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1. Introduction. P. M. Whitman [4] defined an ordering of the set of lattice polynomials generated by a set of unrelated elements. R. P. Dilworth [3] generalized this ordering to apply to the case of lattice polynomials generated by an arbitrary partly ordered set P . Dilworth proved that this ordering gives a lattice isomorphic to the free lattice, $FL(P)$, which is generated by P and which preserves bounds of pairs of elements of P . R. A. Dean [2] considered the ordering of lattice polynomials which preserves order of pairs of elements in P and which leads to the completely free lattice $CF(P)$. He shows that $CF(P)$ and $FL(P)$ are identical in the case in which P is a set of unrelated chains.

This article is a further study of $FL(P)$ where P is a set of unrelated chains. An arbitrary element of P will be denoted by p or q . The set of chains consisting of

$$a_{11} < a_{12} < \cdots < a_{1n_1}; a_{21} < a_{22} < \cdots < a_{2n_2}; \cdots; a_{m1} < a_{m2} < \cdots < a_{mn_m};$$

where a_{ij} and a_{kl} are unrelated when $i \neq k$, will be denoted by $n_1 + n_2 + \cdots + n_m$.

DEFINITION 1. *Lattice polynomials* over P are defined inductively as follows.

- (1) The elements p, q, \dots , of P are lattice polynomials over P .
- (2) If A and B are lattice polynomials over P , then so are $A \cup B$ and $A \cap B$.

DEFINITION 2. The *rank*, $r(A)$, of a lattice polynomial A is defined inductively as follows.

- (1) $r(A) = 0$ if and only if A is in P .
- (2) $r(A \cup B) = r(A \cap B) = r(A) + r(B) + 1$.

DEFINITION 3. The dual polynomial, A' , of a polynomial A of $FL(n_1 + n_2 + \cdots + n_m)$ is defined inductively as follows.

- (1) If $A \equiv a_{ij}$, then $A' \equiv a_i(n_i - j + 1)$.
- (2) If $A \equiv A_1 \cup A_2$, then $A' \equiv A'_1 \cap A'_2$.
- (3) If $A \equiv A_1 \cap A_2$, then $A' \equiv A'_1 \cup A'_2$.

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From Definition 2, Lemma 1, and Lemma 2 of [2], and the fact that $FL(P)$ and $CF(P)$ are identical in the case under consideration, we have the following theorem.

THEOREM 1. *Let P be a partly ordered set consisting of a set of unrelated chains. In $FL(P)$, each relation $A \geq B$ is one of six types. These types and necessary and sufficient conditions which apply to each case are the following.*

- (A) $p \geq q$ if and only if $p \geq q$ in P .
- (B) $p \geq B_1 \cap B_2$ if and only if $p \geq B_1$ or $p \geq B_2$.
- (C) $A_1 \cup A_2 \geq p$ if and only if $A_1 \geq p$ or $A_2 \geq p$.
- (D) $A \geq B_1 \cup B_2$ if and only if $A \geq B_1$ and $A \geq B_2$.
- (E) $A_1 \cap A_2 \geq B$ if and only if $A_1 \geq B$ and $A_2 \geq B$.
- (F) $A_1 \cup A_2 \geq B_1 \cap B_2$ if and only if $A_1 \geq B_1 \cap B_2$ or $A_2 \geq B_1 \cap B_2$ or $A_1 \cup A_2 \geq B_1$ or $A_1 \cup A_2 \geq B_2$.

2. $FL(2+2)$. Let $a_1 < a_2$ and $b_1 < b_2$ be the generators of $FL(2+2)$. The notation of the elements of $FL(2+2)$ is defined in the following recursive manner.

$$A_1 = a_2, \quad B_1 = b_2$$

and, for $n > 1$,

$$\begin{aligned} A_n &= a_2 \cap (a_1 \cup B_{n-1}), & B_n &= b_2 \cap (b_1 \cup A_{n-1}). \\ C_n &= a_1 \cup B_n. \\ D_n &= b_1 \cup A_n. \\ P_n &= A_n \cup B_n. \\ Q_n &= C_n \cap D_n. \\ M_1 &= a_1 \cup b_1. \\ M_2 &= (a_2 \cap b_2) \cup a_1 \cup b_1. \\ V_1 &= b_2 \cap ((a_2 \cap b_2) \cup a_1 \cup b_1). \\ V_2 &= (a_2 \cap b_2) \cup (b_2 \cap (a_1 \cup b_1)). \\ V_3 &= b_2 \cap (a_1 \cup b_1). \\ W_1 &= a_2 \cap ((a_2 \cap b_2) \cup a_1 \cup b_1). \\ W_2 &= (a_2 \cap b_2) \cup (a_2 \cap (a_1 \cup b_1)). \\ W_3 &= a_2 \cap (a_1 \cup b_1). \end{aligned}$$

