In the theory, developed by Dieudonné [1], of determinants of nonsingular square matrices over a noncommutative field K the determinantal values are cosets modulo the commutator subgroup of $K^{\times}$, the multiplicative group of K. Since the matrix groups $M_n^{\times}(K)$ and their commutator subgroups C_n have the property that $M_n^{\times}(K)/C_n$ is independent of n, the latter cosets will serve just as well for determinantal values, at least for theorems involving only the multiplication of determinants.

The rings whose principal right ideal lattices form continuous geometries have many resemblances to matrix rings; in fact, the axioms of Continuous Geometry are satisfied by finite dimensional geometries over a field which are always equivalent to the right ideal lattice of some matrix ring. Irrespective of questions as to the existence or otherwise of fields in connection with a general continuous geometry playing a similar role to that of the field of coordinate values in the finite dimensional case we will show that multiplicative determinantal theorems can be obtained for the more general ring; the determinants will be cosets of the group of invertible ring elements modulo the closure of its commutator subgroup with respect to the rank-distance topology in the ring.

The definition of a complete rank ring is given by von Neumann [3, (iv)]. Essential properties of such a ring $\mathfrak{R}$ and the associated lattice of principal right ideals have been developed by von Neumann [3, 4] and Ehrlich [2]. We will assume throughout that $\mathfrak{R}$ is a complete rank ring, of characteristic not 2; and that if the discrete case (matrices over a field) applies, then the order of the matrices is at least 3.

2. Groups in a complete rank ring. Using a notation similar to that of [2], [3] we denote by $\mathfrak{G}$ the group of invertible ring elements; that is, $u \in \mathfrak{G} \subset \mathfrak{R}$ if and only if the rank $R(u)$ of u is 1.

DEFINITION 1. We denote by $\mathfrak{K}$ the closure of the commutator subgroup of $\mathfrak{G}$ in the rank-distance topology and by $\mathfrak{K}^{\dagger}$ the closure of the group generated by the elements of class 2 in $\mathfrak{G}$.

COROLLARY 1. *$\mathfrak{K}$ and $\mathfrak{K}^{\dagger}$ are groups.*

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Proof. Let $\{t_n;\ t_n \in \mathfrak{S},\ n=1, 2, \cdots\}$ be a converging sequence in $\mathfrak{R}$. Then $\lim_{n,m\to\infty} R(t_n-t_m)=0$ implies

$$\lim_{n,m\to\infty} R(t_n^{-1}-t_m^{-1})=\lim_{n,m\to\infty} R\{t_n^{-1}(t_m-t_n)t_m^{-1}\}=0$$

and hence $\lim_{n\to\infty} t_n^{-1}$ exists in $\mathfrak{R}$. By the continuity of multiplication $(\lim_{n\to\infty} t_n)(\lim_{n\to\infty} t_n^{-1})=1$ so that $\lim_{n\to\infty} t_n^{-1} \in \mathfrak{S}$. The result then follows routinely after the observation that the inverse of a commutator is a commutator and the inverse of the general class 2 element $1+r$ $(r^2=0)$ is $1-r$, also of class 2.

LEMMA 1. *Let $t \in C^2$ (be of class 2), $s \in \mathfrak{S}$. Then $sts^{-1} \in C^2$.*

COROLLARY 2. *Let $t \in C^2$, $s \in \mathfrak{S}$. Then $st=t_1s$ for some $t_1 \in C^2$.*

DEFINITION 2. We writh $u \cong s$ for nonsingular (invertible) u, $s \in \mathfrak{R}$ when $u=ts$ for some $t \in \mathfrak{K}^{\dagger}$.

COROLLARY 3. *The relation $\cong$ is an equivalence relation.*

LEMMA 2. *Let e be any idempotent of rank $1/2$ and s be nonsingular and otherwise arbitrary in $\mathfrak{R}$. Then for some $t \in \mathfrak{R}$*

$$s \cong e+(1-e)t(1-e)\,.$$

Proof. The existence of idempotents of rank $1/2$ is assumed in continuous rings, that is, when the range of R is the unit interval. In the discrete case the result has no meaning if the order of the matrices is odd.

Now suppose the principal left ideal $((1-e)se)_l=(g_1)_l$ where $g_1=eg_1e$, $g_1^2=g_1$ [4, Chapter 15]. By the Pierce decomposition, s is the sum of the quantities in the blocks of

$$\left[\begin{array}{cc|c} g_1sg_1 & g_1s(e-g_1) & \\ (e-g_1)sg_1 & (e-g_1)s(e-g_1) & es(1-e) \\ (1-e)sg_1 & (1-e)s(e-g_1) & (1-e)s(1-e) \end{array}\right]$$

where a matrix notation is used for clarity and to permit the comparison of later processes with standard matrix ones; we will simply equate such a partitioned array to the sum of its members. We have

$$g_1=y_1(1-e)se=y_1(1-e)seg_1=y_1(1-e)sg_1$$

for some $y_1 \in \mathfrak{R}$ so that

$$\{1+g_1(g_1-g_1sg_1)y_1(1-e)\}s$$

$$=\begin{bmatrix} g_1 & g_1s(e-g_1) & g_1s^*(1-e) \\ (e-g_1)sg_1 & (e-g_1)s(e-g_1) & (e-g_1)s(1-e) \\ (1-e)sg_1 & 0 & (1-e)s(1-e) \end{bmatrix}$$

for some $s^* \in \mathfrak{R}$ since

$$g_1sg_1+(g_1-g_1sg_1)y_1(1-e)sg_1=g_1sg_1+g_1-g_1sg_1=g_1$$

and

$$(1-e)s(e-g_1)=(1-e)se-(1-e)sg_1=(1-e)seg_1-(1-e)sg_1=0 .$$

Multiplying on the left by $(1-(1-e)sg_1)(1-(e-g_1)sg_1)$ and on the right by $(1-g_1s(e-g_1))(1-g_1s^*(1-e))$ gives

$$t_1s=\begin{bmatrix} g_1 & 0 & 0 \\ 0 & (e-g_1)s_1(e-g_1) & (e-g_1)s_1(1-e) \\ 0 & (1-e)s_1(e-g_1) & (1-e)s_1(1-e) \end{bmatrix}=s_1$$

for some $s_1 \in \mathfrak{R}$ and some $t_1 \in \mathfrak{K}^\dagger$ by Corollary 2.

Define g_{n+1}, s_{n+1}, t_{n+1} for $n=1, 2, \cdots$ as follows.

Let $((1-e)s_n(e-g_1-\cdots-g_n))_l=(g_{n+1})_l$ where $g_{n+1}^2=g_{n+1}$ and $(e-g_1-\cdots-g_n)g_{n+1}(e-g_1-\cdots-g_n)=g_{n+1}$. We have, similarly to the above, the existence of a $t_{n+1} \in \mathfrak{K}^\dagger$ and an $s_{n+1} \in \mathfrak{R}$ such that

$$t_{n+1}s=\begin{bmatrix} g_1 & & & & \\ & \ddots & & 0 & \\ & & g_n & & \\ & & & g_{n+1} & \\ & 0 & & & \begin{matrix} (e-g_1-\cdots-g_{n+1})s_{n+1}(e-g_1-\cdots-g_{n+1}) & (e-g_1-\cdots-g_{n+1})s_{n+1}(1-e) \\ (1-e)s_{n+1}(e-g_1-\cdots-g_{n+1}) & (1-e)s_{n+1}(1-e) \end{matrix} \end{bmatrix}=s_{n+1} .$$

Now,

$$\frac{1}{2} \geqq R(g_1+\cdots+g_n)=R(g_1)+\cdots+R(g_n)=\sum_{i=1}^{n} R((1-e)s_i(e-g_1-\cdots-g_i))$$

so $\lim_{i\to\infty} R((1-e)s_i(e-g_1-\cdots-g_i))=0$ and in turn

$$(1) \qquad \lim_{i\to\infty}(1-e)s_i(e-g_1-\cdots-g_i)=0$$

More strongly,

$$\lim_{n,p\to\infty} R(g_{n+1}+\cdots+g_{n+p})=\lim_{n,p\to\infty}\{R(g_{n+1})+\cdots+R(g_{n+p})\}=0 .$$

Hence, by [3, (iv), Section 3] $\lim_{n\to\infty}(g_1+\cdots+g_n)=g$, say, exists in $\mathfrak{R}$; also, by the continuity of multiplication, $g=ege$ and g is idempotent, being the limit of a sequence of idempotents.

