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CURVATURE IN HILBERT GEOMETRIES

PAUL JOSEPH KELLY AND ERNST GABOR STRAUS

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For every pair of points, p and q , interior to a simple, closed, convex curve C in the Euclidean plane, the line $\xi = p \times q$ cuts C in a pair of points u and v . If C has at most one segment then the Hilbert distance from p to q , defined by

$$h(p, q) = \left| \log \left(\frac{up}{uq} \cdot \frac{vq}{vp} \right) \right|,$$

is a proper metric (where up denotes the Euclidean distance from u to p), and is invariant under projective transformations. The geometry induced on the interior of C is a Hilbert geometry, and the Hilbert lines are carried by Euclidean lines [2].

We shall be concerned here with curvature at a point defined in a qualitative rather than a quantitative sense (cf. [1, p 237]).

DEFINITION 1. The *curvature at p* is *positive* or *negative* if there exists a neighborhood U of p such that for every x, y in U we have

$$2 h(\bar{x}, \bar{y}) \geq h(x, y),$$

respectively

$$2 h(\bar{x}, \bar{y}) \leq h(x, y),$$

where \bar{x}, \bar{y} are the Hilbert midpoints respectively of the segments from p to x and p to y . If there is neither positive nor negative curvature at a point then the curvature is *indeterminate* at that point. This qualitative curvature is clearly a projective invariant.

In order to state our result we need one more concept.

DEFINITION 2. A point p is a *projective center* of C if there exists a projective transformation, π , of the plane so that πp is the affine center of πC .

A projective center is characterized by the following. Let ξ be a line through p , and let $\xi \cap C = \{u, v\}$, and let p'_ξ be the harmonic conjugate of p with respect to u and v . Finally, let L_p be the locus of all p'_ξ . Then p is a projective center if and only if L_p is a straight line.

Conic sections are characterized by the fact that every point in their interior is a projective center [3]. We can now state our main result, which solves a problem of H. Busemann [1, Problem 34, p. 406].

THEOREM. *If p is a point of determinate curvature then it is*

a projective center of C . In particular, if the curvature is determinate everywhere then C is an ellipse and the Hilbert geometry is hyperbolic.

We first establish some lemmas.

LEMMA 1. *For any point p , interior to C , there exists a line η (possibly the line at infinity) which intersects L_p in at least two points and does not intersect C .*

Proof. There is at least one chord of C which is bisected by p . If ξ_1 is the line of such a chord then ξ_1 intersects L_p at q_1 on the line at infinity. If L_p has a second point at infinity then the line at infinity satisfies the lemma. If L_p has only one point at infinity then L_p is a connected curve. It cannot lie within the strip formed by the two supporting lines of C which are parallel to ξ_1 for then it would intersect C . There is therefore a point q_2 of L_p outside this strip and the line $\eta = q_1 \times q_2$ satisfies the lemma.

COROLLARY. *For every p in the interior of C there exists a projective transformation, π , so that πC is a closed, convex curve, and so that πp is the midpoint of two mutually perpendicular chords of πC whose endpoints are points of differentiability of πC .*

Proof. Since all but a denumerable set of points of C are points of differentiability, we may choose the line η of Lemma 1 so that $\eta \cap L_p$ contains p'_{ξ_1} and p'_{ξ_2} and so that C is differentiable at its points of intersection with ξ_1 and ξ_2 . Now let π_1 be a projective transformation which maps η into the line at infinity, and let π_2 be an affine transformation which maps $\pi_1 \xi_1$ and $\pi_1 \xi_2$ into perpendicular lines. Then $\pi = \pi_2 \pi_1$ has the required properties.

