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UNIFORM CONTINUITY OF CONTINUOUS FUNCTIONS OF METRIC SPACES

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In this paper we intend to find equivalent conditions under which continuous functions of a metric space are always uniformly continuous. Isiwata has attempted to prove a theorem in a recently published paper [3] by a method that has a close relation with ours. Unfortunately he does not accomplish his purpose, so we shall give a correct theorem (Theorem 3) in the last part of this paper and, for this purpose, give a condition for the existence of a uniformly continuous unbounded function in a metric space (Theorem 2).

In this paper the space S, unless otherwise specified, is the metric space with a distance function d(x, y), and, for a positive number α , the α -sphere about a subset $A \{x; d(A, x) < \alpha\}$ is denoted by $S(A, \alpha)$; the function is the real valued continuous mapping.

DEFINITION 1. Let us consider a family of neighborhoods U_n of x_n such that $\{x_n\}$ is a sequence of distinct points and $U_m \cap U_n = \phi$ (=empty) for $m \neq n$. Let $f_n(x)$ be a function such that $f_n(x_n) = n$ and $f_n(x) = 0$ for $x \notin U_n$. Then a mapping constructed from the family is a mapping f(x)defined by $f(x) = f_n(x)$ for x belonging to some U_n and f(x) = 0 for the other x.

LEMMA. Consider a family of neighborhoods U_n of x_n satisfying the following conditions:

- (1) $\{x_n\}$, which consists of distinct points, has no accumulation point,
- (2) $\overline{U}_m \cap \overline{U}_n = \phi, \ m \neq n \ (\overline{U} \ a \ closure \ of \ U), \ and \ U_n \subset S(x_n, \ 1/n),$

(3) there is a sequence of points y_n such that distances of x_n and y_n converge to 0 and y_n does not belong to any U_m ; then the mapping constructed from the family is continuous and not uniformly continuous. When $\{x_n\}$ is a sequence containing infinitely many distinct points and has no accumulation point, there is a family of neighborhoods of x_n satisfying (2); if $\{x_n\}$ further contains infinitely many distinct accumulation points, then the family besides satisfies (3).

Proof. The continuity of the mapping constructed from the family follows from $\overline{\bigcup U_{n_i}} = \bigcup \overline{U_{n_i}}$ for any subsequence $\{n_i\}$ of indices; the mapping is not uniformly continuous by (3). Suppose $\{x_n\}$ consists of distinct accumulation points and has no accumulation point, then, by an inductive process, we have neighborhood V_n of x_n such that $V_n \subset S(x_n, 1/n)$ and $\overline{V_m} \cap \overline{V_n} = \phi$, and have y_n and a neighborhood U_n of x_n

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such that $U_n \not\ni y_n \in V_n$, $U_n \subset V_n$.

DEFINITION 2. Let x be isolated in a metric space, then we write I(x) for a supremum of positive numbers α such that $S(x, \alpha)$ consists of x alone.

THEOREM 1. The following conditions on a metric space S are equivalent

(1) If $\{x_n\}$ is a sequence of points without accumulation point, then all but finitely many members of x_n are isolated and $\inf I(x_n)$ for the isolated points is positive.

(2) If a subset A of S has no accumulation point then all but finitely many points of A are isolated and $\inf I(x)$ for all the isolated points of A is positive.

(3) The set A of all accumulation points in S is compact and $I(x_n)$ is positive for any sequence $\{x_n\}$ in S-A which has no accumulation point (Isiwata [2], Theorem 2).

(4) $\overline{A} \cap B = \phi$ implies $S(A, \alpha) \cap S(B, \alpha) = \phi$ for some α (Nagata [4], Lemma 1).

(5) $\bigcap_{n=1}^{\infty} \overline{A}_n = \phi$ implies $\bigcap_{n=1}^{\infty} S(A_n, \alpha) = \phi$ for some α .

(6) For any function f(x), there is a positive integer n such that every point of $A = \{x; |f(x)| \ge n\}$ is isolated and $\inf_{x \in A} I(x)$ is positive.

(7) All functions of S are uniformly continuous.

(8) All continuous mappings of S into an arbitrary uniform space S' are uniformly continuous.

