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In this paper we intend to find equivalent conditions under which continuous functions of a metric space are always uniformly continuous. Isiwata has attempted to prove a theorem in a recently published paper [3] by a method that has a close relation with ours. Unfortunately he does not accomplish his purpose, so we shall give a correct theorem (Theorem 3) in the last part of this paper and, for this purpose, give a condition for the existence of a uniformly continuous unbounded function in a metric space (Theorem 2).

In this paper the space S, unless otherwise specified, is the metric space with a distance function $d(x, y)$, and, for a positive number α, the α-sphere about a subset A $\{x ; d(A, x) < \alpha\}$ is denoted by $S(A, \alpha)$; the function is the real valued continuous mapping.

DEFINITION 1. Let us consider a family of neighborhoods U_n of x_n such that $\{x_n\}$ is a sequence of distinct points and $U_m \cap U_n = \phi$ ($=$empty) for $m \neq n$. Let $f_n(x)$ be a function such that $f_n(x_n) = n$ and $f_n(x) = 0$ for $x \notin U_n$. Then a *mapping constructed from the family* is a mapping $f(x)$ defined by $f(x) = f_n(x)$ for x belonging to some U_n and $f(x) = 0$ for the other x.

LEMMA. *Consider a family of neighborhoods U_n of x_n satisfying the following conditions*:

(1) *$\{x_n\}$, which consists of distinct points, has no accumulation point,*

(2) *$\overline{U}_m \cap \overline{U}_n = \phi$, $m \neq n$ ($\overline{U}$ a closure of U), and $U_n \subset S(x_n, 1/n)$,*

(3) *there is a sequence of points y_n such that distances of x_n and y_n converge to 0 and y_n does not belong to any U_m; then the mapping constructed from the family is continuous and not uniformly continuous. When $\{x_n\}$ is a sequence containing infinitely many distinct points and has no accumulation point, there is a family of neighborhoods of x_n satisfying (2); if $\{x_n\}$ further contains infinitely many distinct accumulation points, then the family besides satisfies (3).*

Proof. The continuity of the mapping constructed from the family follows from $\overline{\cup U_{n_i}} = \cup \overline{U}_{n_i}$ for any subsequence $\{n_i\}$ of indices; the mapping is not uniformly continuous by (3). Suppose $\{x_n\}$ consists of distinct accumulation points and has no accumulation point, then, by an inductive process, we have neighborhood V_n of x_n such that $V_n \subset S(x_n, 1/n)$ and $\overline{V}_m \cap \overline{V}_n = \phi$, and have y_n and a neighborhood U_n of x_n

such that $U_n \not\ni y_n \in V_n$, $U_n \subset V_n$.

DEFINITION 2. Let x be isolated in a metric space, then we write $I(x)$ for a supremum of positive numbers α such that $S(x, \alpha)$ consists of x alone.

THEOREM 1. *The following conditions on a metric space S are equivalent*

(1) *If $\{x_n\}$ is a sequence of points without accumulation point, then all but finitely many members of x_n are isolated and* inf $I(x_n)$ *for the isolated points is positive.*

(2) *If a subset A of S has no accumulation point then all but finitely many points of A are isolated and* inf $I(x)$ *for all the isolated points of A is positive.*

(3) *The set A of all accumulation points in S is compact and* inf $I(x_n)$ *is positive for any sequence $\{x_n\}$ in $S-A$ which has no accumulation point* (Isiwata [2], Theorem 2).

(4) $\bar{A} \cap \bar{B} = \phi$ *implies $S(A, \alpha) \cap S(B, \alpha) = \phi$ for some α* (Nagata [4], Lemma 1).

(5) $\bigcap\limits_{n=1}^{\infty} \bar{A}_n = \phi$ *implies $\bigcap\limits_{n=1}^{\infty} S(A_n, \alpha) = \phi$ for some α.*

(6) *For any function $f(x)$, there is a positive integer n such that every point of $A = \{x ; |f(x)| \geqq n\}$ is isolated and* $\inf_{x \in A} I(x)$ *is positive.*

(7) *All functions of S are uniformly continuous.*

(8) *All continuous mappings of S into an arbitrary uniform space S' are uniformly continuous.*

Proof. Since the equivalence of (1) and (3) is simple, we shall show (1)→(8)→(7)→(6)→(5)→(4)→(2)→(1).