LEMMA 2. *If a chord of C , of (Euclidean) length $2k$, has p for its midpoint and if q is a neighboring point on the chord at (Euclidean) distance ds from p , then $dS = (2/k) ds + O(ds^3)$, where $dS = h(p, q)$.*

Proof. If the endpoints of the chord are u and v , and the order of the points on the chords is u, p, q, v , then, by definition,

$$\begin{aligned} dS &= \log\left(\frac{up+pq}{up}\right)\left(\frac{vp}{vp-pq}\right) = \log\left(\frac{k+ds}{k}\right)\left(\frac{k}{k-ds}\right) \\ &= \log\left(1+\frac{ds}{k}\right) - \log\left(1-\frac{ds}{k}\right) \\ &= \left[\frac{ds}{k} - \frac{1}{2}\left(\frac{ds}{k}\right)^2 + \frac{1}{3}\left(\frac{ds}{k}\right)^3 - \dots\right] \\ &\quad - \left[-\frac{ds}{k} - \frac{1}{2}\left(\frac{ds}{k}\right)^2 - \frac{1}{3}\left(\frac{ds}{k}\right)^3 - \dots\right] \end{aligned}$$

$$= \frac{2}{k} ds + O(ds^3).$$

LEMMA 3. Let (r, θ) be polar coordinates whose pole p is an interior point of C at which the curvature is determinate. If C is differentiable at the ends of two perpendicular chords which bisect each other at p , then C satisfies the “one-sided” differential relations

$$(1) \quad \begin{aligned} \frac{d}{d\theta} \left(\frac{\csc 2\theta}{r^2} \right) \Big|_{\theta_0^+} &= \frac{d}{d\theta} \left(\frac{\csc 2\theta}{r^2} \right) \Big|_{(\theta_0 + \pi)^+} \\ \frac{d}{d\theta} \left(\frac{\csc 2\theta}{r^2} \right) \Big|_{\theta_0^-} &= \frac{d}{d\theta} \left(\frac{\csc 2\theta}{r^2} \right) \Big|_{(\theta_0 + \pi)^-} \end{aligned}$$

for all θ_0 .

Proof. We first introduce Cartesian coordinates, with origin p , so that the y -axis intersects C at points of second order differentiability, and so that the axes do not coincide with the two given chords bisected by p . The curve C is then given by an “upper” arc $y = y_1(x)$ and a “lower” arc $y = -y_2(x)$. Let the bisected chords lie on the lines $\xi_1: y = ax$ and $\xi_2: y = (1/a)x$ respectively. Let $b_1 = (dx, a dx)$ and $c_1 = (2 dx, 2a dx)$ on ξ_1 , and $b_2 = (dx, -(1/a) dx)$ and $c_2 = (2 dx, -(2/a) dx)$ on ξ_2 , where dx is positive and chosen so that b_1, b_2, c_1 , and c_2 lie inside C . Assume that p is a point of negative curvature. Then.

$$(2) \quad 2 h(m_1, m_2) \leq h(c_1, c_2).$$

where m_i is the Hilbert midpoint of the segment from p to c_i .

To show that $h(m_i, b_i) = O(dx^3)$, we define $dS_1 = h(p, c_1)$ and $ds_1 = pc_1$. With $2k$ representing the Euclidean length of the chord on ξ_1 , it follows from Lemma 2 that $dS_1 = (2/k) ds_1 + O(ds_1^3)$, and hence that

$$(3) \quad h(p, m_1) = \frac{1}{2} dS_1 = \frac{1}{k} ds_1 + O(ds_1^3).$$

Also, from Lemma 2 and the relation $ds_1 = 2 pb_1$, it follows that

$$(4) \quad h(p, b_1) = \frac{2}{k} pb_1 + O[(pb_1)^3] = \frac{1}{k} (ds_1) + O(ds_1^3).$$