In order to prove that $\lim_{n\to\infty} t_n$ exists in $\mathfrak{R}$ and so belongs to $\mathfrak{K}^\dagger$ we note that

$$(2) \qquad \begin{aligned}&(1-(1-e)s_n g_{n+1})(1-(e-g_1-\cdots-g_{n+1})s_n g_{n+1})\\ &\cdot(1+g_{n+1}(g_{n+1}-g_{n+1}s_n g_{n+1})y_{n+1}(1-e))t_n s\\ &\cdot(1-g_{n+1}s_n(e-g_1-\cdots-g_{n+1}))(1-g_{n+1}s_n^*(1-e))=t_{n+1}s\end{aligned}$$

where $s_n^*\in\mathfrak{R}$ and y_{n+1} is defined by the condition $g_{n+1}=y_{n+1}(1-e)s_n e$ The last two factors on the left side of (2) may be transferred after a similarity transformation to the left of $t_n s$, by Corollary 2, giving

$$(1+\Phi(g_{n+1}))t_n s=t_{n+1}s$$

where $\Phi(g_{n+1})$ is an expression involving no more than $2^5-1=31$ terms, each containing g_{n+1} as a factor and so of rank $\leqq R(g_{n+1})$. Hence $t_{n+1}-t_n=\Phi(g_{n+1})t_n$ and

$$R(t_{n+1}-t_n)\leqq R\Phi(g_{n+1})\leqq 31R(g_{n+1}) ,$$

$$\begin{aligned}R(t_{n+p}-t_n)&\leqq\sum_{i=1}^{p}R(t_{n+i}-t_{n+i-1})\\ &\leqq 31\sum_{i=1}^{p}R(g_{n+i})\to 0 \text{ as } n,\ p\to\infty .\end{aligned}$$

[3, (iv), Equation 3, (iii)]

We conclude that

$$\lim_{n\to\infty}(1-g_1-\cdots-g_n)s_n(1-g_1-\cdots-g_n)=\lim_{n\to\infty}(t_n s-(g_1+\cdots+g_n))$$

exists in $\mathfrak{R}$. It equals $(1-g)t(1-g)$ for some $t\in\mathfrak{R}$. Moreover, $(1-e)\cdot t(e-g)=0$ by (1). Then

$$s\cong\begin{bmatrix} g & 0 & 0\\ 0 & (e-g)t(e-g) & (e-g)t(1-e)\\ 0 & 0 & (1-e)t(1-e)\end{bmatrix}$$

where $R((e-g)t(e-g))\leqq 1/2$ and $(e-g)t(e-g)$ has an inverse in the subring $\mathfrak{R}(e-g)$.

By the proof of [4, Lemma 3.6], if $(1-e)h(1-e)=h$ is an idempotent of rank equal to $R(e-g)$, then $e-g$, h define quantities $x, y \in \mathfrak{R}$ such that

$$xh=(e-g)x=x\,, \quad hy=y(e-g)=y\,, \quad xy=e-g\,, \quad yx=h\,.$$

We have that $1+x$, $1+y \in C^2$ since $x^2=xh(e-g)x=0$, $y^2=y(e-g)hy=0$, and so $(1+x)(1-y)(1+x)=1-(e-g)-h+x-y \in \mathfrak{R}^\dagger$ whence

$$s\cong\begin{bmatrix} g & 0 & 0 \\ 0 & 0 & (e-g)t^*(1-e) \\ 0 & -h(e-g)t(e-g) & (1-e)t^*(1-e) \end{bmatrix}$$

for some $t^* \in \mathfrak{R}$. Since

$$R(-h(e-g)t(e-g))=R(e-g)\,,$$

then

$$(-h(e-g)t(e-g))_l=(e-g)_l\,,$$

and by a similar argument to one above we have, for some $t' \in \mathfrak{R}$,

$$s\cong\begin{bmatrix} g & 0 & 0 \\ 0 & e-g & 0 \\ 0 & 0 & (1-e)t'(1-e) \end{bmatrix}.$$

This useful lemma permits us to obtain an analogue in continuous rings for a diagonalization theorem of Dieudonné [1, p. 30].

THEOREM 1. *In a continuous ring $\mathfrak{R}$, let $e^2=e$, $R(e)<1$ and s be nonsingular. Then, for some $t \in \mathfrak{R}$,*

$$s\cong e+(1-e)t(1-e)\,.$$

Proof. If $R(e)<1/2$, a similar proof to that of Lemma 2 yields the result.

We may suppose then, that

$$\sum_{i=1}^{p-1} 2^{-i}\leqq R(e)<\sum_{i=1}^{p} 2^{-i} \qquad \text{for } p>1$$

Let $e_1=ee_1e$ be an idempotent of rank $1/2$. Then, by Lemma 2, $t_1s=e_1+(1-e_1)s_1(1-e_1)$ for some $t_1 \in \mathfrak{R}^\dagger$ and $s_1 \in \mathfrak{R}$. If $p>2$, we let $e_2=(e-e_1)\cdot e_2(e-e_1)$ be an idempotent of rank $1/4$; then e_2 has normalized rank $1/2$ in the continuous ring $\mathfrak{R}(1-e_1)$ and $(1-e_1)s_1(1-e_1)$ is nonsingular in this ring. Hence, there exists t_2 in the group $\mathfrak{K}^\dagger$ of $\mathfrak{R}(1-e_1)$ such that

$$t_2(1-e_1)s_1(1-e_1)=e_2+(1-e_1-e_2)s_2(1-e_1-e_2)$$

where $s_2\in\mathfrak{R}(1-e_1)\subset\mathfrak{R}$. Then

$$(e_1+t_2)(e_1+(1-e_1)s_1(1-e_1))=e_1+e_2+(1-e_1-e_2)s_2(1-e_1-e_2);$$

moreover, $e_1+t_2\in\mathfrak{K}^\dagger$ as can be verified simply.

Proceeding in a similar fashion, we have eventually, for some s_{p-1} and independent idempotents $e_i=ee_ie$ $(i=1,\cdots,p-1)$ with $R(e_i)=2^{-i}$

$$s\cong e_1+\cdots+e_{p-1}+(1-e_1-\cdots-e_{p-1})s_{p-1}(1-e_1-\cdots-e_{p-1}).$$

Application of the first statement of the proof to the idempotent $e-e_1-\cdots-e_{p-1}$ in the subring $\mathfrak{R}(1-e_1-\cdots-e_{p-1})$ gives

$$t_p(1-e_1-\cdots-e_{p-1})s_{p-1}(1-e_1-\cdots-e_{p-1})$$
$$=e-e_1-\cdots-e_{p-1}+(1-e)s_p(1-e)$$

where

$$t_p\in\mathfrak{R}(1-e_1-\cdots-e_{p-1}),\ e_1+\cdots+e_{p-1}+t_p\in\mathfrak{K}^\dagger \text{ and } s_p\in\mathfrak{R}.$$

The result follows.

THEOREM 2. *In a continuous ring* $\mathfrak{K}=\mathfrak{K}^\dagger$.

Proof. The equation $utu^{-1}=t^2$ is satisfied by any $t\in C^2$, for some $u\in\mathfrak{G}$ depending on t [2, Theorem 2.12]. Hence the arbitrary $t\in C^2$ satisfies

$$t=utu^{-1}t^{-1} \tag{3}$$

and $\mathfrak{K}^\dagger\subseteq\mathfrak{K}$.

By Lemma 2, if $a_1,a_2\in\mathfrak{G}$ and e is an idempotent such that $R(e)=1/2$, then $a_1=b_1d_1$, $a_2=b_2d_2$ where b_1, $b_2\in\mathfrak{K}^\dagger$ and

$$d_1=e+(1-e)d_1(1-e),\quad d_2=e+(1-e)d_2(1-e).$$

The commutator $a_1a_2a_1^{-1}a_2^{-1}$ has the form $bd_1d_2d_1^{-1}d_2^{-1}$ with $b\in\mathfrak{K}^\dagger$ by Corollary 2. It is sufficient to show that $d_1d_2d_1^{-1}d_2^{-1}\in\mathfrak{K}^\dagger$ and we need only show that $d_1d_2=b^{(1)}d_2d_1b^{(2)}$ where $b^{(1)}$, $b^{(2)}\in\mathfrak{K}^\dagger$. Write $(1-e)d_1(1-e)=\lambda$, $(1-e)d_2(1-e)=\mu$.