These elements and their dual elements are all the elements of $FL(2+2)$. This is shown by considering the \cap and \cup tables of the above elements and their dual elements. Since the generators of $FL(2+2)$ are among these elements and their duals, in order to show that these

are all the elements of $FL(2+2)$ it is sufficient to show that this set is closed under \cup and \cap . Actually, it is sufficient to show the set consisting of the above elements and their duals is closed under \cup (or \cap). This follows from the fact that the intersection of two elements, $A \cap B$, can be expressed as $A \cap B = (A') \cap (B') = (A' \cup B')$, the latter being found from the \cup table. The diagram of $FL(2+2)$ as shown in Figure 1 is obtained from the relations found in the $\cup(\cap)$ table. Rather than give the entire \cup table, the diagram of $FL(2+2)$ is given and a typical element, $A_i \cup B_j$, of the \cup table is considered. The other parts of the \cup table are obtained in a similar manner. First, we consider the following theorem.

THEOREM 2. *In $FL(2+2)$ we have $A_1 > A_2 > \dots$, $B_1 > B_2 > \dots$, $C_1 > C_2 > \dots$, and $D_1 > D_2 > \dots$.*

Proof. The proof of this theorem is similar to the proof, in §4 of [2], that $FL(2+2)$ contains four infinite chains. In [2] the symbols, A_n , B_n , C_n , and D_n represent the same elements as A_{2n-1} , D_{2n-1} , B_{2n} , and C_{2n} , respectively, of this paper. Thus we conclude from the results of [2] that $A_1 > A_3 > A_5 > \dots$, $B_2 > B_4 > B_6 > \dots$, $C_2 > C_4 > C_6 > \dots$, and $D_1 > D_3 > D_5 > \dots$. The conclusion of this theorem follows in a similar manner.

We now show that

$$A_i \cup B_j = \begin{cases} P_i, & i=j, \\ D_i, & i < j, \\ C_j, & j < i. \end{cases}$$

$A_i \cup B_i = P_i$ by definition. Since $B_j \geq b_1$, it follows by (D) of Theorem 1 that $A_j \cup B_j \geq b_1 \cup A_i = D_i$. Now consider the relation

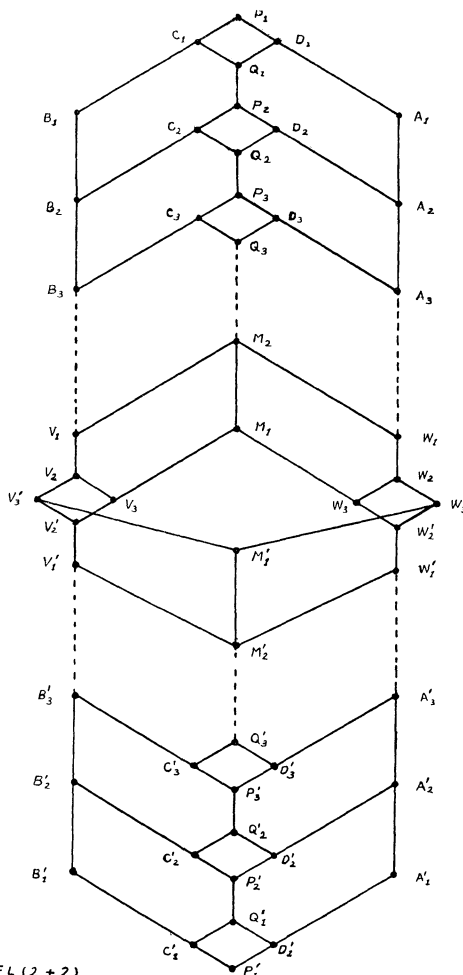


Fig. 1

$$b_1 \cup A_i \geq A_i \cup B_j = (b_2 \cap (b_1 \cup A_{j-1})) \cup A_i, \quad i < j.$$

Since $A_n \geq A_m$, $n \leq m$ we have $b_1 \cup A_i \geq b_1 \cup A_{j-1}$, $i < j$. Hence by (F) of Theorem 1, $b_1 \cup A_i \geq B_j$. Since $b_1 \cup A_i \geq A_i$, it follows by (D) of Theorem 1 that $b_1 \cup A_i \geq B_j \cup A$. This completes the proof that $D_i = b_1 \cup A_i = B_j \cup A_i$, $i < j$.

It follows in a similar manner that $A_i \cup B_j = C_j$, $j < i$.

