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DISTAL TRANSFORMATION GROUPS

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Let X be a topological space and G a group of homeomorphisms of X onto itself. Then G is said to be *distal* if given any three points x, y, z in X and any filter $\mathscr{F}$ on G, then $x\mathscr{F} \to z$ and $y\mathscr{F} \to z$ implies that $x = y$. The above definition of distal is a topological variant of the one given in [2]; the two notions coincide when the underlying space X is compact.

This paper deals with two topics in the study of distal transformation groups. First, a recursive characterization of these groups is given in a general setting, and second it is shown that under suitable restrictions on X and G, distal is a property strong enough to imply equicontinuity of G. In order to make this statement precise a few definitions are needed. For a complete discussion of the following notions, the reader is referred to [2].

Let a, b be functions of X into X and let $x \in X$. Then xa will denote the image of x under a, and ab the composite function first a then b. Under the operation of composition X^X is a semigroup such that the maps $b \to ab\ (b \in X^X)$ are continuous for all $a \in X^X$, and the maps $b \to ba\ (b \in X^X)$ are continuous for all continuous functions a of X into X. The group G may be regarded as a subset of X^X and its closure T formed. One may also consider S the closure of G in the topology of uniform convergence on X. When X is compact, S is a topological group of homeomorphisms of X onto X but is in general not compact, whereas T is compact but is in general not a group. Hence in studying T instead of S the emphasis is on the algebraic rather than the topological structure.

A subset A of G is said to be *syndetic* if there exists a compact subset K of G such that $AK = G$. (If no topology is specified for G, then it is assumed to be provided with the discrete topology.) A point $x \in X$ is an *almost periodic point with respect to* G if given any neighborhood U of x, there exists a syndetic subset A of G such that $xA = [xa \mid a \in A] \subset U$. If every point of X is an almost periodic point with respect to G, then G is said to be *pointwise almost periodic*.

Let I be a set with cardinal number $\alpha > 0$. Then each $g \in G$ induces a homeomorphism $(x_i \mid i \in I) \to (x_i g \mid i \in I)$ of X^α onto X^α which will also be referred to as g. Under this identification G becomes a group of

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homeomorphisms of X^α onto X^α.

The characterization mentioned in the second paragraph is that if $\overline{xG}$ is compact for all $x \in X$ and X is Hausdorff, then G is distal if and only if G is pointwise almost periodic on X^α for all cardinals $\alpha > 0$.

The following lemma is probably well-known but the proof is included for the sake of completeness. For references to the literature see [3].

LEMMA 1. *Let S be a compact Hausdorff space with a semigroup structure such that the maps $s \to ts$ $(s \in S)$ are continuous for all $t \in S$. Then there exists an idempotent $u \in S$.*

Proof. Let $\mathscr{E}$ denote the class of non-null compact subsets E of S such that $E^2 \subset E$. Then $\mathscr{E} \neq \phi$ since $S \in \mathscr{E}$. If $\mathscr{E}$ is ordered by inclusion, an application of Zorn's lemma shows that there is a minimal element A in $\mathscr{E}$. If $r \in A$, then rA is a non-null compact subset of S such that $rA \in \mathscr{E}$ and $rA \subset A$. Hence $rA = A$ since A is minimal. Thus there exists $p \in A$ with $rp = r$. Define $L = [a \mid a \in A$ and $ra = r]$. Then $p \in L$, and L is a compact subset of A. Moreover k, $l \in L$ imply that $rkl = rl = r$; that is $L^2 \subset L$. Thus $L \in \mathscr{E}$ and so $L = A$. Hence $r \in L$; that is $r^2 = r$. The proof is completed.

THEOREM 1. *Let X be a Hausdorff space and G a group of homeomorphisms of X onto X such that $\overline{xG}$ is compact for all $x \in X$. Then the following statements are pairwise equivalent.*

(1) *The closure T of G in X^X is a compact group.*

(2) *For every cardinal $\alpha > 0$, G is pointwise almost periodic on X^α.*

(3) *There exists a cardinal $\alpha > 1$ such that G is pointwise almost periodic on X^α.*