Now e, $1-e$ define a matrix basis s_{ij} with $s_{11}=e$, $s_{22}=1-e$, $s_{12}=es_{12}=s_{12}(1-e)$, $s_{21}=(1-e)s_{21}=s_{21}e$ [4, Chapter 3]. Then

$$(1+s_{12})(1-s_{21})(1+s_{12})=-s_{21}+s_{12}$$

and

$$(-s_{21}+s_{12})^2=-s_{11}-s_{22}=-1$$

belong to $\mathfrak{K}^\dagger$.

Noticing that λ has an inverse in $\mathfrak{R}(1-e)$ we obtain without difficulty

$$d_1d_2=\begin{bmatrix} e & 0 \\ 0 & \lambda\mu \end{bmatrix}\cong\begin{bmatrix} e & s_{12}\mu \\ 0 & \lambda\mu \end{bmatrix}\cong\begin{bmatrix} e & s_{12}\mu \\ -\lambda s_{21} & 0 \end{bmatrix}\cong\begin{bmatrix} 0 & s_{12}\mu \\ -\lambda s_{21} & 0 \end{bmatrix} \tag{4}$$

and on left multiplying the last member of (4) by $-(-s_{21}+s_{12})$

$$d_1d_2\cong\begin{bmatrix} s_{12}\lambda s_{21} & 0 \\ 0 & \mu \end{bmatrix}\cong\begin{bmatrix} 0 & s_{12}\lambda \\ -\mu s_{21} & 0 \end{bmatrix}.$$

Retracting the steps of (4) we obtain the result.

REMARK 1. When $\mathfrak{R}$ is a matrix ring over a field (discrete ring), $\mathfrak{K}$, $\mathfrak{K}^\dagger$ are respectively the commutator group and the group generated by the elements of class 2. Provided the order of the matrices exceeds two, as we assume, (3) holds and again $\mathfrak{K}^\dagger\subseteqq\mathfrak{K}$; also $\mathfrak{K}^\dagger$ contains the group generated by the transvections which is shown by Dieudonné [1, p. 31] to itself contain $\mathfrak{K}$. Hence Theorem 2 holds for rings of matrices of order greater than two.

3. Determinants in a complete rank ring.

DEFINITION 3. Let $\mathfrak{R}$ be a continuous or discrete ring. We define the determinant $\Delta(a)$ $(a\in\mathfrak{G})$ as the coset $\mathfrak{K}_a$.

We now proceed to obtain generalizations of some well-known results in determinants; the restrictions on characteristic and order apply and the determinants, we note, are defined only for nonsingular ring elements. Theorem 2, Remark 1 and the commutativity of the cosets are used freely without additional reference.

(i) *A theorem on minors of the inverse.*

THEOREM 3. *Let c be nonsingular and e any idempotent in $\mathfrak{R}$. Then*

$$\Delta(1-e+ec^{-1}e)\Delta(c)=\Delta(e+(1-e)c(1-e)).$$

Proof. $\Delta(1-e+ec^{-1}e)\Delta(c)=\Delta\{(1+ec^{-1}(1-e))(1-e+ec^{-1}e)\}\Delta(c)$

$$=\Delta((1-e)c+e)$$

$$=\Delta\{(1-(1-e)ce)((1-e)ce+(1-e)c(1-e)+e)\}$$

$$=\Delta(e+(1-e)c(1-e))\,.$$

(ii) *The Laplace development.* (Compare [1, p. 37].)

THEOREM 4. *Let $e^2=e$, $x\in\mathfrak{R}$. If $R(exe)=R(e)$, then*

$$\Delta(x)=\Delta(exe+(1-e))\Delta(e+(1-e)x(1-e)-(1-e)xe\cdot eye\cdot ex(1-e))$$

where eye is the inverse of exe in $\mathfrak{R}(e)$.

Proof.
$$\begin{aligned}\Delta(x)&=\Delta\{(1-(1-e)xe\cdot eye)x\}\\&=\Delta(exe+ex(1-e)+(1-e)x(1-e)-(1-e)xe\cdot eye\cdot ex(1-e))\\&=\Delta\{(exe+ex(1-e)+(1-e)x(1-e)-(1-e)xe\cdot eye\cdot ex(1-e))\\&\quad\cdot(1-eye\cdot ex(1-e))\}\\&=\Delta(exe+(1-e)x(1-e)-(1-e)xe\cdot eye\cdot ex(1-e))\\&=\Delta(exe+(1-e))\cdot\Delta(e+(1-e)x(1-e)\\&\quad-(1-e)xe\cdot eye\cdot ex(1-e))\,.\end{aligned}$$

(iii) *Cramer's rule.*

THEOREM 5. *Let $ax=b$ be satisfied by $a, b, x\in\mathfrak{R}$. Then*

$$\Delta(be+a(1-e))=\Delta(a)\Delta(exe+(1-e))$$

for any idempotent e.

Proof. $ax=b$ implies $axe=be$ and so

$$\begin{aligned}\Delta(be+a(1-e))&=\Delta(axe+a(1-e))\\&=\Delta(a)\Delta(xe+(1-e))\\&=\Delta(a)\Delta\{(exe+(1-e)xe+(1-e))(1-(1-e)xe)\}\\&=\Delta(a)\Delta(exe+(1-e)).\end{aligned}$$

REMARK 2. The fact that Theorem 5 includes Cramer's rule can be seen as follows.

The matrix equation $Ax=b$ with $A=(a_{ij})$ an $n\times n$ matrix and $x=\{x_1,\cdots,x_n\}$, $b=\{b_1,\cdots,b_n\}$, the components being in a field K, can be expressed

$$(a_{ij})\begin{pmatrix} x_1 & & x_1 \\ \vdots & \cdots & \vdots \\ x_n & & x_n \end{pmatrix} = \begin{pmatrix} b_1 & & b_1 \\ \vdots & \cdots & \vdots \\ b_n & & b_n \end{pmatrix}$$

where each vector is replaced by a ring element with identical columns.

Taking $e = e_i = \operatorname{diag}(0, 0, \cdots, 1, \cdots)$ with 1 in the ith place, Theorem 5 gives

$$\Delta\begin{pmatrix} a_{11} & & b_1 & a_{i+1,1} & \\ \vdots & \cdots & \vdots & \vdots & \cdots \\ a_{1n} & & b_n & a_{i+1,n} & \end{pmatrix} = \Delta(A)\Delta\{\operatorname{diag}(1, \cdots, x_i, 1, \cdots)\}\,.$$

If C is the commutator subgroup of $K^\times$, the isomorphism of $M_n^\times(K)/C_n$ and $M^\times/C$ implies the preceding equation holds when we interpret Δ as the Dieudonné determinant (K noncommutative) or as the ordinary determinant (K commutative).

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SUB-QUASIGROUPS OF FINITE QUASIGROUPS

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1. Introduction. Lagrange's theorem for finite groups (that the order of a sub-group divides the order of the group) does not hold for finite quasigroups in general. However, certain relationships can be obtained between the order of the quasigroup and the orders of its sub-quasigroups. This note will give some of these relationships.

DEFINITION. A set of elements Q and a binary operation "$\circ$" form a *quasigroup* $(Q, \circ)$ if and only if the following are satisfied:

I. If $a, b \in Q$ then there exists a unique $c \in Q$ such that $a \circ b = c$.

II. If $a, b \in Q$ then there exist $x, y \in Q$ such that $a \circ x = b$ and $y \circ a = b$.

III. If $a, x, y \in Q$ then either $a \circ x = a \circ y$ or $x \circ a = y \circ a$ implies $x = y$.

If $(Q, \circ)$ is a quasigroup and S is a subset of Q then $(S, \circ)$ is a sub-quasigroup of $(Q, \circ)$ if $(S, \circ)$ is a quasigroup.

Throughout this note the quasigroup operation will be written multiplicatively, that is, "ab" will be written for "$a \circ b$". Also, "Q" will be written to denote the quasigroup "$(Q, \circ)$". By quasigroup will be meant finite quasigroup, since only finite quasigroups will be considered. The order of a finite set X is the number of elements in X. For subsets X and Y of Q the symbols $X \cap Y$, $X \cup Y$ and $X \setminus Y$ will be used to denote the point set intersection, union and relative complement of X with Y, respectively.

The following elementary properties of a finite quasigroup Q will be of use.