(1)→(8): If a continuous mapping $f(x)$ of S is not uniformly continuous, there is an "entourage" V (in the sense of Bourbaki) of S' such that $d(x_n, y_n) < 1/n$ and $(f(x_n), f(y_n)) \notin V$ for any positive integer n and for some x_n and y_n. $\{x_n\}$ contains infinitely many distinct points. If $\{x_n\}$ has an accumulation point x, there are subsequences $\{x_{n_i}\}$ and $\{y_{n_i}\}$ of $\{x_n\}$ and $\{y_n\}$ converging to x, and, since $f(x)$ is continuous, $(f(x), f(x_{n_i})) \in W$ and $(f(x), f(y_{n_i})) \in W$ for W satisfying $W \cdot W \subset V$ (we may assume $W^{-1} = W$) and for all sufficiently large i. Hence we have $(f(x_{n_i}), f(y_{n_i})) \in V$, which is excluded. Consequently $\{x_n\}$ hes no accmulation point and inf $I(x_n) = r > 0$ for all sufficiently large n, which contradicts the first inequality of f for n satisfying $r > 1/n$.

(8)→(7) is obvious.

(7)→(6): If, for some function $f(x)$ and every n, there is an accumulation point x_n such that $|f(x_n)| \geqq n$, $\{x_n\}$ contains infinitely many distinct elements and has no accumulation point, then, by the Lemma, we have

a function which is not uniformly continuous. Suppose that every point of $A = \{x; |f(x)| \geq n\}$ is isolated and inf $I(x) = 0$. Then there is a sequence $\{x_n\}$ in A such that inf $I_n = 0$, $I_n = I(x_n)$. $\{x_n\}$ has no accumulation point, and we may assume $I_n < 1/n$. If distances of distinct points of $\{x_n\}$ are greater than a positive number e, then, for all n satisfying $e > 4I_n$, x_n and y_n ($\neq x_n$, $\in S(x_n, 2I_n)$) satisfy the conditions of the Lemma. In the other case, there are arbitrarily large m and n satisfying $d(x_m, x_n) < e$ for any positive number e, and we have, by an inductive process, a subsequence $\{y_i\}$ of $\{x_n\}$ satisfying $d(y_{2i-1}, y_{2i}) < 1/i$. Then y_{2i-1} and y_{2i} satisfy the conditions of the Lemma.

(6)→(5): Let $\bigcap_n S(A_n, 1/m) \neq \phi$ for every m in spite of $\bigcap_n \overline{A}_n = \phi$. We have a point x_1 contained in $\bigcap_n S(A_n, 1)$ and a point y_1 distinct from x_1 satisfying $d(x_1, y_1) < 1$. Suppose $B_i = \{x_1, \cdots, x_i\}$ consists of distinct points such that $x_j \in \bigcap_n S(A_n, 1/j)$, x_j and y_j are distinct and $d(x_j, y_j) < 1/j$, $j = 1, \cdots, i$. Since, for any point x, $\bigcap_n S(A_n, 1/m)$ does not contain x for a sufficiently large m, $\bigcap_n S(A_n, 1/(i+1))$ contains a point x_{i+1} being not contained in B_i, and some A_n contains y_{i+1} distinct from x_{i+1} satisfying $d(x_{i+1}, y_{i+1}) < 1/(i+1)$. Thus we have a sequence $\{x_n\}$ of distinct points and $\{y_n\}$ such that $x_m \in \bigcap_n S(A_n, 1/m)$, x_n and y_n are distinct, and $d(x_n, y_n) < 1/n$. $\{x_n\}$ has no accumulation point because of $\bigcap_n \overline{A}_n = \phi$. The function obtained from the Lemma does not satisfy the condition (6) whether all but finitely many members of x_n are isolated or not.

(5)→(4) is obvious.

(4)→(2): Suppose A has infinitely many accumulation points x_n, $n = 1, 2, \cdots$. Since $B = \{x_n\}$ has no accumulation point, there is a sequence $C = \{y_n\}$ having no accumulation point such that $d(x_n, y_n) < 1/n$, $B \cap C = \phi$. $\overline{B} \cap \overline{C} = B \cap C = \phi$, and $S(B, \alpha) \cap S(C, \alpha) = \phi$ for no α. If every point of A is isolated and inf $I(x) = 0$, we have a sequence $\{x_n\}$ such that lim $I(x_n) = 0$, and have a sequence $\{y_n\}$ with the same properties as the above.

