

Pacific Journal of Mathematics

LOCALLY COMPACT DIVISION RINGS

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Let K be a division ring with a non-discrete topology T with respect to which both the additive group K^+ and the multiplicative group K^* of K are locally compact topological groups.¹ If m is Haar measure for K^+ and $a \in K$, the function $m'(E) = m(aE)$ is clearly an invariant Borel measure for K^+ . Hence there exists a real number $\phi(a)$ such that $m'(E) = \phi(a)m(E)$ for all Borel subsets E of K^+ . The real-valued function ϕ on K (which is essentially the Radon-Nikodym derivative of m with respect to left-invariant Haar measure on K^*) evidently has the first two of the following three properties.

- (1) $\phi(a) \geq 0$; $\phi(a) = 0$ if and only if $a = 0$.
- (2) $\phi(ab) = \phi(a)\phi(b)$.
- (3) There exists $M > 0$ such that $\phi(a) \leq 1$ implies $\phi(1+a) \leq M$.

We shall show that ϕ satisfies (3) also, i. e., is a valuation for K , and that the topology T_ϕ for K defined by ϕ coincides with T .² The classification of K then follows from known results.

LEMMA 1. ϕ is continuous.

Proof. Let ε be a positive number and let E be a compact set of positive measure. By the regularity of Haar measure we may choose an open set U containing E such that $m(U) - m(E) < \varepsilon m(E)$. Choose a neighborhood V of 1 with $V = V^{-1}$ and $V \cdot E \subset U$. Then for x in V , $\phi(x) = m(xE)/m(E) \leq m(U)/m(E) < 1 + \varepsilon$; since $x^{-1} \in V$, $\phi(x) = (\phi(x^{-1}))^{-1} > (1 + \varepsilon)^{-1}$. Hence $1 - \varepsilon < \phi(x) < 1 + \varepsilon$ and the continuity of ϕ on K^* follows.* Now choose an open set U with $m(U) < \varepsilon m(E)$ and a neighborhood V of 0 with $V \cdot E \subset U$. Then for a in V , $\phi(a) = m(aE)/m(E) \leq m(U)/m(E) < \varepsilon$ and ϕ is continuous at 0.

LEMMA 2. $S = \{a \in K : \phi(a) \leq 1\}$ is compact.

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¹ Continuity of the inverse multiplicative operation need not be assumed; cf. the concluding remark. The continuity of multiplication implies that $a \rightarrow -a = (-1) \cdot a$ is continuous.

² This idea was suggested by some work of Tate, [12].

* Cf. Halmos [3, §60.6, p. 265].

Proof. Let C be a compact neighborhood of 0 and choose a neighborhood V of 0 such that $V \cdot C \subset C$. Let $a \in V \cap C$ such that $0 < \phi(a) < 1$. If $a^n S \subset C$ holds for no $n = 1, 2, \dots$, we select for each n an $s_n \in S$ such that $a^n s_n \notin C$. Since $\phi(a^k) \rightarrow 0$ and all the a^k lie in the compact set C , $a^k \rightarrow 0$ and hence $a^k s_n \in C$ for sufficiently large k . We may therefore choose $k_n \geq n$ such that $a^{k_n} s_n \notin C$ but $a^{k_n+1} s_n \in C$. Then the sequence $\{a^{k_n} s_n\}$ of elements of the compact set $a^{-1}C$ has a cluster point c in $a^{-1}C$. Hence $\phi(a^{k_n} s_n) = \phi(a)^{k_n} \phi(s_n) \leq \phi(a)^{k_n}$ has $\phi(c)$ as a cluster point by the continuity of ϕ ; thus $\phi(c) = 0$ and $c = 0$, which contradicts $a^{k_n} s_n \notin C$. It follows that S is a subset of the compact set $a^{-n}C$ for some n and so, being closed by virtue of the continuity of ϕ , is compact.

COROLLARY. ϕ is a valuation.

Proof. $\phi(1+S)$, the continuous image of the compact set $1+S$, is bounded.

LEMMA 3. $T_\phi = T$.

Proof. Let $V \in T - \{\phi\}$, $a \in V$ and $B_n = \{b \in K : \phi(b-a) < 2^{-n}\}$. Suppose we can choose $b_n \in B_n$ with $b_n \notin V$ for each $n = 1, 2, \dots$. But then the points $b_n - a$, all of which lie in the compact set S , have a cluster point c in S which must be 0 since $\phi(c) = 0$. Hence $b_n \rightarrow a$ contrary to our assumption and it follows that $T \subset T_\phi$. Since the opposite inclusion is an immediate consequence of the continuity of ϕ , the proof is complete.

If K is connected³, it is the real, complex or quaternion field (Pontrjagin [10]); in particular, ϕ is archimedean. Conversely, if ϕ is archimedean, the theorem of Ostrowski [8, p. 278] asserts that the center of K is either the real or complex field and so K , not being totally disconnected, is connected.⁵

If K is totally disconnected, ϕ is non-archimedean (and conversely, according to the above) and results due to van Dantzig [2], Hasse [4], Hasse and Schmidt [5], Jacobson and Taussky [6] and Jacobson [7] assert that K is of one of the following three types;⁴

- (i) the completion of an algebraic number field at a finite prime,
- (ii) the completion of an algebraic function field in one variable

³ K is either connected or totally disconnected: if the component C of 0 contains $a \notin 0$ then $ba^{-1}C$ is a connected set containing 0 and $b \in K$.

⁴ Otake [9] shows that $a \rightarrow a^{-1}$ need not be assumed to be continuous; cf. our final remark in this connection.

⁵ Alternatively, if K is connected, it is not difficult to show that ϕ is archimedean; then K is a vector space over the reals (Ostrowski) with ϕ as a norm, hence is the real, complex or quaternion field (Arens [1] Tornheim [13]), proving Pontrjagin's theorem.

- over a finite field H ,
- (iii) a division ring D obtained from a field F of type (ii) by redefining x . $a = a^\sigma \cdot x$, $a \in H$, σ a fixed non-trivial automorphism of H , the elements of D and F being regarded as power series $\sum_{i=n}^{\infty} a_i x^i$ in an indeterminate x over H with coefficients in H .

REMARK. Continuity of $a \rightarrow a^{-1}$ need not be assumed, for it appears in the connected case only in the proof that K is not compact in the proof of the Pontrjagin theorem [11, p. 173, Theorem 45.]. If K were compact, $\phi(a) = m(aK)/m(K) \leq 1$ for all $a \in K$. But, as in the proof of the continuity of ϕ at 0 in Lemma 1, we can find $a \in K$ such that $0 < \phi(a) < 1$; then $\phi(a^{-1}) > 1$ and it follows that K is not compact. If K is totally disconnected we have only to apply to T, K^* the following unpublished theorem of A. M. Gleason: Let G be a group with a totally disconnected topology T under which the group operation is continuous from $G \times G$ to G . Then T, G is a topological group.

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