3. Order-convergence.

DEFINITION 4. In a lattice, $\{b_i\}$ is said to *order-converge* to b if sequences $\{u_i\}$ and $\{v_i\}$ exist such that

$$v_i \geq v_{i+1} \geq b_{i+1} \geq u_{i+1} \geq u_i$$

for all i , and $\text{lub } \{u_i\} = \text{glb } \{v_i\} = b$.

As seen from Figure 1, or as can be shown directly using Theorem 1, it is clear that $A_n \geq W_1$ for each n , $B_n \geq V_1$ for each n , $C_n \geq M_2$ and $D_n \geq M_2$ for each n . Thus we conclude that W_1 is a lower bound to the set $\{A_n\}$, V_1 is a lower bound to the set $\{B_n\}$, and M_2 is a lower bound to each of the sets $\{C_n\}$ and $\{D_n\}$.

THEOREM 3. In FL(2+2) W_1 is $\text{glb } \{A_n\}$, V_1 is $\text{glb } \{B_n\}$, and M_2 is $\text{glb } \{C_n\}$ and $\text{glb } \{D_n\}$.

Proof. Since each of W_1 , V_1 , and M_2 is a lower bound to the indicated sets, in order to prove the theorem it is sufficient to prove the following four statements.

- (1) If $A_n \geq K$ for each n , then $W_1 \geq K$.
- (2) If $B_n \geq K$ for each n , then $V_1 \geq K$.
- (3) If $C_n \geq K$ for each n , then $M_2 \geq K$.
- (4) If $D_n \geq K$ for each n , then $M_2 \geq K$.

The proof is as follows. Let $r(K) = 0$. If $A_n \geq K$ for each n , then $K \equiv a_1$. In this case $W_1 \geq K$. Similarly if $B_n \geq K$ for each n , then $K \equiv b_1$ and hence $V_1 \geq K$. If $C_n \geq K$ for each n , then $K \equiv a_1$ or $K \equiv b_1$. In either case $M_2 \geq K$. Similarly if $D_n \geq K$ for each n , then $M_2 \geq K$.

Proceeding by induction, we assume, when $r(K) < k$, that the four conditions (1), (2), (3), and (4) each hold. We now consider the cases when $r(K) = k$ and $K \equiv K_1 \cap K_2$ or $K \equiv K_1 \cup K_2$. First, let $K \equiv K_1 \cap K_2$. If $A_n \geq K$ for each n , then $a_2 \geq K$ and $a_1 \cup B_{n-1} \geq K$ for each $n > 1$. The latter is true if and only if one of the following holds.

- (a) $a_1 \geq K_1 \cap K_2$,
- (b) $B_{n-1} \geq K_1 \cap K_2$,

- (c) $C_{n-1} \geq K_1$,
- (d) $C_{n-1} \geq K_2$.

If (a) holds, then $W_1 \geq K_1 \cap K_2$. We now show that if (a) does not hold, then one of (b), (c), or (d) must hold for each $n > 1$. Since $B_m > B_{m+1}$, if $B_m \not\geq K_1 \cap K_2$, then $B_n \not\geq K_1 \cap K_2$, $n > m$. Otherwise $B_m > B_n \geq K_1 \cap K_2$ implies $B_m > K_1 \cap K_2$. Similarly $C_m \not\geq K$ implies $C_n \not\geq K$ when $n > m$. Thus if (b), (c), and (d) fail to hold for some $n = i, j$, or k , respectively, then (b), (c), and (d) fail to hold for $n = \max(i, j, k)$. This result with the assumption that (a) is false contradicts the fact that $a_1 \cup B_{n-1} \geq K_1 \cap K_2$ for each $n > 1$. Thus one of (b), (c), or (d) holds for each $n > 1$ if (a) fails to hold. If (b) is true, then $b_2 \geq K_1 \cap K_2$. This, with $a_2 \geq K_1 \cap K_2$, implies $W_1 \geq K_1 \cap K_2$. By the induction hypothesis, (c) or (d) implies $M_2 \geq K_1$ or K_2 , thus $M_2 \geq K_1 \cap K_2$. This, with $a_2 \geq K_1 \cap K_2$, implies $W_1 \geq K_1 \cap K_2$.

Thus we conclude that $A_n \geq K_1 \cap K_2$ for each n and $r(K_1 \cap K_2) = k$ imply $W_1 \geq K_1 \cap K_2$. Similarly $K \equiv K_1 \cup K_2$ and $A_n \geq K_1 \cup K_2$ for each n imply $W_1 \geq K_1 \cup K_2$. It is shown in a similar manner that $D_n \geq K$ for each n implies $M_2 \geq K$; $B_n \geq K$ for each n implies $V_1 \geq K$; and $C_n \geq K$ for each n implies $M_2 \geq K$ where $r(K) = k$ in each case. Thus, by induction, the proof of the theorem is complete.