(2)→(1) is obvious.

Recently Isiwata has stated a theorem ([3], Theorem 4) which is related to our Theorem 1. However the first step in his proof is wrong. We shall give a correct form of the theorem in Theorem 3. Let us first give a counterexample for the statement "In a connected metric space which is not totally bounded, there exists a sequence $\{x_n\}$ and a uniformly continuous function f such that $f(x_n) = n$".

EXAMPLE. Denoting the points of the plane by polar-coordinate,

we consider the following subsets of the plane:

$$A_m = \{(r, \theta) ; \ 0 \leq r \leq 1, \ \theta = \pi/m\},$$

$$S = \bigcup_{m=1}^{\infty} A_m .$$

We define the distance of the points of S by

$$d((r, \theta), (r', \theta')) = |r - r'| \quad \text{as } \theta = \theta' \text{ or } rr' = 0 ,$$

$$= r + r' \quad \text{as } \theta \neq \theta' ,$$

then S is obviously a connected metric space which is not totally bounded. When $f(x)$, $x \in S$, is a uniformly continuous function of S, there is a positive integer n such that $d(x, y) < 1/n$ implies $|f(x) - f(y)| < 1$. If x is contained in A_m, there are points $y_0 = 0 = \text{pole}$, $y_1, \cdots, y_r = x$, $r \leq n+1$, of A_m such that $d(y_{i-1}, y_i) < 1/n$, $i = 1, \cdots, r$.

$$|f(0) - f(x)| \leq |f(0) - f(y_1)| + \cdots + |f(y_{r-1}) - f(x)| \leq n+1 ;$$

namely $f(x)$ is bounded.

DEFINITION 3. Let e be a positive number, then the finite sequence of points $x_0, x_1, \cdots, x_m$ satisfying $d(x_{i-1}, x_i) < e$, $i = 1, \cdots, m$, is said to be an *e-chain with length m*. If, for any positive number e, there are finitely many points $p_1, \cdots, p_i$ and a positive integer m such that any point of the space can be bound with some p_j, $1 \leq j \leq i$, by an *e*-chain with length m, then the space is said to be *finitely chainable*.

THEOREM 2. *A metric space S admits a uniformly continuous unbounded function if and only if S is not finitely chainable.*

Proof. Verification of "only if" part is analogous to that stated in the above example, hence is passed over. Let S be not finitely chainable, then there is a positive number e such that, for any finitely many points and a positive integer n, there is a point which cannot be bound with any one of points selected above by an *e*-chain with length n. We denote by A_0^n the set of all points which can be bound with a fixed x_0 by an *e*-chain with length n.

(1) When $A_0^n \neq A_0^{n+1}$ for every n, we put

$$f(x) = (n-1)e + d(x, A_0^{n-1})$$

for x belonging to A_0^n and not to A_0^{n-1}, and $f(x) = 0$ for $x \notin A_0 = \bigcup_n A_0^n$ ($f(x) = d(x_0, x)$ for $x \in A_0^1$). Since $S(A_0, e) = A_0$, $f(x)$ is uniformly continuous on S if it is so on A_0. Let $A_0^n \ni x \notin A_0^{n-1}$ and $d(x, y) < e' < e$, then $A_0^{n+1} \ni y \notin A_0^{n-2}$. (i) When y is in A_0^{n-1}, then

$$f(y) = (n-2)e + d(y, A_0^{n-2})$$

and $d(x, A_0^{n-1}) < e'$, $d(y, A_0^{n-2}) < e$, hence $f(y) \leq f(x)$. If $d(y, A_0^{n-2}) < e - e'$, then $d(y, y') < e - e'$ for some y' of A_0^{n-2} and $d(x, y') \leq d(x, y) + d(y, y') < e$, so that x is in A_0^{n-1}, which is excluded. Therefore $d(y, A_0^{n-2}) \geq e - e'$ and

$$|f(x) - f(y)| = f(x) - f(y) = e + d(x, A_0^{n-1}) - d(y, A_0^{n-2})$$
$$< e + e' - (e - e') = 2e' \ .$$

(ii) When y is in A_0^n and not in A_0^{n-1}, then

$$f(y) = (n-1)e + d(y, A_0^{n-1}),$$

and we have

$$|f(x) - f(y)| = |d(x, A_0^{n-1}) - d(y, A_0^{n-1})| \leq d(x, y) < e'$$

(cf. the proof of Prop. 3 of §2, [1]). (iii) The remaining case for y is similar to (i). Consequently $f(x)$ is uniformly continuous on A_0.