Proof. Since the equivalence of (1) and (3) is simple, we shall show $(1) \rightarrow (8) \rightarrow (7) \rightarrow (6) \rightarrow (5) \rightarrow (4) \rightarrow (2) \rightarrow (1)$.

(1) \rightarrow (8): If a continuous mapping f(x) of S is not uniformly continuous, there is an "entourage" V (in the sense of Bourbaki) of S' such that $d(x_n, y_n) < 1/n$ and $(f(x_n), f(y_n)) \notin V$ for any positive integer n and for some x_n and y_n . $\{x_n\}$ contains infinitely many distinct points. If $\{x_n\}$ has an accumulation point x, there are subsequences $\{x_{n_i}\}$ and $\{y_{n_i}\}$ of $\{x_n\}$ and $\{y_n\}$ converging to x, and, since f(x) is continuous, (f(x), $f(x_{n_i})) \in W$ and $(f(x), f(y_{n_i})) \in W$ for W satisfying $W \cdot W \subset V$ (we may assume $W^{-1} = W$) and for all sufficiently large i. Hence we have $(f(x_{n_i}),$ $f(y_{n_i})) \in V$, which is excluded. Consequently $\{x_n\}$ hes no accmulation point and inf $I(x_n) = r > 0$ for all sufficiently large n, which contradicts the first inequality of f for n satisfying r > 1/n.

 $(8) \rightarrow (7)$ is obvious.

 $(7) \rightarrow (6)$: If, for some function f(x) and every *n*, there is an accumulation point x_n such that $|f(x_n)| \ge n$, $\{x_n\}$ contains infinitely many distinct elements and has no accumulation point, then, by the Lemma, we have

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a function which is not uniformly continuous. Suppose that every Then there is a point of $A = \{x; |f(x)| \ge n\}$ is isolated and $\inf I(x) = 0$. sequence $\{x_n\}$ in A such that inf $I_n=0$, $I_n=I(x_n)$. $\{x_n\}$ has no accumulation point, and we may assume $I_n < 1/n$. If distances of distinct points of $\{x_n\}$ are greater than a positive number e, then, for all n satisfying $e > 4I_n$, x_n and y_n ($\neq x_n$, $\in S(x_n, 2I_n)$) satisfy the conditions of the Lemma. In the other case, there are arbitrarily large m and nsatisfying $d(x_m, x_n) < e$ for any positive number e, and we have, by an inductive process, a subsequence $\{y_i\}$ of $\{x_n\}$ satisfying $d(y_{2i-1}, y_{2i}) < 1/i$. Then y_{2i-1} and y_{2i} satisfy the conditions of the Lemma. (6) \rightarrow (5): Let $\bigcap_n S(A_n, 1/m) \neq \phi$ for every *m* in spite of $\bigcap_n \overline{A_n} = \phi$. We have a point x_1 contained in $\bigcap_n S(A_n, 1)$ and a point y_1 distinct from x_1 satisfying $d(x_1, y_1) < 1$. Suppose $B_i = \{x_1, \dots, x_i\}$ consists of distinct points such that $x_j \in \bigcap_{n \in I} S(A_n, 1/j)$, x_j and y_j are distinct and $d(x_j, y_j)$ $<1/j, j=1,\cdots, i$. Since, for any point $x, \bigcap_{n} S(A_n, 1/m)$ does not contain x for a sufficiently large m, $\bigcap_{n} S(A_n, 1/(i+1))$ contains a point x_{i+1} being not contained in B_i , and some A_n contains y_{i+1} distinct from x_{i+1} satisfying $d(x_{i+1}, y_{i+1}) < 1/(i+1)$. Thus we have a sequence $\{x_n\}$ of distinct points and $\{y_n\}$ such that $x_m \in \bigcap_n S(A_n, 1/m)$, x_n and y_n are distinct, and $d(x_n, y_n) < 1/n$. $\{x_n\}$ has no accumulation point because of $\cap \overline{A}_n = \phi$. The function obtained from the Lemma does not satisfy the condition (6) whether all but finitely many members of x_n are isolated or not. $(5) \rightarrow (4)$ is obvious.

