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REMARKS ON THE MAXIMUM PRINCIPLE FOR PARABOLIC EQUATIONS AND ITS APPLICATIONS

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Introduction. In [3] Nirenberg has proved maximum principles, both weak and strong, for parabolic equations. In § 1 of this paper we give a generalization of his strong maximum principle (Theorem 1). Hopf [2] and Olainik [4] have proved that if $Lu \geq 0$ and L is a linear elliptic operator of the second order, if the coefficient of u in L is non-positive, and if u ($\neq \text{const.}$) assumes its positive maximum at a point P' (which necessarily belongs to the boundary) then $\partial u / \partial \nu < 0$, where ν is the inwardly directed normal. In § 2 we extend this result to parabolic operators (Theorem 2). A further discussion of the assumptions made in Theorem 2 is given in § 3. Application of Theorem 2 to the Neumann problem is given in § 4. In § 5 we apply the weak maximum principle to prove a uniqueness theorem for certain nonlinear parabolic equations with nonlinear boundary conditions, and thus extend the special case considered by Ficken [1]. An even more special case arises in the theory of diffusion (for references, see [1]).

1. Consider the operator

$$(1) \quad Lu \equiv \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i(x, t) \frac{\partial u}{\partial x_i} + a(x, t)u - \frac{\partial u}{\partial t}$$

with $a(x, t) \leq 0$. Here, $(x, t) = (x_1, \dots, x_n, t)$ varies in the closure \bar{D} of a given $(n+1)$ -dimensional domain D . Assume that L is parabolic in \bar{D} , that is, for every real vector $\lambda \neq 0$ and for every $(x, t) \in \bar{D}$ we have

$$\sum a_{ij}(x, t) \lambda_i \lambda_j > 0.$$

All the coefficients of L are assumed to be continuous in \bar{D} and u is assumed to be continuous in \bar{D} and to have a continuous t -derivative and continuous second x -derivatives in D . From [3; Th. 5] it follows that, under the above assumptions, *if $Lu \geq 0$ and if u assumes its positive maximum at an interior point P^0 , then $u \equiv \text{const.}$ in $S(P^0)$.* Here, $S(P^0)$ denotes the set of all points Q in D which can be connected to P^0 by a simple continuous curve in D along which the coordinate t is non-decreasing from Q to P^0 . In the following theorem we consider the case

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in which P^0 is a boundary point of D . We may assume that P^0 is the origin. Let $t=\varphi(x)$ be the equation of the boundary of D near P^0 . Assume that $t=0$ is the tangent hyperplane to the boundary of D at P^0 . Therefore $\partial\varphi/\partial x_i|_{P^0}=0$. Let D be on the side $t<\varphi(x)$.

THEOREM 1. *If $Lu\geq 0$ in D , if u assumes its positive maximum M at P^0 , if*

$$(2) \quad \lim_{P \rightarrow P^0} \frac{\partial u(P)}{\partial x_i} = 0, \quad \lambda = \lim_{P \rightarrow P^0} \sum a_{ij}(P) \frac{\partial^2 u(P)}{\partial x_i \partial x_j} \leq 0 \quad P \in D$$

and if

$$(3) \quad 1 + \sum a_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \bigg|_{P^0} > 0 \quad \varphi \in C''$$

then $u \equiv M$ in $S(P^0)$.

REMARK 1. Without making any use of (3) one can deduce the following:

Put $\mu = \limsup_{P \rightarrow P^0} \frac{\partial u(P)}{\partial t}$ ($P \in D$), then $\mu \geq 0$ since $\mu < 0$ will contradict $u(P^0) \geq u(P)$. Letting $P \rightarrow P^0$ in $Lu(P) \geq 0$ and using (2), we obtain $\lambda + a(P^0)M - \mu \geq 0$, from which it follows that $\lambda \geq 0$. Since, by (2), $\lambda \leq 0$, we conclude that $\lambda = 0$. Hence $a(P^0)M - \mu \geq 0$, from which it follows that $\mu \leq 0$ and, therefore, (since $\mu \geq 0$) $\mu = 0$. We also get $a(P^0) = 0$.