P1. *If $X \subset Q$ and $a \in Q$ then X, aX and Xa have the same order.*

P2. *If $S \subset Q$ and S satisfies I then S is a sub-quasigroup of Q.*

Proof. To prove II, let $a, b \in S$. Since S satisfies I, $aS \subset S$ and by P1, $aS = S$. Thus, since $b \in S$ there exists an $x \in S$ such that $ax = b$. III is inherited from Q.

P3. *If S is a sub-quasigroup of Q then $a \in S$ and $b \notin S$ imply $ab \notin S$.*

2. Relationship of the order of any sub-quasigroup to the order of the quasigroup. The order of a sub-quasigroup need not divide the order of the quasigroup; in fact, these orders may be relatively prime. An example is given by Garrison [1, page 476] of a quasigroup of order 5 with a sub-quasigroup of order 2.

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THEOREM 1. *If Q is a quasigroup of order n and S is a sub-quasigroup of order S then $2s \leqq n$.*

Proof. Let $x \in Q \setminus S$. If $y \in S$ then $xy \in Q \setminus S$. Thus $xS \subset Q \setminus S$. But, by P1, xS has order s and since $Q \setminus S$ has order $n-s$ this implies that $s \leqq n-s$ or $2s \leqq n$.

This shows that the order of a sub-quasigroup is equal to or less than one half the order of the quasigroup. The quasigroup with two elements gives the simplest example in which the equality holds.

3. Relationship between the order of a quasigroup and the orders of two of its sub-quasigroups. Let Q be a quasigroup of order n and R and S be two proper sub-quasigroups of orders r and s, respectively. Assume that R and S intersect. Then $P = R \cap S$ is a sub-quasigroup of Q. Denote the order of P by p. Note that the subsets $R \setminus P$, $S \setminus P$, and $R \cup S$ are of orders $r-p$, $s-p$, and $r+s-p$, respectively.

THEOREM 2. $n \geqq r+s+\max(r, s)-2p$.

Proof. 1. Suppose $S \subset R$. Then $R \cap S = S$ and hence $p=s$, $s \leqq r$ and $\max(r, s)=r$. Thus,

$$r+s+\max(r, s)-2p = 2r-s \leqq 2r\ .$$

But by Theorem 1, $2r \leqq n$ and so $r+s+\max(r, s)-2p \leqq n$.

2. Assume $R \setminus P$ and $S \setminus P$ are non-null. If $x \in R \setminus P$ and $y \in S \setminus P$ then $xy \notin R \cup S$. Thus, for $x \in R \setminus P$, $x(S \setminus P) \subset Q \setminus (R \cup S)$. But $x(S \setminus P)$ is of order $s-p$ and $Q \setminus (R \cup S)$ is of order $n-(r+s-p)$. Therefore, $s-p < n-(r+s-p)$. Similary, if $y \in S \setminus P$ then $y(R \setminus P) \subset Q \setminus (R \cup S)$ and thus, $r-p \leqq n-(r+s-p)$. Therefore,

$$n-(r+s-p) \geqq \max(r-p, s-p) = \max(r, s)-p$$

and so, $n \geqq r+s+\max(r, s)-2p$.

COROLLARY. *If $r=s$ then $n \geqq 3r-2p$.*

THEOREM 3. *If $n=r+s+\max(r, s)-2p$ then $r=s$ if and only if $T = P \cup [Q \setminus (R \cup S)]$ is a sub-quasigronp of Q.*

Proof. A. Assume $r=s$. Then R and S are sub-quasigroups of order r and T is a subset of order r. By P2, to show that T is a sub-quasigroup it suffices to show that if $x \in T$ and $y \in T$ then $xy \in T$.

(1) Let $x \in P$. Then if $y \in P$ then $xy \in P$ since P is a sub-quasigroup. If $y \in T \setminus P$ then $y \in Q \setminus (R \cup S)$ and hence $y \notin R$ and $y \notin S$. Hence $xy \notin R$, $xy \notin S$ and so $xy \in Q \setminus (R \cup S) = T \setminus P$. Thus if $x \in P$ and $y \in T$ then $xy \in T$.

(2) Let $x \in T \setminus P$ and $a \in R \setminus P$. First note that $xa \notin R$. For $b \in S \setminus P$, $ba \notin R \cup S$ and so $(S \setminus P)a \subset Q \setminus (R \cup S) = T \setminus P$. But $(S \setminus P)a$ and $T \setminus P$ are both of order $r-p$. Thus, $(S \setminus P)a = T \setminus P$ and since $x \notin S \setminus P$ this implies $xa \notin T \setminus P$ by III. Thus xa is in neither R nor $T \setminus P$ and so

$$xa \in Q \setminus [R \cup (T \setminus P)] = S \setminus P .$$

Thus, for $x \in T \setminus P$ it follows that $x(R \setminus P) \subset S \setminus P$. But $x(R \setminus P)$ and $S \setminus P$ are both of order $r-p$ and so $x(R \setminus P) = S \setminus P$. Similarly, it can be shown that $x(S \setminus P) = R \setminus P$. Thus, for

$$x \in T \setminus P, x[(R \setminus P) \cup (S \setminus P)] = [(R \setminus P) \cup (S \setminus P)] .$$

By noting that $T = Q \setminus [(R \setminus P) \cup (S \setminus P)]$ and by use of III, it follows that if $x \in T \setminus P$ and $z \in T$ then $xz \in T$. Combining parts (1.) and (2.), it follows that if $x \in T$ and $y \in T$ then $xy \in T$ and thus, T is a sub-quasigroup of Q.

B. Assume that T is a sub-quasigroup. T is of order $\max(r, s)$. Either $r > s$, $r < s$, or $r = s$. Assume $r > s$. Then $\max(r, s) = r$ and T and R are two sub-quasigroups of order r. Thus, by the Corollary to Theorem 2, $n \geqq 3r - 2p$. But, by hypothesis,

$$n = r + s + \max(r, s) - 2p = 2r + s - 2p .$$

Thus, $2r + s - 2p \geqq 3r - 2p$ and so $s \geqq r$, which is contrary to the assumption that $s < r$. Thus $r \not> s$. Similarly, $s \not> r$ and so $r = s$.

For the case in which R and S do not intersect the following results can be obtained.

THEOREM 2′. $n \geqq r + s + \max(r, s)$.

COROLLARY. *If* $r = s$ *then* $n \geqq 3s$.

THEOREM 3′. *If* $n = r + s + \max(r, s)$ *then* $r = s$ *if and only if* $Q \setminus (R \cup S)$ *is a sub-quasigroup of* Q.

An example of a group satisfying the hypothesis of Theorem 3 is the four group which has 3 subgroups of order 2 which intersect pairwise in the identity element. The following are examples of quasigroups satisfying the hypothesis of Theorem 3.

	a	b	c	d	e	f	g	h
a	a	b	c	d	f	e	g	h
b	b	a	d	c	e	f	h	g
c	c	d	a	b	g	h	f	e
d	d	c	b	a	h	g	e	f
e	f	e	h	g	b	a	d	c
f	e	f	g	h	a	b	c	d
g	h	g	e	f	c	d	a	b
h	g	h	f	e	d	c	b	a

Example 1.

	a	b	c	d	e	f
a	a	b	c	d	f	e
b	b	a	e	f	c	d
c	c	e	d	a	b	f
d	d	f	a	c	e	b
e	e	d	f	b	a	c
f	f	c	b	e	d	a

Example 2.

In Example 1, let $P=\{a, b\}$, $R=\{a, b, c, d\}$, $S=\{a, b, e, f\}$ and $T=\{a, b, g, h\}$. The hypothesis of Theorem 3 is satisfied and $r=s$ and T is a sub-quasigroup.

In Example 2, let $P=\{a\}$, $R=\{a, b\}$, $S=\{a, c, d\}$ and $T=\{a, e, f\}$. In this case $r\neq s$ and T is not a sub-quasigroup.

Counterexamples to many of the possible generalizations to more than two sub-quasigroups can be constructed. For example, it has been proved that (1) if Q is of order n with a subquasigroup of order s then $n\geqq 2s$ and (2) if Q is of order with two non-intersecting sub-quasigroups of order s then $n\geqq 3s$. Thus, it might be conjectured that for any positive integer m, if Q contains m mutually disjoint sub-quasigroups of order s then $n\geqq(m+1)s$. However, this fails for $m=3$ since it is possible to construct a quasigroup of order $3s$ with three disjoint sub-quasigroups of order s. In another direction, it is possible to construct a quasigroup of order $4s$ containing three disjoint sub-quasigroups of order s, in which the remaining s elements do not form a sub-quasigroup.