 $(4) \rightarrow (2)$: Suppose A has infinitely many accumulation points x_n , $n = 1, 2, \dots$. Since $B = \{x_n\}$ has no accumulation point, there is a sequence $C = \{y_n\}$ having no accumulation point such that $d(x_n, y_n) < 1/n$, $B \cap C = \phi$. $\overline{B} \cap \overline{C} = B \cap C = \phi$, and $S(B, \alpha) \cap S(C, \alpha) = \phi$ for no α . If every point of A is isolated and inf I(x)=0, we have a sequence $\{x_n\}$ such that lim $I(x_n)=0$, and have a sequence $\{y_n\}$ with the same properties as the above.

 $(2) \rightarrow (1)$ is obvious.

Recently Isiwata has stated a theorem ([3], Theorem 4) which is related to our Theorem 1. However the first step in his proof is wrong. We shall give a correct form of the theorem in Theorem 3. Let us first give a counterexample for the statement "In a connected metric space which is not totally bounded, there exists a sequence $\{x_n\}$ and a uniformly continuous function f such that $f(x_n)=n$ ".

EXAMPLE. Denoting the points of the plane by polar-coordinate,

we consider the following subsets of the plane:

$$A_m = \{(r, \ heta); \ 0 \leq r \leq 1, \ heta = \pi/m\},$$

 $S = \bigcup_{m=1}^{\infty} A_m.$

We define the distance of the points of S by

$$\begin{aligned} d((r, \ \theta), \ (r', \ \theta')) &= |r - r'| & \text{as } \theta = \theta' \text{ or } rr' = 0 , \\ &= r + r' & \text{as } \theta \neq \theta' , \end{aligned}$$

then S is obviously a connected metric space which is not totally bounded. When f(x), $x \in S$, is a uniformly continuous function of S, there is a positive integer n such that d(x, y) < 1/n implies |f(x)-f(y)| < 1. If x is contained in A_m , there are points $y_0=0=$ pole, $y_1, \dots, y_r = x$, $r \leq n+1$, of A_m such that $d(y_{i-1}, y_i) < 1/n$, $i=1, \dots, r$.

$$|f(0)-f(x)| \leq |f(0)-f(y_1)| + \cdots + |f(y_{r-1})-f(x)| \leq n+1;$$

namely f(x) is bounded.

DEFINITION 3. Let e be a positive number, then the finite sequence of points x_0, x_1, \dots, x_m satisfying $d(x_{i-1}, x_i) < e, i=1, \dots, m$, is said to be an *e*-chain with length m. If, for any positive number e, there are finitely many points p_1, \dots, p_i and a positive integer m such that any point of the space can be bound with some $p_j, 1 \le j \le i$, by an *e*-chain with length m, then the space is said to be finitely chainable.

THEOREM 2. A metric space S admits a uniformly continuous unbounded function if and only if S is not finitely chainable.

Proof. Verification of "only if" part is analogous to that stated in the above example, hence is passed over. Let S be not finitely chainable, then there is a positive number e such that, for any finitely many points and a positive integer n, there is a point which cannot be bound with any one of points selected above by an e-chain with length n. We denote by A_0^n the set of all points which can be bound with a fixed x_0 by an e-chain with length n.

(1) When $A_0^n \neq A_0^{n+1}$ for every *n*, we put

$$f(x) = (n-1)e + d(x, A_0^{n-1})$$

for x belonging to A_0^n and not to A_0^{n-1} , and f(x)=0 for $x \notin A_0 = \bigcup_n A_0^n(f(x))$ = $d(x_0, x)$ for $x \in A_0^1$). Since $S(A_0, e) = A_0$, f(x) is uniformly continuous on S if it is so on A_0 . Let $A_0^n \ni x \notin A_0^{n-1}$ and d(x, y) < e' < e, then $A_0^{n+1} \ni y \notin A_0^{n-2}$. (i) When y is in A_0^{n-1} , then

$$f(y) = (n-2)e + d(y, A_0^{n-2})$$

and $d(x, A_0^{n-1}) < e', d(y, A_0^{n-2}) < e$, hence $f(y) \le f(x)$. If $d(y, A_0^{n-2}) < e - e'$, then d(y, y') < e - e' for some y' of A_0^{n-2} and $d(x, y') \le d(x, y) + d(y, y') < e$, so that x is in A_0^{n-1} , which is excluded. Therefore $d(y, A_0^{n-2}) \ge e - e'$ and

$$egin{aligned} |f(x)-f(y)| =& f(x)-f(y) = e + d(x, A_0^{n-1}) - d(y, A_0^{n-2}) \ &< e + e' - (e - e') = 2e' \ . \end{aligned}$$