REMARK 2. The assumptions (2) and (3) can be verified if we assume that $\varphi(x) = o(|x|^2)$ and that u belongs to C'' in the closure of the domain $V \cap \{t < 0\}$, where V is some neighborhood of P^0 . Indeed, by making an appropriate orthogonal transformation we can assume that $a_{ij}(P^0) = \delta_{ij}$. By the mean value theorem we have

$$u(x, t) - u(0, 0) = \sum x_i \frac{\partial}{\partial x_i} u(\tilde{x}, \tilde{t}) + t \frac{\partial}{\partial t} u(\tilde{x}, \tilde{t}).$$

Taking $(x, t) \in \bar{D} \cap V \cap \{t < 0\}$ such that $|t| = o(|x|)$ and noting that $u(x, t) \leq u(0, 0)$, one can show that $\partial u(P^0)/\partial x_i = 0$. Noting that $\varphi(x) = o(|x|^2)$ and expanding $[u(x, t) - u(0, 0)]$ in terms of the first and second derivatives of u , one can show that $\partial^2 u(P^0)/\partial x_i^2 \leq 0$, and (2) is thereby proved. The proof of (3) is immediate.

PROOF OF THEOREM 1. For simplicity we shall prove the theorem only in case $n=1$; the proof of the general case is analogous. Lu takes the form

$$(4) \quad Lu \equiv A \frac{\partial^2 u}{\partial x^2} + a \frac{\partial u}{\partial x} + cu - \frac{\partial u}{\partial t} \quad c \leq 0, A > 0.$$

From the strong maximum principle [3; Th. 5] it follows that all we need to prove is that $u(P) \equiv M$ if $P \in V' \cap S(P^0)$ where V' is some neighborhood of P^0 .

There are two possibilities: Either there exists a sequence $\{P^k\}$ such that $P^k \in S(P^0)$, $P^k \rightarrow P^0$, $u(P^k) = M$, or there exists a neighborhood $V = \{x^2 + t^2 < R^2\}$ of P^0 such that $u(P) < M$ for all $P \in V \cap S(P^0)$, $P \neq P^0$. In the first case we can use [3; Th. 5] to conclude that $u(P) \equiv M$ if $P \in V' \cap S(P^0)$ where V' is some neighborhood of P^0 (since $u(P) = M$ for all $P \in S(P^k)$).

It remains therefore to consider the case in which $u(P) < M$ for all $P \in V \cap S(P^0)$, $P \neq P^0$. We shall prove that this case cannot occur by deriving a contradiction. Writing

$$\varphi(x) = Kx^2 + o(x^2),$$

we define a domain D_δ ($\delta > 0$) as the intersection of $S(P^0)$ with the set of points (x, t) in V for which

$$t < \tilde{\varphi}(x) = (K - \delta)x^2.$$

If $K < 0$ then, because of (3), we can choose δ sufficiently small such that

$$(5) \quad 1 + A \frac{\partial^2}{\partial x^2} \tilde{\varphi}(x)|_{x=0} > 0.$$

If $K \geq 0$, we can obviously take δ such that $K - \delta < 0$ and such that (5) holds.

We now can take R sufficiently small such that $\tilde{\varphi}(x) < \min(0, \varphi(x))$ for all (x, t) in D_δ , $x \neq 0$. Consequently, $u(x, t) < M$ if $t = \tilde{\varphi}(x)$, $x \neq 0$. The function $h(x, t) = -t + \tilde{\varphi}(x)$ vanishes on $t = \tilde{\varphi}(x)$ and is positive in D_δ . Therefore, if $\varepsilon > 0$ is sufficiently small, then $v = u + \varepsilon h$ is smaller than M at all points on the boundary of D_δ with the exception of P^0 , where $v(P^0) = M$. Noting that $\tilde{\varphi}'(0) = 0$ and using (5), we conclude that

$$Lh = 1 + A\tilde{\varphi}''(x) + a\tilde{\varphi}'(x) + ch > 0$$

if R has been chosen sufficiently small. Hence, $Lv = Lu + \varepsilon Lh > 0$. It follows that v cannot assume its positive maximum at interior points of D_δ and, therefore, it assumes its maximum M at P^0 . We thus obtain $\partial v / \partial t \geq 0$ at P^0 and, consequently,

