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A NOTE ON A PAPER OF L. GUTTMAN

ANCEL CLYDE MEWBORN

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A. C. MEWBORN

In a recent paper L. Guttman [2] obtained, using a result of von Neumann on the theory of games, lower bounds for the largest characteristic root of the matrix AA' where A is a real matrix of order $m \times n$. As Guttman points out his bounds are non-trivial only if some row or column of A has only positive or only negative elements. I wish to show that Guttman's results, and even a better result, are an immediate corollary of a well known theorem on Hermitian matrices: that each diagonal element lies between the smallest and largest characteristic roots (see e.g. [1]). Moreover, if AA' be replaced by AA^* then A can be real or complex and a non-trivial result is always obtained.

THEOREM 1. *Let $A=(a_{ij})$ be an $m \times n$ matrix with real or complex elements. Let λ be the largest characteristic root of the $m \times m$ non-negative definite Hermitian matrix $B=AA^*=(b_{ij})$. Then*

$$\lambda \geqq \max_i \sum_{j=1}^{n} |a_{ij}|^2 \tag{1}$$

$$\lambda \geqq \max_j \sum_{i=1}^{m} |a_{ij}|^2 \tag{2}$$

Proof. Let b_{rr} be the largest diagonal element of B. Then

$$\lambda \geqq b_{rr} = \sum_{j=1}^{n} |a_{rj}|^2 = \max_i \sum_{j=1}^{n} |a_{ij}|^2 ,$$

and (1) is proved. Now the non-zero characteristic roots of AA^* are the same as those of A^*A. Then (2) follows as above if we consider A^*A instead of AA^*.

The bounds in (1) and (2) can be replaced by the weaker bounds

$$\lambda \geqq n \cdot \max_i \left(\min_j |a_{ij}|^2 \right) \tag{3}$$

$$\lambda \geqq m \cdot \max_j \left(\min_i |a_{ij}|^2 \right) \tag{4}$$

respectively, and even these bounds are obviously better than Guttman's.

Theorem 1 can be improved further.

THEOREM 2. *Under the hypotheses of Theorem* 1 *we have*

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$$(5)\quad 2\lambda \geqq \max_{i,j}\left[\sum_{\nu=1}^{n}(|a_{i\nu}|^2+|a_{j\nu}|^2)+\left\{\left[\sum_{\nu=1}^{n}(|a_{i\nu}|^2-|a_{j\nu}|^2)\right]^2+4\left|\sum_{\nu=1}^{n}a_{i\nu}\bar{a}_{j\nu}\right|^2\right\}^{1/2}\right]$$

$$(6)\quad 2\lambda \geqq \max_{i,j}\left[\sum_{\nu=1}^{m}(|a_{\nu i}|^2+|a_{\nu j}|^2)+\left\{\left[\sum_{\nu=1}^{m}(|a_{\nu i}|^2-|a_{\nu j}|^2)\right]^2+4\left|\sum_{\nu=1}^{m}\bar{a}_{\nu i}a_{\nu j}\right|^2\right\}^{1/2}\right]$$

Proof. It was shown in [1] that the largest root of an Hermitian matrix is greater than or equal to the larger of the two roots of any principal minor of order two of the matrix. Suppose the principal minor or order two of B having the largest root lies in the r, s rows and columns of B. Then

$$\begin{aligned}2\lambda &\geqq b_{rr}+b_{ss}+[(b_{rr}-b_{ss})^2+4|b_{rs}|^2]^{1/2}\\ &=\sum_{\nu=1}^{n}(|a_{r\nu}|^2+|a_{s\nu}|^2)+\left\{\left[\sum_{\nu=1}^{n}(|a_{r\nu}|^2-|a_{s\nu}|^2)\right]^2+4\left|\sum_{\nu=1}^{n}a_{r\nu}\bar{a}_{s\nu}\right|^2\right\}^{1/2}\end{aligned}$$

and (3) follows. (4) is proved similarly by considering A^*A instead of B.

References

1. Alfred Brauer and A. C. Mewborn, *Intervals for the characteristic roots of an Hermitian matrix*, Elisha Mitchell Scien. Soc., **73** (1957), 247-254.

2. Louis Guttman, *Some inequalities between latent roots and minimax (maximin) elements of real matrices*, Pacific J. Math., **7** (1957), 897-902.

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Pacific Journal of Mathematics
Vol. 8, No. 2 April, 1958

John Herbert Barrett, *Second order complex differential equations with a real independent variable* 187
Avner Friedman, *Remarks on the maximum principle for parabolic equations and its applications* 201
Richard Robinson Goldberg, *An inversion of the Stieltjes transform* 213
Olavi Hellman, *On the periodicity of the solution of a certain nonlinear integral equation* 219
Gilbert Helmberg, *A theorem on equidistribution on compact groups* 227
Lloyd Kenneth Jackson, *Subfunctions and the Dirichlet problem* 243
Naoki Kimura, *The structure of idempotent semigroups. I* 257
Stephen Kulik, *A method of approximating the complex roots of equations* 277
Ancel Clyde Mewborn, *A note on a paper of L. Guttman* 283
Zeev Nehari, *On the principal frequency of a membrane* 285
G. Pólya and I. J. Schoenberg, *Remarks on de la Vallée Poussin means and convex conformal maps of the circle* 295
B. M. Stewart, *Asymmetry of a plane convex set with respect to its centroid* 335
Hans F. Weinberger, *Lower bounds for higher eigenvalues by finite difference methods* 339
Edwin Weiss and Neal Zierler, *Locally compact division rings* 369
Bertram Yood, *Homomorphisms on normed algebras* 373