(ii) When y is in A_0^n and not in A_0^{n-1} , then

$$f(y) = (n-1)e + d(y, A_0^{n-1}),$$

and we have

$$|f(x) - f(y)| = |d(x, A_0^{n-1}) - d(y, A_0^{n-1})| \le d(x, y) < e^{-1}$$

(cf. the proof of Prop. 3 of §2, [1]). (iii) The remaining case for y is similar to (i). Consequently f(x) is uniformly continuous on A_0 .

(2) When $A_0^n = A_0^{n+1}$ for some *n*, then $A_0^n = A_0^n$ for every $m \ge n$, and, in the similar way to (1), $A_1 = \bigcup A_1^n$ is obtained from a point of $S-A_0$. If we can make an unbounded function which is uniformly continuous on A_1 , our proof will be complete.

(3) When we cannot, for every $m(0 \le m \le n)$, construct a desired function on A_m obtained in the same way as (2), A_0, \dots, A_n cannot cover the space, because the space is not finitely chainable; namely we have a sequence of infinitely many subsets A_0, A_1, \dots when our proof is not complete in the similar way to (2). Then we put f(x)=nfor x of A_n and f(x)=0 for x which is not in any A_n . Then, since $S(A_m, e) \cap A_n = \phi$ for any $m \ne n$ and $S(\cup A_n, e) = \cup A_n$, f(x) is uniformly continuous.

THEOREM 3. If S is a connected metric space which is not finitely chainable, then the set of all uniformly continuous functions of S does not form a ring.

Proof. The following verification is essentially due to Isiwata [3]. There is, by Theorem 2, a uniformly continuous unbounded function f(x) of the space, and we have a sequence $A = \{x_n; n=1, 2, \dots\}$ such that $f(x_n) = a_n$, $a_{n+1} - a_n \ge 1$, $a_1 \ge 1$; A has no accumulation point. For some positive number α , $d(x, y) < \alpha$ imlies |f(x) - f(y)| < 1/3, and so $S(x_m, \alpha) \cap S(x_n, \alpha) = \phi$ for $m \ne n$. We put

$$h(x) = 1 - d(A, x)/\alpha$$
 and $G = \bigcup_n S(x_n, \alpha)$

and

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$$f'(x) = \begin{cases} h(x) & \text{ for } x \in G , \\ 0 & \text{ for } x \notin G . \end{cases}$$

h(x) is uniformly continuous on the space, because d(A, x) is so (cf. Prop. 3 of §2, [1]). h(x)>0 and $h(y)\leq 0$ for x of G and y of S-G respectively, so we have

$$|h(x)-h(y)| = h(x)-h(y) \ge h(x) = |f'(x)-f'(y)|$$
.

Hence f'(x) is uniformly continuous on the space. g(x)=f(x)f'(x) is not uniformly continuous. In fact, if it is uniformly continuous, $d(x, y) < \beta$ implies

(*)
$$|g(x)-g(y)| < 1$$
 and $|f(x)-f(y)| < 1$

for some β ($\leq \alpha$). We select a positive integer *n* such that a_n is greater than $1+4\alpha/\beta$, and take a point *y* such that $\beta/2 \leq d(x_n, y) < \beta$ (it is possible to take such a point because of the connectedness of the space). Then, by (*), we have $|a_n - f(y)| < 1$, $f(y) > a_n - 1 \geq 0$, and

$$egin{aligned} |g(x_n)-g(y)| &= |a_n-(1-d(A,\ y)/lpha)f(y)| = |a_n-f(y)+d(x_n,\ y)f(y)/lpha| \ &\geq |d(x_n,\ y)f(y)/lpha| - |a_n-f(y)| > d(x_n,\ y)f(y)/lpha-1 \ &> eta(a_n-1)/2lpha-1 > eta(1+4lpha/eta-1)/2lpha-1 = 1 \ , \end{aligned}$$

which contradicts (*).

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