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} - \varepsilon \frac{\partial h}{\partial t} \geq \varepsilon > 0$$

(Here

$$\frac{\partial g}{\partial t} = \lim_{t \rightarrow 0} \inf \frac{g(0, 0) - g(0, t)}{-t}.$$

On the other hand, letting in (4) $P \rightarrow P^0$ in an appropriate way and using (2) and the inequality $Lu(P) \geq 0$, we get

$$0 \leq \lim A(P) \frac{\partial^2 u(P)}{\partial x^2} + \lim a(P) \frac{\partial u(P)}{\partial x} + C(P^0)M - \limsup \frac{\partial u(P)}{\partial t} \leq -\limsup \frac{\partial u(P)}{\partial t}.$$

We have thus obtained

$$\limsup_{P \rightarrow P^0} \partial u(P) / \partial t \leq 0 < \varepsilon \leq \partial u / \partial t.$$

This is however a contradiction (since

$$\frac{\partial u}{\partial t} = \lim_{t_k \rightarrow 0} \frac{\partial u(0, t_k)}{\partial t} \leq \limsup_{P \rightarrow P^0} \frac{\partial u(P)}{\partial t}$$

for an appropriate sequence $\{t_k\}$), and the proof is completed.

REMARK (a) Consider the following example: $n=1$, $P^0=(0, 0)$ and D defined by

$$x^2 + t^2 < R, \quad t < \gamma_1 x, \quad t < \gamma_2 x \quad \gamma_1 > 0 > \gamma_2.$$

The function $u(x, t) = (t - \gamma_1 x)(\gamma_2 x - t)$ satisfies the following properties: $u < 0$ in D , $u = 0$ at P^0 , and

$$Lu \equiv A \frac{\partial^2 u}{\partial x^2} + a \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = -2A\gamma_1\gamma_2 + 0(|x| + |t|) \geq 0,$$

provided R is sufficiently small. Consequently, (3) and the second assumption in (2) are not satisfied and also the assertion of Theorem 1 is false.

REMARK (b). Consider now the case in which the tangent hyperplane at P^0 is not of the form $t = \text{const.}$. We shall prove that in this case Theorem 1 is false. Take $n=1$ and consider first the case in which D is defined by

$$x > 0, \quad x^2 + t^2 < R^2.$$

If $Lu \equiv \partial^2 u / \partial x^2 - \partial u / \partial t$, then the function $u(x, t) = -x$ takes its maximum in \bar{D} at $P^0 = (0, 0)$, $Lu = 0$, but $u \neq 0$ in $S(P^0)$.

Consider next the case in which \bar{D} is defined by

$$x > \alpha t, \quad x^2 + t^2 < R^2.$$

and take $Lu = \partial^2 u / \partial x^2 - \alpha \partial u / \partial x - \partial u / \partial t$.

The transformation $t'=t, x'=x-\alpha t$ carries the present case into the previous one.

Note that if the tangent hyperplane H at P^0 is not the plane $t=0$ and the axes are rotated so as to give H the equation $t'=0$ (in new x', t' coordinate), then Lu loses the form (1), for $u_{x' t'}$ and $u_{t' t'}$ will appear in it.

REMARK (c). If in Theorem 1 the domain D is on the side $t > \varphi(x)$, then the theorem is false. Indeed, as a counter-example take $u = -t$, and D bounded from below by $t=0$.