Since $h(m_1, b_1) = |h(p, m_1) - h(p, b_1)|$, equations (3) and (4) imply that $h(m_1, b_1) = O(ds_1^3)$. But $ds_1 = dx(1 + a^2)^{1/2} = O(dx)$, hence $h(m_1, b_1) = O(dx^3)$. Similarly, $h(m_2, b_2) = O(dx^3)$, and therefore

$$(5) \quad h(m_1, b_1) + h(m_2, b_2) = O(dx^3).$$

From the triangle inequality,

$$(6) \quad h(m_1, m_2) \geq h(b_1, b_2) - h(m_1, b_1) - h(m_2, b_2) .$$

This, together with (5), yields

$$(7) \quad h(m_1, m_2) \geq h(b_1, b_2) - O(dx^3) ,$$

and from (1) and (7) we obtain

$$(8) \quad 2 h(b_1, b_2) < h(c_1, c_2) + O(dx^3) .$$

We now wish to calculate the distances in (8). First, we have

$$(9) \quad \begin{aligned} h(b_1, b_2) &= h[(dx, a dx), (dx, -\frac{1}{a} dx)] \\ &= \log \left[\frac{y_1(dx) + \frac{1}{a} dx}{y_1(dx) - a dx} \cdot \frac{y_2(dx) + a dx}{y_2(dx) - \frac{1}{a} dx} \right] \\ &= \log \left[1 + \frac{dx}{a y_1(dx)} \right] + \log \left[1 + \frac{a dx}{y_2(dx)} \right] \\ &\quad - \log \left[1 - \frac{a dx}{y_1(dx)} \right] - \log \left[1 - \frac{dx}{a y_2(dx)} \right] . \end{aligned}$$

Using the Maclaurin expansion of the logarithms, and collecting first and second degree terms, we obtain

$$(10) \quad \begin{aligned} h(b_1, b_2) &= dx \left(a + \frac{1}{a} \right) \left[\frac{1}{y_1(dx)} + \frac{1}{y_2(dx)} \right] \\ &\quad + \frac{dx^2}{2} \left(a^2 - \frac{1}{a^2} \right) \left[\frac{1}{y_1^2(dx)} - \frac{1}{y_2^2(dx)} \right] + O(dx^3) . \end{aligned}$$

Because both of the functions $y_1(x)$ and $y_2(x)$ are convex and have second derivatives at $x=0$, they can be represented in the form

$$(11) \quad y_i(dx) = y_i(0) + y_i'(0)dx + O(dx^2) , \quad i=1, 2,$$

and hence

$$(12) \quad \begin{aligned} \frac{1}{y_i(dx)} &= \frac{1}{y_i(0)} - \frac{y_i'(0)}{y_i^2(0)} dx + O(dx^2) \\ \frac{1}{y_i^2(dx)} &= \frac{1}{y_i^2(0)} + O(dx) . \end{aligned}$$

The substitution of (12) in (10) gives

$$(13) \quad \begin{aligned} h(b_1, b_2) &= dx \left(a + \frac{1}{a} \right) \left[\frac{1}{y_1} - \frac{y_1' dx}{y_1^2} + \frac{1}{y_2} - \frac{y_2' dx}{y_2^2} \right] \\ &\quad + \frac{dx^2}{2} \left(a^2 - \frac{1}{a^2} \right) \left(\frac{1}{y_1^2} - \frac{1}{y_2^2} \right) + O(dx^3) , \end{aligned}$$

where $y_i=y_i(0)$. Hence

$$(14) \quad 2 h(b_1, b_2) = 2 dx \left(a + \frac{1}{a} \right) \left[\frac{1}{y_1} + \frac{1}{y_2} - \frac{y_1' dx}{y_1^2} - \frac{y_2' dx}{y_2^2} \right. \\ \left. + \frac{dx}{2} \left(a - \frac{1}{a} \right) \left(\frac{1}{y_1^2} - \frac{1}{y_2^2} \right) \right] + O(dx^3).$$

By the substitution of $2 dx$ for dx we obtain

$$(15) \quad h(c_1, c_2) = 2 dx \left(a + \frac{1}{a} \right) \left[\frac{1}{y_1} + \frac{1}{y_2} - \frac{2 y_1' dx}{y_1^2} - \frac{2 y_2' dx}{y_2^2} \right. \\ \left. + dx \left(a - \frac{1}{a} \right) \left(\frac{1}{y_1^2} - \frac{1}{y_2^2} \right) \right] + O(dx^3).$$