COROLLARY. *In FL(2+2) the sequence $\{A_n\}$ order-converges to W_1 , $\{B_n\}$ order-converges to V_1 , $\{C_n\}$ and $\{D_n\}$ each order-converge to M_2 .*

Proof. In the case of $\{A_n\}$ we let $u_n = W_1$ and $v_n = A_n$. Then each of the conditions of Definition 4 is satisfied where $\text{lub } \{u_n\} = \text{glb } \{v_n\} = W_1$. Thus $\{A_n\}$ order-converges to W_1 . The other conclusions of the corollary follow in like manner.

We may generalize these results in the following manner. Let $n_1 + n_2 + \dots + n_m$ be a set of chains in which two chains each have two or more elements. From each of these two chains take the least elements, a_{i1}, a_{i2} and a_{j1}, a_{j2} . If we replace a_r with a_{ir} and b_r with a_{jr} , $r = 1, 2$, in $A_n, B_n, C_n, D_n, W_1, V_1$, and M_2 , the resulting elements will be mutually related in the same manner as $A_n, B_n, C_n, D_n, W_1, V_1$, and M_2 since the set $a_{i1}, a_{i2}, a_{j1}, a_{j2}$ is isomorphic to $2+2$.

If we substitute a_{ir} and a_{jr} in A_n, B_n , etc. as indicated above, and if we designate the resulting elements by the same symbols as the symbols from which they are obtained, we obtain the following theorem.

THEOREM 4. *In FL($n_1 + n_2 + \dots + n_m$), where $n_i \geq 2$ and $n_j \geq 2$ for some unequal i, j , the set $\{A_n\}$ order-converges to W_1 ; $\{B_n\}$ order-converges to V_1 ; $\{C_n\}$ and $\{D_n\}$ each order-converge to M_2 .*

4. FL(4+1). The notation for the elements of FL(4+1) is defined recursively in the following manner.

$$A_1 = a_3, \quad B_1 = a_1 \cup b, \quad A_2 = a_3 \cap (a_2 \cup B_1), \quad B_2 = (a_1 \cup b) \cap a_4,$$

and for $n > 2$,

$$A_n = a_3 \cap (a_2 \cup B_{n-1}), \quad B_n = B_1 \cap ((a_4 \cap b) \cup A_{n-2}).$$

$$C_n = (a_4 \cap b) \cup A_n.$$

$$D_n = a_2 \cup B_n.$$

$$F_n = A_n \cup B_n.$$

$$G_n = A_n \cup B_{n+1}.$$

$$H_1 = a_4 \cap D_1 \text{ and, for } n > 1, H_n = C_{n-1} \cap D_n.$$

$$E_1 = a_4 \text{ and, for } n > 1, E_n = C_{n-1} \cap D_{n-1}.$$

$$S_n = A_n \cup H_n.$$

$$T_n = D_n \cap G_n.$$

$$P_1 = a_4 \cap F_1 \text{ and, for } n > 1, P_n = C_{n-1} \cap F_n.$$

$$Q_n = B_n \cup E_n.$$

$$V_1 = a_3 \cap ((a_3 \cap (a_1 \cup b)) \cup a_2 \cup (a_4 \cap b)).$$

$$V_2 = (a_3 \cap (a_1 \cup b)) \cup (a_3 \cap (a_2 \cup (a_4 \cap b))).$$

$$V_3 = a_3 \cap (a_2 \cup (a_4 \cap b)).$$

$$W_1 = (a_1 \cup b) \cap ((a_3 \cap (a_1 \cup b)) \cup a_2 \cup (a_4 \cap b)).$$

$$W_2 = (a_3 \cap (a_1 \cup b)) \cup ((a_1 \cup b) \cap (a_2 \cup (a_4 \cap b))).$$

$$W_3 = (a_1 \cup b) \cap (a_2 \cup (a_4 \cap b)).$$

$$M_1 = a_2 \cup (a_4 \cap b).$$

$$M_2 = (a_3 \cap (a_1 \cup b)) \cup a_2 \cup (a_4 \cap b).$$

b .

As in the case of $FL(2+2)$, that these elements and their dual elements are all the elements of $FL(4+1)$ follows from the fact that they include the generators of $FL(4+1)$ and are closed under \cup and \cap . The relations between the elements of $FL(4+1)$ as shown by the diagram in Figure 2 are proved similar to the way the relations of the elements of $FL(2+2)$ are proved. The following results are stated without proof since the proofs are similar to the proofs of the corresponding statements regarding $FL(2+2)$.