2. Consider the linear operator

$$(6) \quad \begin{aligned} L'u = & \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i,j=1}^m b_{ij}(x, t) \frac{\partial^2 u}{\partial t_i \partial t_j} + \sum_{i=1}^n a_i(x, t) \frac{\partial u}{\partial x_i} \\ & + \sum_{i=1}^m b_i(x, t) \frac{\partial u}{\partial t_i} + a(x, t)u \end{aligned} \quad a(x, t) \leq 0,$$

where $x = (x_1, \dots, x_n)$ and $t = (t_1, \dots, t_m)$ vary in the closure of a given $(n+m)$ -dimensional domain D . We assume that L' is elliptic in the variables x and parabolic in the variables t , that is, for every real vector $\lambda \neq 0$,

$$(7) \quad \sum a_{ij} \lambda_i \lambda_j > 0, \quad \sum b_{ij} \lambda_i \lambda_j \geq 0.$$

All the coefficients appearing in (6) are assumed to be continuous in \bar{D} and u is assumed to be continuous in \bar{D} and to have a continuous t -derivative and continuous second x -derivatives in D . Under these assumptions, Nirenberg [3; Th. 2] has proved a weak maximum principle from which it follows that, if $L'u \geq 0$ in D then u must assume its positive maximum on the boundary.

Let $P^0 = (x^0, t^0)$ be a point on the boundary of D such that $u(P^0) = M > 0$ is the maximum of u in \bar{D} . Assume that there exists a neighborhood $V: |x - x^0|^2 + |t - t^0|^2 < R_0^2$ of P^0 such that $u(x, t) < M$ in $V \cap D$. We then can prove the following theorem.

THEOREM 2. *If there exists a sphere $S: |x - x'|^2 + |t - t'|^2 < R^2$ passing through P^0 and contained in \bar{D} , and if $x^0 \neq x'$ then, under the assumptions made above (in particular, $L'u \geq 0, u(x, t) < M$ in $V \cap D$), every non-tangential derivative $\partial u / \partial \tau$ at (x^0, t^0) , understood as the limit inferior of $\Delta u / \Delta \tau$ along a non-tangential direction τ , is negative.*

By a non-tangential direction we mean a direction from P^0 into the interior of the sphere S .

REMARK (a). If $a(x, t) \equiv 0$ then the assumption $M > 0$ is superfluous.

REMARK (b). In § 3 we shall show that the assumption $x^0 \neq x'$ is essential. We shall also discuss the case in which $u(x, t)$ is not smaller than M at all the points of $V \cap D$.

Proof. For simplicity we give the proof in the case $m = n = 1$, so that

$$(8) \quad L'u \equiv A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial t^2} + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial t} + cu \quad A > 0, B \geq 0, c \leq 0;$$

the proof of the general case is quite similar. Without loss of generality we can take $(x', t') = (0, 0)$ and $x^0 > 0$. Furthermore, we may assume that, with the exception of P^0 , S lies in $V \cap D$, so that $u(x, t) < M$ in $S - P^0$. Denote by C the intersection of S with the plane $x > \delta$, where $0 < \delta < x^0$. The function

$$h(x, t) = \exp(-\alpha(x^2 + t^2)) - \exp(-\alpha R^2)$$

satisfies the following properties: $h = 0$ on the boundary of S , $h \geq 0$ in C ; if α is large enough, then

$$L'h = \exp(-\alpha(x^2 + t^2))[4\alpha^2(Ax^2 + Bt^2) - 2\alpha(A + B + ax + bt) + c] - c \exp(-\alpha R^2) > 0.$$

(Here we used $x > \delta > 0$, $c \leq 0$.)

If ε is sufficiently small, then the function $v = u + \varepsilon h$ is smaller than M at all points of the boundary of C with the exception of P^0 , where $v(P^0) = M$. Since $L'v = L'u + \varepsilon L'h > 0$, v cannot assume its positive maximum in \bar{C} at the interior of C (since, otherwise, at such interior points $L'v$ would be non-positive). Hence, v assumes its maximum at P^0 and, consequently, $\partial v / \partial \tau = \liminf (\Delta v / \Delta \tau) \leq 0$. Since along the normal ν (i. e., along the radius through P^0) $\partial h / \partial \nu > 0$ and since along the tangential direction σ $\partial h / \partial \sigma = 0$, it follows that $\partial h / \partial \tau > 0$. Using the definition of v , we conclude that $\partial u / \partial \tau = \partial v / \partial \tau - \varepsilon \partial h / \partial \tau < 0$, and the proof is completed.