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MONOTONE COMPLETENESS OF NORMED SEMI-ORDERED LINEAR SPACES

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Introduction. Let R be a continuous semi-ordered linear space, namely, a semi-ordered linear space where, for any sequence $x_\nu \geqq 0$ $(\nu=1,2,\cdots)$, $\bigcap_{\nu=1}^{\infty} x_\nu$ exists.[1] R is said to be a normed semi-ordered linear space, if a norm $\|x\|(x \in R)$ is defined and satisfies the condition:

$$|x| \leqq |y| \qquad \text{implies} \qquad \|x\| \leqq \|y\|$$

in addition to the usual conditions.

A norm $\|x\|(x \in R)$ on a normed semi-ordered linear space is said to be *monotone complete*, if, when $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty}$ and $\sup_{\nu \geqq 1} \|x_\nu\| < +\infty$, there exists $\bigcup_{\nu=1}^{\infty} x_\nu$.

A norm on R is said to be *continuous*, if $x_\nu \downarrow_{\nu=1}^{\infty} 0$ implies $\lim_{\nu\to\infty} \|x_\nu\| = 0$ and *semi-continuous*, if $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty} x$ implies $\sup_{\nu \geqq 1} \|x_\nu\| = \|x\|$. It is clear that continuity implies semi-continuity.

Kantorovitch [4] has proved that, if a norm on R is monotone complete and continuous, then it is complete, namely, R is a Banach lattice. Nakano [5; Theorem 31.7] has proved that, if a norm on R is monotone complete and semi-continuous, then the norm is complete, and, recently, Amemiya [1] has proved that, if a norm on R is monotone complete, it is complete.[2] In this connection, see also [2].

In this paper, we will consider several problems concerning monotone completeness and completeness of normed semi-ordered linear spaces and Nakano spaces.

1. Monotone completeness of normed semi-ordered linear spaces. In this section, we will consider two problems.

As usual, let (c_0) be the set of all null-sequences of real numbers. This is a normed semi-ordered linear space by the usual ordering and

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[1] Namely, a conditionally σ-complete vector lattice. In this paper we use the terminology and notation of [5].

[2] In this paper, Amemiya also proved the following lemma: Let R be a monotone complete normed semi-ordered linear space. Then there exists a number $\gamma > 0$ such that $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty} x$ implies $\gamma\|x\| \leqq \mathrm{sus}_{\nu \geqq 1} \|x_\nu\|$.

the norm: $\|x\|=\sup_{\nu=1} |\xi_\nu|$ for $x=(\xi_\nu)\in(c_0)$. The fact that this norm is complete is well known. But, it is not monotone complete, because, for the sequence of elements:

$$e_1=(1,0,0,\cdots),\quad e_2=(1,1,0,\cdots),\quad e_3=(1,1,1,0,\cdots),\ \cdots,$$

we have $0\leqq e_\nu\uparrow_{\nu=1}^{\infty}$ and $\sup_{\nu\geqq 1}\|e_\nu\|\leqq 1$, but $\bigcup_{\nu=1}^{\infty} e_\nu$ does not exist in the space (c_0).

Among function spaces, we can also find an example of this type. Let $L^{f}_{1/t}$ be the set of all measurable functions $x(t)$ $(0\leqq t\leqq 1)$ such that

$$\int_0^1 |\xi x(t)|^{1/t}\,dt<+\infty \qquad \text{for all} \qquad \xi>0\,.$$

Then $L^{f}_{1/t}$ is a Banach lattice by the norm:

$$\|x\|=\inf_{m(\xi x)\leqq 1}\frac{1}{|\xi|} \qquad \text{where} \qquad m(x)=\int_0^1 |x(t)|^{1/t}\,dt\,,$$

but this norm is not monotone complete.

In § 1.1, we will state a necessary and sufficient condition in order that a complete norm be monotone complete.

It is well known that every (norm) closed subset of a Banach lattice is also complete. But, we have a monotone complete semi-ordered linear space which contains a closed, but not monotone complete subspace. Namely, let $L_{1/t}$ be the set of all measurable functions $x(t)$ $(0\leqq t\leqq 1)$ such that

$$\int_0^1 |\xi x(t)|^{1/t}\,dt<+\infty \qquad \text{for some} \qquad \xi>0\,.$$

This is a monotone complete normed semi-ordered linear space and $L^{f}_{1/t}$ is a (norm) closed subspace of $L_{1/t}$.

In § 1.2, we will state a necessary and sufficient condition in order that every closed subspace of a monotone complete semi-ordered linear space be monotone complete.

1.1. Let R be a continuous semi-ordered linear space. A sequence x_ν $(\nu=1,2,\cdots)$ is said to be *bounded*, if there exists an element $x\in R$ such that $x_\nu\leqq x$ $(\nu=1,2,\cdots)$. If $0\leqq x_\nu\uparrow_{\nu=1}^{\infty}$ and this sequence is not bounded, then we write $0\leqq x_\nu\uparrow_{\nu=1}^{\infty}+\infty$.

DEFINITION. R is said to be *K-bounded* (bounded in the sense of Kantorovitch), if $0\leqq x_\nu\uparrow_{\nu=1}^{\infty}+\infty$ implies we can find a sequence of real numbers ξ_ν ($\nu=1, 2, \cdots$) such that $\xi_\nu \downarrow_{\nu=1}^{\infty} 0$ and the sequence $\xi_\nu x_\nu$ ($\nu=1, 2, \cdots$) is not bounded.

DEFINITION. R is said to be *K^2-bounded*, if $0 \leqq x_{\mu,\nu} \uparrow_{\mu=1}^{\infty} +\infty$ for every ν implies we can find a sequence of indices μ_ν ($\nu=1, 2, \cdots$) such that the sequence $x_{\mu_\nu,\nu}$ ($\nu=1, 2, \cdots$) is not bounded.

These concepts were introduced by Kantorovitch [4]. It is easily seen that K^2-boundedness implies K-boundedness. If R is reflexive in the sense of [5] § 24, then it is easily seen that R is K-bounded. Therefore, for any R, its conjugate space is always K-bounded.

The K-boundedness can be expressed in other ways, namely, *the following three conditions are mutually equivalent*:

(1) *R is K-bounded*;

(2) *if* $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty}$ *and* $\sum_{\nu=1}^{\infty} \xi_\nu x_\nu$ *is order-convergent for all sequences* (ξ_ν) *with* $\sum_{\nu=1}^{\infty} |\xi_\nu| < +\infty$, *then the sequence* x_ν ($\nu=1, 2, \cdots$) *is bounded*;

(3) *if* $x_\nu \geqq 0$ *and* $\sum_{\nu=1}^{\infty} \xi_\nu x_\nu$ *is order-convergent for all sequences* (ξ_ν) *with* $\xi_\nu \downarrow_{\nu=1}^{\infty} 0$, *then* $\sum_{\nu=1}^{\infty} x_\nu$ *is order-convergent.*

For example, we will prove that (1) implies (2). Let $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty} +\infty$. Then there exists a sequence of real numbers $\xi_\nu \downarrow_{\nu=1}^{\infty} 0$ such that $\xi_\nu x_\nu$ ($\nu=1, 2, \cdots$) are not bounded. Since

$$x_\nu = \sum_{\mu=2}^{\nu} (x_\mu - x_{\mu-1}) + x_1 \uparrow_{\nu=1}^{\infty} +\infty ,$$

and

$$\xi_\nu x_\nu \leqq \sum_{\mu=2}^{\nu} \xi_\mu (x_\mu - x_{\mu-1}) + \xi_1 x_1 ,$$

the sequence:

$$\sum_{\mu=1}^{\nu} (\xi_\mu - \xi_{\mu+1}) x_\mu \qquad (\nu=1, 2, \cdots)$$

is not bounded and

$$\sum_{\mu=1}^{\infty} |\xi_\mu - \xi_{\mu+1}| < +\infty .$$

This is inconsistent with the hypothesis of (2).