(4) *The group G is distal.*

Proof. (1) implies (2). Let α be a cardinal > 0, and let I be a set of cardinal α. Let $x = (x_i \mid i \in I) \in X^I$ and U a neighborhood of x. Then there exists a finite subset J of I and open subsets V_i $(i \in J)$ of X such that $x \in W = \times(W_i \mid i \in I) \subset U$ where $W_i = V_i$ $(i \in J)$ and $W_i = X$ $(i \in I - J)$. Let $N = [t \mid t \in T$ and $x_i t \in V_i$ $(i \in J)]$. Then N is an open neighborhood of the identity e of T. Let $t \in T$. Since the map $r \to rs$ $(r \in T)$ of T onto T is a homeomorphism for all $s \in T$, $t^{-1}N$ is a non-null open subset of T. Hence there exists $g \in G$ such that $g \in t^{-1}N$; that is $t \in Ng^{-1} \subset NG$. Thus $T \subset NG$, and so $T \subset NK$ for some finite subset K of G. Since G is a subgroup of T and $K \subset G$, $G \subset (N \cap G)K$. Thus $A = N \cap G$ is a syndetic subset of G with $xA \subset U$.

That (2) implies (3) is clear.

(3) implies (4). Let x, y, $z \in X$ and let $\mathscr{F}$ be a filter on G such

that $x\mathscr{F}\to z$ and $y\mathscr{F}\to z$. Let $\mathfrak{a}$ be a cardinal >1, I a set of cardinal $\mathfrak{a}$, and i and j two distinct elements of I. Let $w=(w_k|k\in I)\in X^{\mathfrak{a}}$ such that $w_k=x$, $k\neq j$ and $w_j=y$. Then $w\mathscr{F}\to u=(u_k|k\in I)$ where $u_k=z(k\in I)$. Hence $u\in\overline{wG}\subset\times(\overline{w_kG}|k\in I)$. Thus $u\in\overline{wG}$ which is a compact set on which G is pointwise almost periodic. Therefore by [2; 4.07] $w\in\overline{uG}$. Consequently there exists a filter $\mathscr{G}$ on G such that $u\mathscr{G}\to w$; that is $z\mathscr{G}\to x$ and $z\mathscr{G}\to y$. Thus $x=y$.

(4) implies (1). Since $T\subset\times(xG|x\in X)$, T is a compact subset of X^X. That $T^2\subset T$ follows directly from the definition of T and the fact that the maps $t\to st\,(t\in T)$ and $t\to tg\,(t\in T)$ of T into T are continuous for all $s\in T$ and $g\in G$. It remains to be shown that given $t\in T$ then it is invertible and that $t^{-1}\in T$.

To this end let $t\in T$. Then tT is a compact subset of T such that $(tT)(tT)\subset tTT\subset tT$. Hence by Lemma 1 there exists $u\in tT$ such that $u^2=u$. Let $x\in X$ and $\mathscr{F}$ a filter on G such that $\mathscr{F}\to u$. Let $y=xu$. Then $x\mathscr{F}\to xu=y$, and $y\mathscr{F}\to yu=xu^2=xu=y$. Hence $x=y$ since G is assumed distal. Thus $xu=x\,(x\in X)$; that is $u=e$ the identity of T.

Since $e\in tT$, there exists $s\in T$ such that $ts=e$. A similar argument applied to s instead of t produces $r\in T$ with $sr=e$. Hence $t=te=tsr=r$; in other words $ts=st=e$. The proof is completed.

REMARK. Let X be a Hausdorff space, and let G be a distal group of homeomorphisms of X onto X such that $\overline{xG}$ is compact for all $x\in X$. Then G is pointwise almost periodic on X.

A topological group G is said to be *generative* provided that G is abelian and is generated by some compact neighborhood of the identity. The remainder of this paper will be concerned with the transformation group (X, G, π) where X is a Hausdorff space and the group G is generative.

THEOREM 2. *Let X be locally compact zero-dimensional, let G be distal, and let xG be compact for all $x\in X$. Then G is equicontinuous.*

Proof. By Theorem 1. G is pointwise almost periodic on $X\times X$. Hence G is locally weakly almost periodic on $X\times X$ [2; 7.07], and so $\overline{[(x,\ y)G|x,\ y\in X]}$ is a star closed decomposition of $X\times X$[2; 4.16]. Let $x\in X$ and α an index on X. Then α is a neighborhood of $\overline{(x,\ x)G}$, and therefore there exists a neighborhood V of x such that $(V\times V)G\subset\alpha$; that is G is equicontinuous at x. The proof is completed.

The group G is said to be *regularly almost periodic at the point* $x\in X$ if given any neighborhood U of x there exists a syndetic subgroup H of G with $xH\subset U$. If G is regularly almost periodic at x for all

$x \in X$, then G is called *pointwise regularly almost periodic on* X.