Added in proof. Theorem 2 was recently and independently proved also by R. Viborni, *On properties of solutions of some boundary value problems for equations of parabolic type*, Doklody Akad. Nauk SSSR, 117 (1957), 563-565.

3. From now on we shall consider only parabolic operators of the form (1). Suppose the assumption $u < M$ in $V \cap D$, made in Theorem 2, is replaced by $u \leq M$. If there exists a sequence of points $\{P^k\}$ such

that $P^k \rightarrow P^0$, $P^k \in D$, $P^k = (x^k, t^k)$ and $t^k \geq t^0$, $u(P^k) = M$, then, by [3; Th. 5], $u \equiv M$ in $S(P^k)$. Hence, if the boundary of D near P^0 is sufficiently smooth, $u \equiv M$ in some set $V' \cap D$ where V' is some neighborhood of P^0 . Consequently $\partial u / \partial \tau = 0$ for every τ .

If $u(P) \leq M$ for all $P \in V \cap D$, if $u(P)$ is not strictly smaller than M for all $P \in V \cap D$, $P \neq P^0$, and if the previous situation does not arise, then one and only one of the following cases must occur:

(i) $u < M$ at all points (x, t) in $V \cap D$ with $t \geq t^0$. Using [3; Th. 5] one can easily conclude that there exists a neighborhood V' of P such that $u < M$ in $V' \cap D$, and Theorem 2 remains true.

(ii) $u < M$ at all points (x, t) in $V \cap D$ with $t > t_0$ and $u \equiv M$ at all points (x, t) in $V \cap D$ with $t \leq t_0$. We then consider only those directions τ along which $u < M$. We claim that *Theorem 2 is not true for the present situation*. To prove this, consider the following simple counter-example:

$$P^0 = (0, 0), M = 0, Lu = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}, u(x, t) = \begin{cases} -t^2 & \text{if } t > 0 \\ 0 & \text{if } t < 0. \end{cases}$$

u satisfies $Lu \geq 0$ and assumes its maximum 0 for $t \leq 0$. But, the derivative $\partial u / \partial \tau$ at $P^0 = (0, 0)$, along any direction τ , is zero.

As another counter-example (with $Lu = 0$) one can take a fundamental solution of the heat equation.

Note that the preceding counter-examples are valid without any assumptions on the behavior of the boundary of D near P^0 .

We shall now consider the case $x^1 = x^0$ which was excluded by the assumptions of Theorem 2. We shall assume that at $P^0 = (0, 0)$ there passes a tangent hyperplane $t = 0$. If D is above this hyperplane, then the preceding counter-examples show that Theorem 2 is not true. It remains to consider the case in which D is "essentially" below $t = 0$, that is, if we denote by $t = \varphi(x)$ the equation of the boundary of D near P^0 , then D is on the side $t < \varphi(x)$. In this case, however, Theorem 1 tells us that in general we cannot assume both $u(P^0) = \max u(P) > 0$ ($P \in \bar{D}$) and $u < u(P^0)$ in $V \cap D$.

The example in § 1 Remark (a) can also serve as a counter-example to Theorem 2 in case P^0 is a vertex-point. Indeed, along the t -direction

$$\frac{\partial u}{\partial t} \Big|_{P^0} = \frac{\partial}{\partial t} [(t - \gamma_1 x)(\gamma_2 x - t)] \Big|_{x=0, t=0} = 0.$$

By a small modification of this counter-example one can get a counter-example to the analogue of Theorem 2 for elliptic operators [2] [4] in case P^0 is a vertex. Indeed, define D by

$$x^2 + y^2 < R^2, y < \gamma_1 x, y > \gamma_2 x \quad \gamma_1 > 0 > \gamma_2,$$

and take $Lu=\partial^2u/\partial x^2+A\partial^2u/\partial y^2$, where $A>|\gamma_1\gamma_2|$. The function $u(x,y)=(y-\gamma_1x)(y-\gamma_2x)$ satisfies: $u<0$ in D , $u=0$ at the origin, $Lu=2\gamma_1\gamma_2+2A>0$. But along any direction τ within D , $\partial u/\partial \tau|_{x=0,y=0}=0$.