THEOREM 1.1. *Let R be a normed semi-ordered linear space. Then the following three conditions are mutually equivalent;*

(1) *The norm on R is monotone complete;*

(2) *the norm is complete and R is K-bounded;*

(3) *the norm is complete and R is K^2-bounded.*

Proof. We have only to prove that (2) implies (1).
Let $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty}$ and $\sup_{\nu \geqq 1} \|x_\nu\| < +\infty$. Then, for any sequence of numbers $\xi_\nu > 0$ ($\nu = 1, 2, \cdots$) such that $\sum_{\nu=1}^{\infty} \xi_\nu < +\infty$, we have $\sum_{\nu=1}^{\infty} \xi_\nu \|x_\nu\| < +\infty$. Since the norm is complete by assumption, $\sum_{\nu=1}^{\infty} \xi_\nu x_\nu$ is convergent in norm, and so, in order convergence. Therefore, x_ν ($\nu = 1, 2, \cdots$) is bounded, because R is K-bounded.

1.2. Let R be a continuous semi-ordered linear space. For any element $p \geqq 0$ and for all $x \geqq 0$, the *projector* $[p]$ is defined as

$$[p]x = \bigcup_{\nu=1}^{\infty} (x \cap \nu p) .$$

$[p] \geqq [q]$ means $[p]x \geqq [q]x$ for any $x \geqq 0$.

Let R be a normed semi-ordered linear space. *A norm $\|x\|$ on R is continuous if and only if $x \geqq 0$ and $[p_\nu] \downarrow_{\nu=1}^{\infty} 0$ implies $\lim_{\nu \to \infty} \|[p_\nu]x\| = 0$* ([Nakano] Theorem 30.8) We will call a subset A of R *monotone complete,* if $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty}$ and $\sup_{\nu \geqq 1} \|x_\nu\| < +\infty$ for $x_\nu \in A$ implies $\bigcup_{\nu=1}^{\infty} x_\nu \in A$.

If a norm on R is monotone complete and continuous, then every (norm) closed subset is monotone complete in the sense described above. Here, we will prove the converse. A subset A is said to be *semi-normal,* if $x \in A$, $|y| \leqq |x|$ implies $y \in A$.

THEOREM 1.2. *Let R be a normed semi-ordered linear space and suppose every (norm) closed, semi-normal subset of R is monotone complete. Then the norm is continuous.*

Proof. Let us assume that there exist $[p_\nu]$ ($\nu = 1, 2, \cdots$) and $x_0 \in R$ such that $[p_\nu] \downarrow_{\nu=1}^{\infty} 0$ and $\lim_{\nu \to \infty} \|[p_\nu]x_0\| \geqq \varepsilon$ for some $\varepsilon > 0$. Then the least closed set A containing all $x \in R$ such that $\lim_{\nu \to \infty} \|[p_\nu]x\| = 0$ is semi-normal and $(1 - [p_\nu])x_0 \in A$. On the other hand,

$$0 \leqq (1 - [p_\nu])x_0 \uparrow_{\nu=1}^{\infty} x_0 \qquad \text{and} \qquad \|(1 - [p_\nu])x_0\| \leqq \|x_0\| .$$

Therefore, since A is monotone complete, $x_0 \in A$. This is inconsistent with the definition of A.

2. Monotone completeness of Nakano spaces. It will be necessary to state here the definition and several properties of Nakano spaces.

A semi-ordered linear space is said to be *universally continuous*, if for any system of positive elements x_λ $(\lambda \in \Lambda)$ there exists $\bigcap_{\lambda \in \Lambda} x_\lambda$. A *Nakano space* is a universally continuous semi-ordered linear space where a functional $m(x)$ $(x \in R)$ is defined and satisfies the following conditions:

(1) $0 \leqq m(x) \leqq +\infty \, (x \in R)$;

(2) for any $x \in R$ we can find a number $\xi > 0$ such that $m(\xi x) < +\infty$;

(3) if $m(\xi x) = 0$ for every $\xi > 0$, then $x = 0$;

(4) $|x| \leqq |y|$ implies $m(x) \leqq m(y)$;

(5) $m\left(\frac{\xi+\eta}{2} x\right) \leqq \frac{1}{2}\{m(\xi x) + m(\xi y)\}$ for numbers $\xi, \eta > 0$ and for every element $x \in R$;

(6) $|x| \frown |y| = 0$ implies $m(x+y) = m(x) + m(y)$;

(7) $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda} x$ implies $m(x) = \sup_{\lambda \in \Lambda} m(x_\lambda)$.

This functional $m(x)$ is called a *modular* on the Nakano space R. In the Nakano space R, we can define two kinds of norms:

$$\text{the first norm:} \qquad \|x\| = \inf_{\xi > 0} \frac{1 + m(\xi x)}{\xi};$$

$$\text{the second norm:} \qquad \|\!|x|\!\| = \inf_{m(\xi x) \leqq 1} \frac{1}{|\xi|}.$$

It is easily seen that $\|\!|x|\!\| \leqq \|x\| \leqq 2\|\!|x|\!\|$. The modular is said to be complete or monotone complete, if these norms are complete or monotone complete. Namely, a modular m on R is said to be monotone complete, if, when $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty}$ and $\sup_{\nu \geqq 1} m(x_\nu) < +\infty$, then there exists $\bigcup_{\nu=1}^{\infty} x_\nu$.

A modular m is said to be *simple*, if $m(x) = 0$ implies $x = 0$. If m is simple, we can define in R a convergence by this modular. Namely, a sequence x_ν $(\nu = 1, 2, \cdots)$ is said to be *modular-convergent* to $x \in R$, if $\lim_{\nu \to \infty} m(x_\nu - x) = 0$. If a sequence x_ν $(\nu = 1, 2, \cdots)$ is convergent to $x \in R$ by the norms defined above, then it is modular-convergent to the same limit. But the converse is not always true. In order that the modular-convergence be equivalent to the norm convergence, it is necessary and sufficient that the modular is *uniformly simple*: $\inf_{0 \neq x \in R} m\left(\xi \frac{x}{\|\!|x|\!\|}\right) > 0$ for any $\xi > 0$ ([5] Theorem 48.1)

The norms defined above are not always continuous. If the modular is *finite*, namely, $m(x)<+\infty$ for every $x\in R$, then the norms are continuous ([5] Theorem 44.4)

A modular m is said to be *uniformly finite*, if $\sup_{0\neq x\in R} m\left(\xi\frac{x}{\|\|x\|\|}\right)<+\infty$ for every $\xi>0$. It is clear that uniform finiteness is stronger than finiteness.[3]

2.1. In this section, we will consider the relations between monotone completeness and completeness of Nakano spaces. In the sequel, let R be a Nakano space and $m(x)$ $(x\in R)$ be its modular.

The following lemma is a generalization of the essential part of Kalugyna's results [3].

LEMMA 2.1. *If m is monotone complete, simple, and its norms are continuous, then m is uniformly simple.*

Proof. If m is not uniformly simple, we can find a sequence $x_\nu\geqq 0$ $(\nu=1, 2, \cdots)$ such that $\lim_{\nu\to\infty} m(x_\nu)=0$ and $\|x_\nu\|\geqq\varepsilon>0$ for all ν. Hence, we can select a subsequence x_{ν_μ} $(\mu=1, 2, \cdots)$ such that

$$m(x_{\nu_\mu})\leqq 1/2^\mu .$$

Then, for the elements:

$$y_{\mu,\lambda}=x_{\nu_\mu}\smile x_{\nu_{\mu+1}}\smile\cdots\smile x_{\nu_{\mu+\lambda}} ,$$

we have

$$m(y_{\mu,\lambda})\leqq m(x_{\nu_\mu})+m(x_{\nu_{\mu+1}})+\cdots+m(x_{\nu_{\mu+\lambda}})$$
$$\leqq\frac{1}{2^\mu}+\cdots+\frac{1}{2^{\mu+\lambda}} .$$

Namely, we have $y_{\mu,\lambda}\uparrow_{\lambda=1}^{\infty}$ and $\sup_{\lambda\geqq 1} m(y_{\mu,\lambda})<+\infty$. Since m is monotone complete, there exist y_μ $(\mu=1,2,\cdots)$ such that $y_\mu=\bigcup_{\lambda=1}^{\infty} y_{\mu,\lambda}$ and $m(y_\mu)\leqq 1/2^{\mu+1}$. It is clear that $y_\mu\downarrow_{\mu=1}^{\infty}$. On the other hand, for any $x\geqq 0$ such that $x\leqq y_\mu$ $(\mu=1, 2, \cdots)$, we have

$$m(y_\mu-x)\leqq m(y_\mu) , \qquad \text{thus,} \qquad \lim_{\mu\to\infty} m(y_\mu-x)=0 .$$

[3] More details of the theory of Nakano spaces are given in [**5**]. As examples of Nakano spaces, we cite two representative types. The first is an Orlicz space. The second is the space $L_{p(t)}(p(t)\geqq 1)$, namely, the set of measurable functions $x(t)(0\leqq t\leqq 1)$ such that $\int_0^1|\xi x(t)|^{p(t)}dt$ is finite for some $\xi>0$. Here $p(t)$ is a measurable function on $0\leqq t\leqq 1$.