Substituting this and (14) in (8) we have

$$(16) \quad 2 dx \left(a + \frac{1}{a} \right) \left[\frac{1}{y_1} + \frac{1}{y_2} - \frac{y_1' dx}{y_1^2} - \frac{y_2' dx}{y_2^2} \right. \\ \left. + \frac{dx}{2} \left(a - \frac{1}{a} \right) \left(\frac{1}{y_1^2} - \frac{1}{y_2^2} \right) \right] \\ < 2 dx \left(a + \frac{1}{a} \right) \left[\frac{1}{y_1} + \frac{1}{y_2} - \frac{2 y_1' dx}{y_1^2} - \frac{2 y_2' dx}{y_2^2} \right. \\ \left. + dx \left(a - \frac{1}{a} \right) \left(\frac{1}{y_1^2} - \frac{1}{y_2^2} \right) \right] + O(dx^3).$$

By dividing both sides of this inequality by $2 dx(a+1/a)$, and then rearranging the terms, we obtain

$$(17) \quad dx \left(\frac{y_1'}{y_1} + \frac{y_2'}{y_2} \right) - \frac{dx}{2} \left(a - \frac{1}{a} \right) \left(\frac{1}{y_1^2} - \frac{1}{y_2^2} \right) < O(dx^2).$$

Division of both sides of (17) by dx yields a new inequality whose right side is $O(dx)$ but whose left side is independent of dx . From this it follows that

$$(18) \quad \frac{y_1'}{y_1} + \frac{y_2'}{y_2} - \frac{1}{2} \left(a - \frac{1}{a} \right) \left(\frac{1}{y_1^2} - \frac{1}{y_2^2} \right) \leq 0.$$

Consider now a reflection in the y -axis taking C into a curve \bar{C} which is divided by the x -axis into an "upper" arc $z=z_1(x)$ and a "lower" arc $z=-z_2(x)$. With the lines $z=(1/a)$ and $z=-ax$ playing the roles of ξ_1 and ξ_2 , and with $\bar{b}_1, \bar{c}_1, \bar{b}_2, \bar{c}_2$ defined respectively by $(dx, (1/a) dx)$, $(2 dx, (2/a) dx)$, $(dx, -a dx)$, and $(2 dx, -2a dx)$, a repetition of the former argument leads to

$$(19) \quad \frac{z_1'}{z_1^2} + \frac{z_2'}{z_2^2} - \frac{dx}{2} \left(\frac{1}{a} - a \right) \left(\frac{1}{z_1^2} - \frac{1}{z_2^2} \right) \leq 0.$$

Since $z_i=y_i$ and $z_i'=-y_i'$, (19) is also

$$(20) \quad -\frac{y'_1}{y_1^2} - \frac{y'_2}{y_2^2} + \frac{dx}{2} \left(a - \frac{1}{a} \right) \left(\frac{1}{y_1^2} - \frac{1}{y_2^2} \right) \leq 0 .$$

Combining the opposite inequalities (18) and (20), we obtain

$$(21) \quad \frac{y'_1}{y_1^2} + \frac{y'_2}{y_2^2} - \frac{1}{2} \left(a - \frac{1}{a} \right) \left(\frac{1}{y_1^2} - \frac{1}{y_2^2} \right) = 0 .$$

Since (21) is an equality, it is clear that the same result would have been obtained if all preceding inequalities has been reversed. In other words (21) holds if p is a point of determinate curvature.