4. Let D be a domain bounded by the two hyperplanes $t=0, t=T>0$ and a surface B between them. Assume that the intersection $\{t=T\}\cap \bar{D}$ is the closure of an open set on $t=T$, and denote by A the boundary of D on $t=0$. The Neumann problem for the parabolic equation $Lu=0$ consists in finding a solution to the equation $Lu=0$ which satisfies the following initial and boundary conditions :

$$u=f \text{ on } A, \quad \frac{\partial u}{\partial \nu}=g \text{ on } B$$

(f, g are given functions).

From Theorem 2 and from the strong maximum principle [3 ; Th. 5] we conclude: *If for every point $P^0=(x^0, t^0)$ of B (i) there exists a sphere with center (x', t') , $x'\neq x^0$, passing through P^0 and contained in \bar{D} , and (ii) $\overline{S(P^0)}$ contains interior points of A , then the Neumann problem has at most one solution.* Clearly, this uniqueness property holds also for the more general problem where $\partial u/\partial \nu$ is replaced by $\partial u/\partial \tau$ and τ is a non-tangential direction which varies on B .

As another application to Theorem 2, one can deduce the positivity of $\partial G/\partial \nu$, where G is the Green's function of $Lu=0$.

5. Let D be a domain bounded by $t=0, t=T(0<T\leq \infty)$ and surfaces $\Gamma_k, 0\leq k\leq m, \Gamma_0$ being the outer boundary. Suppose further that the intersection of each Γ_k with $t=t_0(0\leq t_0<T)$ is a simple closed curve $\gamma_k(t_0)$ which belongs to $C^{(3)}$ and does not reduce to a single point. Write $u_{x_i}=\partial u/\partial x_i, u_t=\partial u/\partial t$. We shall consider the following problem P :

$$(9) \qquad \sum_{i,j=1}^n a_{ij}(x,t)u_{x_ix_j}-u_t=c(x,t,u,\nabla u)$$

(where ∇u denotes the vector $\partial u/\partial x_i$),

$$(10) \qquad \frac{\partial u}{\partial \tau} \equiv \sum_{i=1}^n \alpha_i(x,t)u_{x_i} + \alpha(x,t)u_t = \varphi(x,t,u) \quad (x,t) \in \Gamma = \sum_{k=0}^m \Gamma_k$$

$$(11) \qquad u(x,0)=\psi(x) \text{ on } A \qquad A=\bar{D} \cap \{t=0\}$$

We make the following assumptions :

- (a) $a_{ij}(x,t)$ is continuous in \bar{D} ; $c(x,t,u,\nabla u)$ and it first derivatives with respect to $u, \nabla u$ are continuous for $(x,t) \in \bar{D}$ and for all values of $u, \nabla u$.

- (b) φ and $\partial\varphi/\partial u$ are continuous for all $(x, t) \in \Gamma$ and for all u .
 (c) $\alpha_i(x, t)$, $\alpha(x, t)$ are continuous for $(x, t) \in \Gamma$; $\psi(x)$ is continuous in A .
 (d) (9) is parabolic in \bar{D} , that is, there exists a positive constant δ such that

$$(12) \quad \sum a_{ij}(x, t) \xi_i \xi_j \geq \delta \sum \xi_i^2$$

holds for all real ξ and for all $(x, t) \in \bar{D}$.

- (e) On each surface Γ_k ($k=0, 1, \dots, m$) either all the directions $\tau=(\alpha_i, \alpha)$ are exterior or all are interior, and in the exterior case $\alpha \geq 0$ and the directions $(\alpha_i, 0)$ are exterior while in the interior case $\alpha \leq 0$ and the directions $(\alpha_i, 0)$ are interior.