THEOREM 5. $FL(4+1)$ contains the infinite chains $A_1 > A_2 > \dots$, $B_1 > B_2 > \dots$, $C_1 > C_2 > \dots$, and $D_1 > D_2 > \dots$.

THEOREM 6. In $FL(4+1)$, $\{A_n\}$ order-converges to V_1 , $\{B_n\}$ order-converges to W_1 , $\{C_n\}$ and $\{D_n\}$ each order-converge to M_2 .

Theorem 6 can be generalized in the same manner as was the

corollary to Theorem 3. Let $n_1+n_2+\dots+n_m$ be a set of two or more chains in which one chain contains four or more elements. From the chain containing four elements, take the four least elements $a_{i1}, a_{i2}, a_{i3},$ and a_{i4} . From another chain, take the least element a_{j1} . If we substitute $a_{i,r}$ for $a_r, r=1, 2, 3, 4,$ and a_{j1} for b in $A_n, B_n, C_n, D_n, M_2, W_1,$ and $V_1,$ and if we designate the resulting elements by the same symbols as the symbols from which they are obtained then we get the following corollary in the same way as Theorem 4 was obtained.

COROLLARY. *In $FL(n_1+n_2+\dots+n_m)$, where $n_i \geq 4$ for some i and $m \geq 2$, $\{A_n\}$ order-converges to V_1 , $\{B_n\}$ order-converges to W_1 , $\{C_n\}$ and $\{D_n\}$ each order-converge to M_2 .*

5. $FL(1+1+1)$ as a sublattice of $FL(n_1+n_2)$, $n_1 \geq 3$ and $n_2 \geq 2$, or $n_1 \geq 5$ and $n_2 \geq 1$. From Theorem 4 and Theorem 6 of [2] we have the following theorem.

THEOREM 7. *Let U be a subset of $FL(n'_1+n'_2+\dots+n'_m)$ and let $U=\{u_{ij}\}$ be isomorphic to $n_1+n_2+\dots+n_m$. Let $u_{ij} \geq u_{pq}$ if and only if $i=p$ and $j \geq q$. $FL(U)$ is isomorphic to $FL(n_1+n_2+\dots+n_m)$ if and only if $u_{ij} \geq u_{ab}$ implies $i=a$ and $j \geq b$ for some $i, j,$ and dually.*

THEOREM 8. $FL(n_1+n_2)$, $n_1 \geq 3$ and $n_2 \geq 2$, contains a sublattice isomorphic to $FL(1+1+1)$.

Proof. In $FL(n_1+n_2)$ let $u_{11}=a_2, u_{21}=a_3 \cap (a_1 \cup b_1)$, and $u_{31}=a_1 \cup (a_3 \cap b_2)$. In order to show that the sublattice generated by $u_{11}, u_{21},$ and u_{31} is isomorphic to $FL(1+1+1)$ it is sufficient to show, by Theorem 7, that the u_{i1} are unrelated and that $u_{i1} \cup u_{j1} \not\geq u_{k1}$ and $u_{i1} \not\geq u_{j1} \cap u_{k1}$ when $i, j,$ and k are all different.

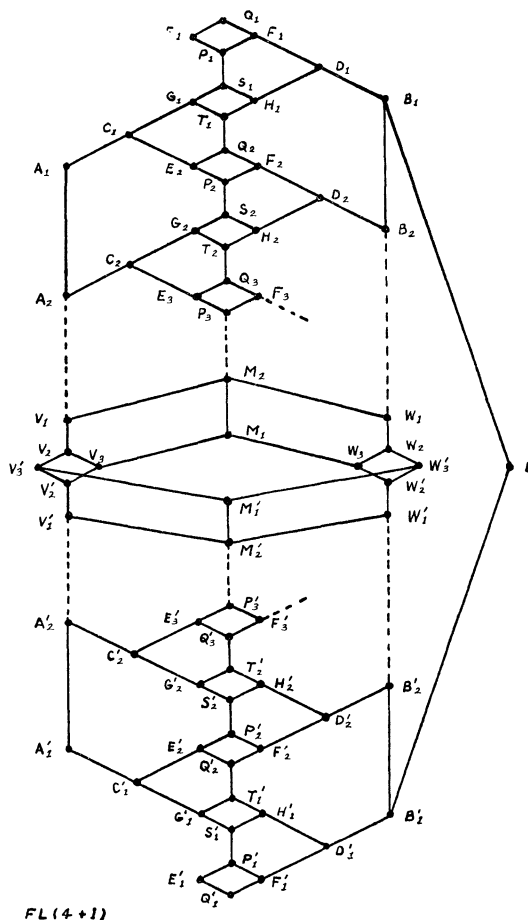


Fig. 2

A direct application of Theorem 1 shows $u_{i1} \not\geq u_{j1}$ and $u_{j1} \not\geq u_{i1}$ when $i \neq j$, thus the u_{i1} form an unrelated set. A straightforward application of Theorem 1 also shows that $u_{i1} \cup u_{j1} \not\geq u_{k1}$ and $u_{i1} \not\geq u_{j1} \cap u_{k1}$ when i, j , and k are all different. Hence $FL(u_{i1}, u_{21}, u_{31})$ is isomorphic to $FL(1+1+1)$.