To express (21) in polar coordinates, let the polar axis be ξ_1 and let θ_0 designate the angle between the polar axis and the upper half-line of the y -axis. The angles of inclination to the x -axis of the tangent lines to C at $(0, y_1)$ and $(0, y_2)$ are α_1 and α_2 respectively and the clockwise angles from the radius vectors to the tangent lines at these points are ω_1 and ω_2 . From the standard relationships between polar and Cartesian coordinates, it follows that

$$(22) \quad \begin{aligned} y'_1(0) &= \tan \alpha_1 = -\cot \omega_1 = \left[-\frac{1}{r} \frac{dr}{d\theta} \right]_{\theta_0} \\ y'_2(0) &= -\tan \alpha_2 = \cot \omega_2 = \left[\frac{1}{r} \frac{dr}{d\theta} \right]_{\theta_0 + \pi} \end{aligned}$$

Also, by definition, $a = \cot \theta_0$ so $\frac{1}{2} \left(a - \frac{1}{a} \right) = \cot 2\theta_0$. Substituting this and

(22) in (21) we obtain

$$(23) \quad \left[-\frac{1}{r^3} \frac{dr}{d\theta} \right]_{\theta_0} + \left[\frac{1}{r^3} \frac{dr}{d\theta} \right]_{\theta_0 + \pi} - (\cot 2\theta_0) \left[\frac{1}{r^2(\theta_0)} - \frac{1}{r^2(\theta_0 + \pi)} \right] = 0 ,$$

and hence

$$(24) \quad \left[\frac{1}{r^3} \frac{dr}{d\theta} + \frac{1}{r^2} \cot 2\theta \right]_{\theta_0} = \left[\frac{1}{r^3} \frac{dr}{d\theta} + \frac{1}{r^2} 2\theta \right]_{\theta_0 + \pi} .$$

Multiplying both sides of (24) by $2 \csc 2\theta_0 = 2 \csc 2(\theta_0 + \pi)$ we have

$$(25) \quad \frac{d}{d\theta} \left(\frac{\csc 2\theta}{r^2} \right) \Big|_{\theta_0} = \frac{d}{d\theta} \left(\frac{\csc 2\theta}{r^2} \right) \Big|_{\theta_0 + \pi} .$$

Since (25) involves only first derivatives, it holds for all θ_0 for which r is differentiable at both θ_0 and $\theta_0 + \pi$. Since the one-sided derivative exists everywhere, we get the desired relations in (1), for all θ_0 , from the semi-continuity of the one sided derivative.

Proof of the Theorem. According to the corollary of Lemma 1 there is always a projective transformation such that, after the transformation,

p satisfies the conditions of Lemma 3. From (1) we obtain

$$(26) \quad \int_{\theta_0}^{\theta} d\left(\frac{\csc 2\theta}{r^2}\right) = \int_{\theta_0+\pi}^{\theta+\pi} d\left(\frac{\csc 2\theta}{r^2}\right),$$

where the integrals are Stieltjes integrals and the interval (θ_0, θ) does not contain a multiple of $\pi/2$. Hence

$$(27) \quad \frac{1}{r^2(\theta)} = -\frac{1}{r^2(\theta+\pi)} + k_j \sin 2\theta, \quad k_j = \text{constant}$$

where
$$(j-1)\frac{\pi}{2} \leq \theta \leq j\frac{\pi}{2}, (j=1, 2, 3, 4).$$

Since r is differentiable at the points for which $\theta=0, \pi/2, \pi, 3\pi/2$, we obtain from (27), upon differentiation at these points, the relations $k_1=k_2=k_3=k_4$. On the other hand, if we replace θ by $\theta+\pi$ in (27) we obtain the relations $k_1=-k_3$, and $k_2=-k_4$. In other words, $k_j=0$ and $r(\theta)=r(\theta+\pi)$. Since this shows p to be a metric center, it was initially a projective center.

The last statement in the theorem is well known (see [3] and e.g. [2, p.164]).

If a Hilbert metric is defined in the interior of an n -dimensional, convex surface S , the definitions for curvature and projective centers are unchanged. The metric for the space induces, on any plane through an interior point p , a two-dimensional Hilbert geometry. If p is a point of determinate curvature, it is a two-dimensional projective center for every plane through it. Since the L_p locus for every plane section is a line, it is easily seen that the total L_p locus must be a plane and hence that p is a projective center of S . If curvature is determinate everywhere then S is an ellipsoid and the geometry is hyperbolic.

It seems probable that a Hilbert geometry can contain no points of positive curvature.

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