Denote by \sum the class of functions $u(x, t)$ defined and continuous in \bar{D} and satisfying the following conditions:

- (α) $\partial u/\partial t$, $\partial u/\partial x_i$, $\partial^2 u/\partial x_i \partial x_j$ are continuous in D ;
 (β) For every $R > 0$, $\partial u/\partial x_i$ is bounded in $D \cap \{|x|^2 + t^2 < R^2\}$.

THEOREM 3. *Under the assumptions (a)–(e) the problem P cannot have two different solutions in the class \sum .*

We shall need the following lemma.

LEMMA. *There exists a function $\zeta(x)$ defined in A and having the following properties: (i) ζ has continuous first derivatives in A and continuous second derivatives in the interior of A ; (ii) $\partial\zeta/\partial\nu = -1$ and $\partial\zeta/\partial\mu = 0$ on $\gamma_0(0), \dots, \gamma_m(0)$, where $\partial/\partial\nu$ and $\partial/\partial\mu$ denote the derivatives with respect to the interior normal and to any tangential direction, respectively.*

PROOF OF THE LEMMA. It will be enough to construct a function $\chi_0(x)$ which is C'' in A , which vanishes in a neighborhood of $\gamma_i(0)$ ($i=1, \dots, m$) and for which $\partial\chi_0/\partial\nu = -1$, $\partial\chi_0/\partial\mu = 0$ along $\gamma_0(0)$; constructing $\gamma_1(x)$ in a similar manner, we can then take $\zeta(x) = \sum \chi_i(x)$. Since $\gamma_0(0)$ belongs to $C^{(3)}$, the normals issuing from $\gamma_0(0)$ and inwardly directed cover in a one-to-one manner a small inner neighborhood of $\gamma_0(0)$, call it A_0 . To each point x in A_0 there corresponds a unique point x^0 on the boundary of $\gamma_0(0)$, such that x lies on the normal through x^0 . Denote by $\sigma(x)$ the distance $|x - x^0|$. It is elementary to show that $\sigma(x)$ has continuous second derivatives in A_0 . Denote by A_1 the domain $0 \leq \sigma \leq \varepsilon_0$, where $\varepsilon_0 > 0$ is small enough so that $A_1 \subset A_0$. The function

$$\chi_0(x) = \begin{cases} \frac{1}{3\varepsilon_0^2}(\varepsilon_0 - \sigma(x))^3 & \text{if } x \in \bar{A}_1 \\ 0 & \text{if } x \in A - A_1 \end{cases}$$

belongs to C'' in A and satisfies: $\partial\chi_0/\partial\nu = \partial\chi_0/\partial\sigma = -1$ and $\partial\chi_0/\partial\mu = 0$ on $\gamma_0(0)$, and χ_0 vanishes near $\gamma_k(0)$, ($1 \leq k \leq m$); the proof is completed.

PROOF OF THEOREM 3. We first consider the case $n > 1$. We may suppose that the vectors (α_i, α) are exterior directions on $\Gamma_0, \dots, \Gamma_q$ and that (α_i, α) are interior directions on $\Gamma_{q+1}, \dots, \Gamma_m$. Suppose now that u and v are two solutions in Σ of the problem P , and define $w = v - u$. Writing

$$\begin{aligned} C(x, t, u, v) &= \int_0^1 \frac{\partial}{\partial u} c(x, t, u + \lambda w, \nabla u + \lambda \nabla w) d\lambda \\ C_i(x, t, u, v) &= \int_0^1 \frac{\partial}{\partial u_{x_i}} c(x, t, u + \lambda w, \nabla u + \lambda \nabla w) d\lambda \\ \Phi(x, t, u, v) &= \int_0^1 \frac{\partial}{\partial u} \varphi(x, t, u + \lambda w) d\lambda \end{aligned}$$

and using (9), (10) and (11), we obtain for w the following system :

$$(13) \quad \sum a_{ij} w_{x_i x_j} - w_t = Cw + \sum C_i w_{x_i}$$

$$(14) \quad \frac{\partial w}{\partial \tau} = \sum \alpha_i w_{x_i} + \alpha w_t = \Phi w$$