THEOREM 9. $FL(n_1+n_2)$, $n_1 \geq 5$ and $n_2 \geq 1$, contains a sublattice isomorphic to $FL(1+1+1)$.

Proof. A proof similar to the proof of Theorem 8 shows that the sublattice of $FL(n_1+n_2)$ generated by $u_{11}=a_3$, $u_{21}=a_4 \cap (a_2 \cup (a_5 \cap b))$, and $u_{31}=a_2 \cup (a_1 \cap (a_1 \cup b))$ is isomorphic to $FL(1+1+1)$.

6. $FL(n_1+n_2+\dots+n_m)$ as a sublattice of $FL(1+1+1)$. In $FL(1+1+1)$, with generators x_1, x_2, x_3 , define $u_0=x_1$, and for $n \geq 1$,

$$u_n = x_1 \cup (x_3 \cap (x_2 \cup (x_1 \cap (x_3 \cup (x_2 \cap u_{n-1}))))).$$

Whitman has shown [5, p. 112] that $u_0 < u_1 < u_2 < \dots$ (In his notation $u_i \equiv t_{6i+1}$).

THEOREM 10. *The free lattice generated by $3m$ unrelated elements, $FL(1+1+\dots+1)$, contains a sublattice isomorphic to $FL(n_1+n_2+\dots+n_m)$.*

Proof. Denote the generators of $FL(1+1+\dots+1)$ by x_1, x_2, \dots, x_{3m} and choose m sets of elements of $FL(n_1+n_2+\dots+n_m)$ in the following manner. For each $i, i=1, 2, \dots, m$, let $u_{i0}=x_{3i-2}$, and for $j \geq 1$,

$$u_{ij} = x_{3i-2} \cup (x_{3i} \cap (x_{3i-1} \cup (x_{3i-2} \cap (x_{3i} \cup (x_{3i-1} \cap u_{i,j-1}))))).$$

We note that the polynomials of each set u_{ij}, i fixed and $j=0, 1, 2, \dots, n_i$, are the same, except for the subscripts of the x 's, as the polynomials u_j defined immediately before this theorem. Since the x 's are unrelated, the reasoning that led to the conclusion $u_0 < u_1 < u_2 < \dots$ applies to the u_{ij} . We then conclude that $u_{i0} < u_{i1} < u_{i2} < \dots, i=1, 2, \dots, m$.

Since $x_{3i-2} \not\geq x_{3p-2}$ and $x_{3i} \not\geq x_{3p-2}$ when $i \neq p, u_{ij} \not\geq u_{pq}$. Similarly $u_{pq} \not\geq u_{ij}$ when $i \neq p$. Thus u_{ij} is unrelated to u_{pq} when $i \neq p$. Letting U denote the set of polynomials $u_{ij}, i=1, 2, \dots, m$ and $j=1, 2, \dots, n_i$ for each i , we see that U is isomorphic to $n_1+n_2+\dots+n_m$. By means of Theorem 7, we shall show that the sublattice generated by U is isomorphic to $FL(n_1+n_2+\dots+n_m)$.

If $u_{ab} \geq \cap u_{ij}$, then it is necessary that one of the following holds.

- (1) $x_{3a-2} \geq \cap u_{ij}$,
- (2) $x_{3a} \cap (x_{3a-1} \cup (x_{3a-2} \cap (x_{3a} \cup (x_{3a-1} \cap u_{ab-1})))) \geq \cap u_{ij}$,
- (3) $u_{ab} \geq u_{ij}$ for some i, j .

Condition (1) is true if and only if $x_{3a-2} \geq \text{some } u_{ij}$. Since $j \neq 0$, this is false; hence (1) cannot hold. Similarly (2) is false since $x_{3a} \not\geq \text{some } u_{ij}$. Hence (3) must hold, but this is true if and only if $a=i$ and $b \geq j$. Thus $u_{ab} \geq \cap u_{ij}$ implies $a=i$ and $b \geq j$.