Therefore,

$$m\left(\frac{1}{2}x\right)=m\left(\frac{1}{2}(y_\mu-x)+\frac{1}{2}y_\mu\right)$$

$$\leqq\frac{1}{2}\left\{m(y_\mu-x)+m(y_\mu)\right\}\to 0 \qquad (\mu\to\infty),$$

that is to say, $m\left(\frac{1}{2}x\right)=0$. Since m is simple, $x=0$. This means that $y_\mu\downarrow_{\mu=1}^{\infty}0$. As the norm is continuous, we have $\lim_{\mu\to\infty}\|y_\mu\|=0$, which contradicts the assumption, because

$$\|y_\mu\|\geqq\|x_{\nu_\mu}\|\geqq\varepsilon .$$

Therefore, m is uniformly simple.

The next two lemmas constitute the converse of the above.

LEMMA 2.2. *If m is uniformly simple, then its norms are continuous.*

Proof. Let $x_\nu\downarrow_{\nu=1}^{\infty}0$. Then there exists a number $\xi>0$ such that $m(\xi x_\nu)<+\infty$ for all ν. For the elements $y_\nu=\xi x_1-\xi x_\nu$, since $y_\nu\geqq 0$ and $\xi x_\nu\geqq 0$, we have

$$m(y_\nu+\xi x_\nu)\geqq m(y_\nu)+m(\xi x_\nu),$$

so,

$$m(\xi x_\nu)\leqq m(y_\nu+\xi x_\nu)-m(y_\nu)=m(\xi x_1)-m(y_\nu).$$

On the other hand, we have $m(\xi x_1)=\sup_{\nu\geqq 1}m(y_\nu)$, because $0\leqq y_\nu\uparrow_{\nu=1}^{\infty}\xi x_1$. Therefore, $\lim_{\nu\to\infty}m(\xi x_\nu)=0$, and hence it follows that $\lim_{\nu\to\infty}\|x_\nu\|=0$, because m is uniformly simple.

LEMMA 2.3. *If m is uniformly simple and its norms are complete, then m is monotone complete.*

Proof. Let $0\leqq x_\nu\uparrow_{\nu=1}^{\infty}$ and $\sup_{\nu\geqq 1}m(x_\nu)<+\infty$. Then

$$m(x_\nu-x_\mu)\leqq m(x_\nu)-m(x_\mu) \qquad (\nu\geqq\mu),$$

and hence, we have

$$\lim_{\nu,\mu\to\infty}m(x_\nu-x_\mu)=0 .$$

Since m is uniformly simple, we have

$$\lim_{\nu,\mu\to\infty} \|x_\nu - x_\mu\| = 0 ,$$

so that there exists an element $x \in R$ such that $\lim_{\nu\to\infty} \|x_\nu - x\| = 0$. For this x, it is easily seen that $x = \bigcup_{\nu=1}^{\infty} x_\nu$, which shows that m is monotone complete.

From these lemmas, we obtain the following theorem:

THEOREM 2.1. *A modular on a Nakano space is monotone complete, simple, and its norms are continuous, if and only if it is uniformly simple and complete.*

Next, we will consider the case when m is finite.

DEFINITION. A modular $m(x)$ $(x \in R)$ is said to be totally finite, if $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty}$ and $\sup_{\nu\geqq 1} m(x_\nu) < +\infty$ implies $\sup_{\nu\geqq 1} m(\xi x_\nu) < +\infty$ for every $\xi > 0$.

LEMMA 2.4. *If m is monotone complete and finite, then it is totally finite.*

Proof. $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty}$ and $\sup_{\nu\geqq 1} m(x_\nu) < +\infty$. Then, since m is monotone complete, there exists $x \in R$ such that $x = \bigcup_{\nu=1}^{\infty} x_\nu$. Therefore $\xi x = \bigcup_{\nu=1}^{\infty} \xi x_\nu$ for every $\xi > 0$. Hence it follows that $m(\xi x) = \sup_{\nu\geqq 1} m(\xi x_\nu) < +\infty$, because m is finite.

LEMMA 2.5. *If m is totally finite and complete, then it is monotone complete.*

Proof. Let $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty}$ and $\sup_{\nu\geqq 1} m(x_\nu) < +\infty$. Then, by the assumption, we have $\sup_{\nu\geqq 1} m(\xi x_\nu) < +\infty$ for every $\xi > 0$. Since

$$m(\xi x_\nu - \xi x_\mu) \leqq m(\xi x_\nu) - m(\xi x_\mu) \qquad (\nu \geqq \mu) ,$$

we have $\lim_{\nu,\mu\to\infty} m(\xi x_\nu - \xi x_\mu) = 0$ for every $\xi > 0$, therefore we have

$$\lim_{\nu,\mu\to\infty} \|x_\nu - x_\mu\| = 0 .$$

Hence, there exists an element $x \in R$ such that $\lim_{\nu\to\infty} \|x_\nu - x\| = 0$. Therefore, we have $x = \bigcup_{\nu=1}^{\infty} x_\nu$, which shows that m is monotone complete.

Thus we obtain the following theorem:

THEOREM 2.2. *A modular on a Nakano space is monotone complete and finite if and only if it is totally finite and complete.*

REMARK. It is easily seen that uniform finiteness implies total finiteness and the latter implies finiteness. The converses are not always true. In fact, $L_{1/t}^{f}$ is a finite Nakano space by the following modular:

$$m(x)=\int_0^1 |x(t)|^{1/t}\,dt \qquad \text{for} \qquad x(t)\in L_{1/t}^{f}\,.$$

But, this is not totally finite, because, if it were totally finite, then, by Theorem 2.2, it would be monotone complete, which is impossible. Next, let $f_\nu(\xi)$ $(\nu=1, 2, \cdots)$ be a sequence of convex functions such that

$$f_\nu(\xi)=\begin{cases}\xi & \text{if} \quad 0\leqq\xi\leqq 1\,;\\ \nu(\xi-1)+1 & \text{if} \quad \xi>1\,.\end{cases}$$

Then, *the space*[4] $l(f_1, f_2, \cdots)$ with the modular

$$m(x)=\sum_{\nu=1}^{\infty} f_\nu(|\xi_\nu|) \qquad \text{for} \qquad x=(\xi_\nu)$$

is totally finite, but not uniformly finite. To see this, we need only take the elements:

$$e_1=(1, 0, 0, \cdots)\,, \quad e_2=(0, 1, 0, \cdots)\,, \quad e_3=(0, 0, 1, 0, \cdots)\,, \quad \cdots\,.$$

It is easily proved that $|||e_\nu|||=1$ and $m(2e_\nu)=\nu+1\to+\infty$. But, this sequence space is uniformly simple by Theorem 2.1. The relations between uniform simplicity and uniform finiteness were considered by my colleagues. If a modular on a Nakano space is uniformly finite and simple, then, by considering the monotone completion and applying Theorem 2.1, we can prove that it is uniformly simple. On the other hand, T. Shimogaki has proved in an unpublished paper that, if a modular is uniformly simple and the space has no atomic elements, then it is uniformly finite.

2.2. In this section, we will consider relations between monotone completeness and finiteness.

An element x is said to be *finite*, if $m(\xi x)<+\infty$ for every $\xi>0$. The set of all finite elements is called a *finite manifold* of R and denoted by F. F is a (norm) closed subspace of R and the norms are continuous in F ([5] Theorem 44.5.). If the norms are continuous in R and m is monotone complete, then F is *universally monotone complete*, that is, if $0\leqq x_\lambda\uparrow_{\lambda\in\Lambda}$ and $\sup_{\lambda\in\Lambda} m(x_\lambda)<+\infty$ then there exists $\bigcup_{\lambda\in\Lambda} x_\lambda$.

m is said to be *almost finite*, if F is complete in R (that is, if $|x|\frown|y|=0$ for all $y\in F$, then $x=0$).

[4] For the definition of this sequence space, see [**6**].