THEOREM 3. *Let* G *be distal and let* $\overline{xG}$ *be compact zero-dimensional for all* $x \in X$. *Then* G *is pointwise regularly almost periodic on* X.

Proof. Let $x \in X$, U a neighberhood of x, and consider the action of G on the invariant subset $Y = \overline{xG}$ of X. Let $G_x = [g \mid g \in G$ and $xg = x]$. Then $xhg = xgh = xh (h \in G, g \in G_x)$. Hence by continuity $yg = y (y \in Y, g \in G_x)$. For $k \in K = G/G_x$ and $y \in Y$ set $yk = yg$ where $k = gG_x$. Then K may be regarded as a group of homeomorphisms of Y onto Y such that $\overline{xK} = Y$. By Theorem 2, K is equicontinuous, therefore T = closure of K in Y^Y is a group of homeomorphisms of Y onto Y. Hence T is a topological group.

Let t, $s \in T$ such that $xt = xs$. Then since all the maps involved are continuous and K is commutative, $xkt = xtk = xsk = xks \, (k \in K)$, hence $yt = ys (y \in Y)$, i.e. $t = s$. Consequently, the map $t \rightarrow xt (t \in T)$ of T onto Y is continuous and one-to-one, hence a homeomorphism. Thus T is compact zero-dimensional.

Now let $V = U \cap Y$. Then $N = [t \mid t \in T$ and $xt \in V]$ is a neighborhood of the identity of T. Hence there is an open closed invariant subgroup L of T with $L \subset N$. Since L is open, there exists a finite subset F of K with $T = LF$. Set $M = K \cap L$. Then $K = MF$ and M is a syndetic subgroup of K, such that $xM \subset V$. Consequently H, the inverse image of M under the projection of K onto G/G_x is the required syndetic subgroup of G. The proof if completed.

THEOREM 4. *Let* X *be locally compact metric, let* G *be distal, let* $\overline{xG}$ *be compact zero-dimensional for all* $x \in X$, *and suppose* G *contains only countably many subgroups. Then the set of points* R *at which* G *is equicontinuous is a residual subset of* X.

As an example of the type of group being considered in Theorem 4, let f be a homeomorphism of X onto X and set $G = [f^n \mid n = 0, \pm 1, \cdots]$.

Proof. Let $[H_n \mid n = 1, 2, \cdots]$ be the set of syndetic subgroups of G, and let α be a metric on X. For m, n positive integers set $E(n, m) = [x \mid xH_n \subset S(x, 1/m)]$ where $S(x, 1/m) = [y \mid \alpha(x, y) \leqq 1/m]$, Then $E(n, m)$ is a closed subset of X for all positive integers n, m, and $\cup [E(n, m) \mid n = 1, \cdots] = X$ by Theorem 3. Hence $E(m) = \cup [\text{int } E(n, m) \mid n = 1, \cdots]$ is an everywhere dense open subset of X. Let $E = \cap [E(m) \mid m = 1, \cdots]$. Then E is a residual subset of X. Moreover, from the definition of E, it follows that given any neighborhood U of $x \in E$ there exist a neighborhood V of x and a syndetic subgroup A of G such that $VA \subset U$. Assume U compact and let K be a compact subset of G such that $AK = G$. Then

$(V\times V)G=(V\times V)AK\subset(U\times U)K\subset(U\times U)G$ shows that $(\bar{V}\times V)G$ is compact and that $\cap[\overline{(V\times V)G}|V$ a neighborhood of x contained in $U]=\cap[(V\times V)G|V$ a neighborhood of x contained in $U]$. The proof that G is equicontinuous at x is now completed as in Lemma 1 [1]. Thus $E\subset R$.

The theorems in the second part of the paper suggest the following problems:

(1) Can the assumption that G is generative be dropped in any of these theorems?

(2) To what extent can the condition of zero-dimensionality be relaxed in Theorem 2?

The example [1] of a ring of concentric circles rotating at different rates about their common center shows that zero-dimensionality must be replaced by some other condition i.e. cannot be dispensed with entirely even if X is compact. It is conjectured that a sufficient condition would be that X be minimal under G; that is that $xG=X$ for all $x\in X$. If this were true then in the general case where all that is assumed is that xG is compact for all $x\in X$, the group G would be distal if and only if G is an equicontinuous family of maps of xG onto xG for all $x\in X$.

The notion of distal was considered by Hilbert see [4] in an attempt to give a topological characterization of the concept of a rigid group of motions. According to the above conjecture and Theorem 1 this would be adequate if X were compact and there existed a point $x\in X$ with $xG=X$.

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