If $\cup u_{ij} \geq u_{ab}$, then it is necessary that

$$\cup u_{ij} \geq x_{3a} \cap (x_{3a-1} \cup (x_{3a-2} \cap (x_{3a} \cup (x_{3a-1} \cap u_{ab-1})))) .$$

This is true if and only if one of the following holds.

- (a) $\cup u_{ij} \geq x_{3a}$,
- (b) $\cup u_{ij} \geq x_{3a-1} \cup (x_{3a-2} \cap (x_{3a} \cup (x_{3a-1} \cap u_{ab-1})))$,
- (c) some $u_{ij} \geq x_{3a} \cap (x_{3a-1} \cup (x_{3a-2} \cap (x_{3a} \cup (x_{3a-1} \cap u_{ab-1}))))$.

Conditions (a) and (b) are false, respectively, since neither $u_{ij} \geq x_{3a}$ nor $u_{ij} \geq x_{3a-1}$ is ever true. Thus (c) must hold. If $i=a$ and $j < b$, since $u_{aj} \geq x_{3a-2}$, it follows that (c) must be false, otherwise it implies $u_{aj} \geq u_{ab}$ when $j < b$ contrary to the known relationship $u_{ab} > u_{aj}$, $j < b$. If $a \neq i$, (c) implies at least one of the following.

- (1) $x_{3i-2} \geq x_{3a} \cap (x_{3a-1} \cup (x_{3a-2} \cap (x_{3a} \cup (x_{3a-1} \cap u_{ab-1}))))$,
- (2) $x_{3i} \geq x_{3a} \cap (x_{3a-1} \cup (x_{3a-2} \cap (x_{3a} \cup (x_{3a-1} \cap u_{ab-1}))))$,
- (3) $u_{ij} \geq x_{3a}$,
- (4) $u_{ij} \geq x_{3a-1}$.

Since $i \neq a$, each of these four conditions is false. Thus $i \neq a$ contradicts (c). We then conclude that if (c) is true, $i=a$ and $j \geq b$. Furthermore, we conclude that $\cap u_{ij} \geq u_{ab}$ implies that $i=a$ and $j \geq b$ for some i, j , and dually. By Theorem 7, it follows that the sublattice generated by U is isomorphic to $FL(n_1+n_2+\dots+n_m)$.

COROLLARY 1. $FL(1+1+1)$ contains a sublattice isomorphic to $FL(n_1+n_2+\dots+n_m)$.

Proof. $FL(1+1+1)$ contains a sublattice isomorphic to $FL(M)$, where M is a set of $3m$ unrelated elements, [5, Theorem 6], and $FL(M)$ in turn contains a sublattice isomorphic to $FL(n_1+n_2+\dots+n_m)$.

COROLLARY 2. $FL(m_1+m_2)$, $m_1 \geq 3$ and $m_2 \geq 2$, or $m_1 \geq 5$ and $m_2 \geq 1$, contains a sublattice isomorphic to $FL(n_1+n_2+\dots+n_m)$.

Proof. By Theorems 8 and 9, $FL(m_1+m_2)$ contains a sublattice isomorphic to $FL(1+1+1)$. In turn, Corollary 1 implies that $FL(1+1+1)$ contains a sublattice isomorphic to $FL(n_1+n_2+\dots+n_m)$.

We note that the reasoning in the proof of Theorem 10 is valid if m is any cardinal number and each chain contains a finite or countable

number of elements. In the corollaries to Theorem 10 m must be countable since $\text{FL}(1+1+1)$ contains only a countable number of elements.

7. Order-convergence in $\text{FL}(n_1+n_2+\dots+n_m)$. By Theorem 4 and the corollary to Theorem 6, we see that $\text{FL}(n_1+n_2+\dots+n_m)$, where $n_i \geq 2$ and $n_j \geq 2$ for some distinct i, j , or some $n_i \geq 4$ and $m \geq 2$, contains an infinite subset that order-converges. We now show that in case $m \geq 3$ there exists an infinite subset that order-converges. We summarize this in the following theorem and prove the case $m \geq 3$ immediately following.

THEOREM 11. $\text{FL}(n_1+n_2+\dots+n_m)$, where $n_i \geq 2$ and $n_j \geq 2$ for some distinct i and j ; or some $n_i \geq 4$ and $m \geq 2$; or $m \geq 3$, contains an infinite subset that order-converges.