(2) When $A_0^n = A_0^{n+1}$ for some n, then $A_0^m = A_0^n$ for every $m \geq n$, and, in the similar way to (1), $A_1 = \cup A_1^n$ is obtained from a point of $S - A_0$. If we can make an unbounded function which is uniformly continuous on A_1, our proof will be complete.

(3) When we cannot, for every $m(0 \leq m \leq n)$, construct a desired function on A_m obtained in the same way as (2), $A_0, \cdots, A_n$ cannot cover the space, because the space is not finitely chainable ; namely we have a sequence of infinitely many subsets $A_0, A_1, \cdots$ when our proof is not complete in the similar way to (2). Then we put $f(x) = n$ for x of A_n and $f(x) = 0$ for x which is not in any A_n. Then, since $S(A_m, e) \cap A_n = \phi$ for any $m \neq n$ and $S(\cup A_n, e) = \cup A_n$, $f(x)$ is uniformly continuous.

THEOREM 3.   *If S is a connected metric space which is not finitely chainable, then the set of all uniformly continuous functions of S does not form a ring.*

Proof. The following verification is essentially due to Isiwata [3]. There is, by Theorem 2, a uniformly continuous unbounded function $f(x)$ of the space, and we have a sequence $A = \{x_n ; n = 1, 2, \cdots\}$ such that $f(x_n) = a_n$, $a_{n+1} - a_n \geq 1$, $a_1 \geq 1$; A has no accumulation point. For some positive number α, $d(x, y) < \alpha$ imlies $|f(x) - f(y)| < 1/3$, and so $S(x_m, \alpha) \cap S(x_n, \alpha) = \phi$ for $m \neq n$. We put

$$h(x) = 1 - d(A, x)/\alpha \qquad \text{and} \qquad G = \bigcup_n S(x_n, \alpha)$$

and

$$f'(x) = \begin{cases} h(x) & \text{for } x \in G \text{,} \\ 0 & \text{for } x \notin G \text{.} \end{cases}$$

$h(x)$ is uniformly continuous on the space, because $d(A, x)$ is so (cf. Prop. 3 of §2, [1]). $h(x) > 0$ and $h(y) \leq 0$ for x of G and y of $S - G$ respectively, so we have

$$|h(x) - h(y)| = h(x) - h(y) \geq h(x) = |f'(x) - f'(y)| \text{ .}$$

Hence $f'(x)$ is uniformly continuous on the space. $g(x) = f(x)f'(x)$ is not uniformly continuous. In fact, if it is uniformly continuous, $d(x, y) < \beta$ implies

(*) $|g(x) - g(y)| < 1$ and $|f(x) - f(y)| < 1$

for some β ($\leq \alpha$). We select a positive integer n such that a_n is greater than $1 + 4\alpha/\beta$, and take a point y such that $\beta/2 \leq d(x_n, y) < \beta$ (it is possible to take such a point because of the connectedness of the space). Then, by (*), we have $|a_n - f(y)| < 1$, $f(y) > a_n - 1 \geq 0$, and

$$\begin{aligned} |g(x_n) - g(y)| &= |a_n - (1 - d(A, y)/\alpha)f(y)| = |a_n - f(y) + d(x_n, y)f(y)/\alpha| \\ &\geq |d(x_n, y)f(y)/\alpha| - |a_n - f(y)| > d(x_n, y)f(y)/\alpha - 1 \\ &> \beta(a_n - 1)/2\alpha - 1 > \beta(1 + 4\alpha/\beta - 1)/2\alpha - 1 = 1 \text{ ,} \end{aligned}$$

which contradicts (*).

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