THEOREM 2.3. *If m is almost finite and monotone complete, then m is finite if and only if F is, as a space, universally monotone complete.*

Proof. We need only prove the sufficiency. For any $x \in R$, since m is almost finite, there exists a system of projectors $[p_\lambda]\uparrow_{\lambda\in\Lambda}[x]$ such that $[p_\lambda]x \in F$, and there exists a number $\xi>0$ such that $m(\xi x)<+\infty$. Therefore we have

$$\bigcup_{\lambda\in\Lambda} \xi[p_\lambda]x=\xi x \in F ,$$

since m is monotone complete. Hence it follows that m is finite.

THEOREM 2.4. *If m is almost finite, monotone complete and separable in its norm topology, then m is finite.*

Proof. It is well known that if m is almost finite and norms are continuous, then m is finite. Therefore, we need only prove that if m is monotone complete and separable, then its norms are continuous.

For this purpose, let us suppose that there exists an element $x \geqq 0$ and a sequence of projectors $[p_\nu]\downarrow_{\nu=1}^{\infty} 0$ such that

$$\inf_{\nu\geqq 1} \|[p_\nu]x\|>\varepsilon \qquad \text{for some} \qquad \varepsilon>0 .$$

Then, by Amemiya's lemma, we can find a number $\xi>0$ such that

$$\lim_{\nu\to\infty} \|[p_\mu]a-[p_\nu]a\| \geqq \xi\|[p_\mu]a\|>\xi\varepsilon>0 ,$$

and here, we can select μ_ν ($\nu=1, 2, \cdots$) such that

$$\|[p_{\mu_\nu}]x-[p_{\mu_{\nu+1}}]x\|>\xi\varepsilon .$$

Putting $p_\nu=[p_{\mu_\nu}]x-[p_{\mu_{\nu+1}}]x$ ($\nu=1, 2, \cdots$), we see easily that

$$p_\nu \geqq 0 , \qquad p_\nu \cap p_\lambda=0(\nu\neq\lambda) \qquad \text{and} \qquad \|p_\nu\|>\xi\varepsilon ,$$

and, for any subsequence p_{ν_λ} ($\lambda=1, 2, \cdots$), we have

$$\sum_{\lambda=1}^{\infty} p_{\nu_\lambda} \leqq x .$$

Moreover, the set of all such sequences is not denumerable and

$$\|\sum_{\lambda=1}^{\infty} p_{\nu_\lambda}-\sum_{\rho=1}^{\infty} p_{\nu_\rho}\|>\xi\varepsilon$$

for different sequences $\{p_{\nu_\lambda}\}$ and $\{p_{\nu_\rho}\}$. This contradicts the separability. Therefore, norms are continuous and the proof is established.

REMARK. In order that m be finite, it is necessary and sufficient that its norms be continuous and all atomic elements belong to F.

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HOKKAIDÔ UNIVERSITY AND
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SIMPLIFIED PROOFS OF "SOME TAUBERIAN THEOREMS" OF JAKIMOVSKI: ADDENDUM AND CORRIGENDUM

C. T. RAJAGOPAL

Dr. B. Kuttner has kindly drawn my attention to a paper by F. Hausdorff, *Die Äquivalenz der Hölderschen und Cesaròschen Grenzwerte negativer Ordnung*, Math. Z., 31 (1930), 186–196, which contains a generalization of Jakimovski's fundamental theorem discussed in § 2 of my paper (this volume, pp. 955–960) and Szász's product-theorem referred to in § 3 of my paper, under numbers VI and III respectively in the list of numbered results I–VIII. There is a close connection between Hausdorff's paper and mine, as shown, for instance, by a comparison of Lemmas 1, 3 in the latter with the interpretation of Γ_{-k} and the result numbered VII in the former (pp. 195–6). It is unfortunate that I should have been ignorant of Hausdorff's paper and that the paper should have escaped mention in the lists of references provided by such works as G. H. Hardy's *Divergent series* and O. Szász's *Introduction to the theory of divergent series.*

Dr. Kuttner has also been good enough to call my attention to the fact that my step numbered (6) in p. 958 is not a valid deduction from my Lemma 2. For the convenience of the reader, I add that my incorrect argument may be replaced by the following, after the deletion of the last two lines of p. 957 and the lines 1, 2, 6, 7, 8, 9 of p. 958.

Since, if $k=1$ we infer at once from Lemma 2 that $s_n=o(1)$, we suppose that $k \geqq 2$ and reduce this case to the case $k=1$. When $k \geqq 2$, (7) in p. 958 shows that

$$\sum_{r=0}^{\infty} \Delta^k s_{r-k} x^r = o(1)\,, \qquad x \to 1-0\,,$$

that is, that the series $\sum \Delta^k s_{r-k}$ is summable (A) to 0. In this series, the nth term $\Delta^k s_{n-k}=o(n^{-k})$, $n \to \infty$, by hypothesis, so that the series is convergent and necessarily to 0. Therefore

$$\Delta^{k-1} s_{n-k+1} = -\sum_{r=0}^{n} \Delta^k s_{r-k} = \sum_{r=n+1}^{\infty} \Delta^k s_{r-k} = \sum_{r=n+1}^{\infty} o(r^{-k}) = o(n^{-k+1})\,, \qquad n \to \infty\,.$$

By repetitions of this argument (if necessary), we reduce $k \cdots$.

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CORRECTION TO THE PAPER "EXTENSION OF UNIFORMLY CONTINUOUS TRANSFORMATIONS IN HYPERCONVEX METRIC SPACES" BY N. ARONSZAJIN AND P. PANITCHPAKDI, PACIFIC JOURNAL OF MATHEMATICS, 6 (1956), 405–439.

Due to an oversight the authors make a statement (page 422) which amounts to saying that every generalized absolute retract is an absolute G_δ. This statement is not true, see for instance, Dugundji, Pacific J. Math., 1951. The authors wish to thank E. Michael for drawing their attention to this error. The statement as it was made was in the nature of a side remark and has no influence on the developments of the paper. However a change is made necessary in Problem V at the end of the paper where the problem should read "If an absolute G_δ space $\mathscr{E}$ is a generalized absolute retract, is it possible to define a metric in $\mathscr{E}$ which induces the given topology on $\mathscr{E}$, and makes it into a hyperconvex space?

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CORRECTION TO THE PAPER "THE REFLECTION PRINCIPLE FOR POLYHARMONIC FUNCTIONS"

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Dr. Avner Friedman kindly drew our attention to an error in *The reflection principle for polyharmonic function* (this Journal 5 (1955), 433–439). On p. 436 we stated that the operator (2.1) transforms $x_1^{\nu_1}x_2^{\nu_2}\cdots x_n^{\nu_n}$ into $(-1)^{\nu_1}x_1^{\nu_1}x_2^{\nu_2}\cdots x_n^{\nu_n}$ for $p\leqq\nu_1\leqq 2p-1$. Counterexamples show that this is not generally true. In our proof we had overlooked the fact that the formula on p. 437 does not represent σ if $2k^*>2p-1-\nu_1$.

Correction. The statement is valid under the additional hypothesis that $\nu_1+\nu_2+\cdots+\nu_n\leqq 2p-1$. Indeed, then a direct verification yields $\sigma=0$ in the case $2k^*>2p-1-\nu_1$.

In order to close the gap which now appears in the proof of the theorem we first observe that the operator (2.1) transforms $x_1^{\nu_1}x_2^{\nu_2}\cdots x_n^{\nu_n}$ into a sum of terms of degree $\nu_1+\nu_2+\cdots+\nu_n$. From this and the above assertion we infer that (3.8) is true if

$$\text{(A)} \qquad p\leqq\nu_1\leqq 2p-1 \qquad \text{and} \qquad \nu_1+\nu_2+\cdots+\nu_n\leqq 2p-1\,.$$

Hence, under the same assumptions,

$$\text{(B)} \qquad \frac{\partial^{\nu_1+\nu_2+\cdots+\nu_n}w(-x_1, x_2, \cdots, x_n)}{\partial x_1^{\nu_1}\partial x_2^{\nu_2}\cdots\partial x_n^{\nu_n}} = \frac{\partial^{\nu_1+\nu_2+\cdots+\nu_n}v(x_1, x_2, \cdots, x_n)}{\partial x_1^{\nu_1}\partial x_2^{\nu_2}\cdots\partial x_n^{\nu_n}}\,,$$

everywhere on S. We conclude that (B) and (3.8) remain valid if the second condition (A) is dropped. Now we can follow the previous reasoning.

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