$$(15) \quad w(x, 0) = 0.$$

Substituting $w(x, t) = z(x, t) \exp(Kt + M\zeta(x))$, where $\zeta(x)$ is the function constructed in the lemma and K, M are constant to be determined later, we get for z the following system :

$$\begin{aligned} (13') \quad \sum a_{ij} z_{x_i x_j} - z_t &= -M \sum a_{ij} \zeta_{x_i x_j} z - M^2 \sum a_{ij} \zeta_{x_i} \zeta_{x_j} z \\ &\quad - 2M \sum a_{ij} \zeta_{x_i} z_{x_j} + Kz + Cz + M \sum C_i \zeta_{x_i} z + \sum C_i z_{x_i} \end{aligned}$$

$$(14) \quad \frac{\partial z}{\partial \tau} = \sum \alpha_i z_{x_i} + \alpha z_t = -M \sum \alpha_i \zeta_{x_i} z - \alpha Kz + \Phi z$$

$$(15') \quad z(x, 0) = 0.$$

If $0 \leq k \leq q$, $\alpha \geq 0$ and $\sum \alpha_i(x, 0) \zeta_{x_i}(x) > 0$ on $\gamma_k(0)$, since the angle between the vectors (α_i) and $\text{grad } \zeta$ is $< \pi/2$. By continuity we get $\sum \alpha_i(x, t) \zeta_{x_i}(x) \geq \eta > 0$ on $\gamma_k(t)$, provided $0 \leq t \leq T'$ and T' is sufficiently small. Hence, we can choose M sufficiently large such that

$$(16) \quad -M \sum \alpha_i \zeta_{x_i} - \alpha K + \Phi < 0$$

holds on $\gamma_k(t)$, provided $K \geq 0$ and $0 \leq t \leq T'$.

If $q+1 \leq k \leq m$, $\alpha \leq 0$ and $\sum \alpha_i(x, 0) \zeta_{x_i}(x) < 0$, since the angle between (α_i) and $-\text{grad } \zeta$ is $< \pi/2$. Again, if $K \geq 0$ and M is sufficiently large, then

$$(17) \quad -M \sum \alpha_i \zeta_{x_i} - \alpha K + \Phi > 0$$

on $\gamma_k(t)$, $0 \leq t \leq T'$.

Having fixed M , we now choose K sufficiently large so that the coefficient of z on the right side of (13') becomes positive in the domain $D_{T'} = D \cup \{0 < t < T'\}$. We claim that $z \equiv 0$ in $D_{T'}$. Indeed, if this is not the case then, using (15') and the weak maximum principle [3; Th. 2] we conclude that z assumes either its positive maximum or its negative minimum on the boundary $\sum_{k=0}^m \gamma_k(t)$, $0 \leq t \leq T'$, of $D_{T'}$. It will be enough to consider the case in which z assumes its positive maximum at a point P^0 on $\gamma_k(t)$. If $0 \leq k \leq q$, then $\partial z / \partial \tau \geq 0$ since τ is outwardly directed. On the other hand, using (14') and (16) we get $\partial z / \partial \tau < 0$, which is a contradiction. If $q+1 \leq k \leq m$, then $\partial z / \partial \tau \leq 0$ since τ is inwardly directed. On the other hand, using (14') and (17) we get $\partial z / \partial \tau > 0$ which is a contradiction. We have thus proved that $z \equiv w \equiv 0$ in $D_{T'}$. We can now apply a classical procedure of continuation and thus complete the proof of the theorem for the case $n > 1$.

In the case $n=1$, $\Gamma = \Gamma_0$ is composed of two curves Γ_{01} and Γ_{02} . Suppose Γ_{0k} intersects $t=0$ at a_k , $a_1 < a_2$. The function

$$\zeta(x) = \frac{(x-a_1)(x-a_2)}{a_2-a_1}$$

can be used in the preceding proof. Note that it is not necessary to make any assumptions on the smoothness of the curves Γ_{0k} .

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