Proof. Denote the least elements of three different chains of $\text{FL}(n_1+n_2+\dots+n_m)$ by x_1, x_2, x_3 and define

$$\begin{aligned} u_0 &= x_1, \\ u_n &= x_1 \cup (x_3 \cap (x_2 \cup (x_1 \cap (x_3 \cup (x_2 \cap u_{n-1}))))), & n \geq 1, \\ v_0 &= x_1, \\ v_n &= x_1 \cup (x_2 \cap (x_3 \cup (x_1 \cap (x_2 \cup (x_3 \cap v_{n-1}))))), & n \geq 1. \end{aligned}$$

As mentioned previously, Whitman has shown $u_0 < u_1 < u_2 < \dots$ [5]. Similarly $v_0 < v_1 < v_2 < \dots$.

We now define the following elements in $\text{FL}(n_1+n_2+\dots+n_m)$.

$$\begin{aligned} a_n &= (x_2 \cup (x_1 \cap x_3)) \cap u_n, & n=1, 2. \\ b_n &= (x_3 \cup (x_1 \cap x_2)) \cap v_n, & n=1, 2. \end{aligned}$$

$A_1 = a_2, B_1 = b_2$, and, for $n > 1$,

$$\begin{aligned} A_n &= a_2 \cap (a_1 \cup B_{n-1}), & B_n &= b_2 \cap (b_1 \cup A_{n-1}). \\ C_n &= a_1 \cup B_n. \\ D_n &= b_1 \cup A_n. \\ W_1 &= a_2 \cap ((a_2 \cap b_2) \cup a_1 \cup b_1). \\ V_1 &= b_2 \cap ((a_2 \cap b_2) \cup a_1 \cup b_1). \\ M_2 &= (a_2 \cap b_2) \cup a_1 \cup b_1. \end{aligned}$$

These elements correspond to the elements of $\text{FL}(2+2)$ designated by the same symbols. By means of Theorem 1, in a rather tedious but straightforward manner, it is shown that the above elements are related in the same manner as their corresponding elements in $\text{FL}(2+2)$. Thus

$$\begin{aligned}
 A_1 &> A_2 > \cdots > W_1, \\
 B_1 &> B_2 > \cdots > V_1, \\
 C_1 &> C_2 > \cdots > M_2, \\
 D_1 &> D_2 > \cdots > M_2.
 \end{aligned}$$

A proof similar to the proof of Theorem 3, although more tedious, shows that $W_1 = \text{glb } \{A_n\}$, $V_1 = \text{glb } \{B_n\}$, and $M_2 = \text{glb } \{C_n\}$ and $\{D_n\}$. The first step of the induction is vacuously true. If $A_n \geq K$ for each n where $r(K) = 0$, then it is necessary that $a_2 \geq K$. In turn this implies $x_2 \cup (x_1 \cap x_3) \geq K$ and $u_2 \geq K$. Since K is an element of $n_1 + n_2 + \cdots + n_m$, it follows from these two relations and Theorem 1, that at least two of x_1, x_2, x_3 must be $\geq K$. Since each x_i is from a different unrelated chain, this is false. Hence $A_n \geq K$, for each n and $r(K) = 0$, vacuously implies $W_1 \geq K$. Similarly, statements (2), (3), and (4) at the beginning of the proof of Theorem 3 are vacuously true when $r(K) = 0$. The remainder of the proof is similar to the proof of Theorem 3.

We have answered, in the affirmative, the question posed by Whitman [5], "Does some infinite set in $\text{FL}(1+1+\cdots+1)$ order-converge?" Theorem 11 states that each infinite free lattice generated by a set of chains contains an infinite subset that order-converges.

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Pacific Journal of Mathematics

Vol. 8, No. 3

May, 1958

Michael Israel Aissen, <i>A set function defined for convex plane domains</i>	383
Robert Ellis, <i>Distal transformation groups</i>	401
Ciprian Foias, <i>On a commutative extension of a commutative Banach algebra</i>	407
Jerry William Gaddum, <i>Linear inequalities and quadratic forms</i>	411
Allen A. Goldstein and Elliott Ward Cheney, Jr., <i>A finite algorithm for the solution of consistent linear equations and inequalities and for the Tchebycheff approximation of inconsistent linear equations</i>	415
William L. Hart and T. S. Motzkin, <i>Proof of the fundamental theorem on implicit functions by use of composite gradient corrections</i>	429
Henry Berge Helson, <i>Conjugate series and a theorem of Paley</i>	437
Wu-Chung Hsiang, <i>Abelian groups characterized by their independent subsets</i>	447
John W. Lamperti, <i>On the isometries of certain function-spaces</i>	459
Karel DeLeeuw and Walter Rudin, <i>Extreme points and extremum problems in H_1</i>	467
Eugene Lukacs, <i>Some extensions of a theorem of Marcinkiewicz</i>	487
George W. Mackey, <i>Multiplicity free representations of finite groups</i>	503
Eben Matlis, <i>Injective modules over Noetherian rings</i>	511
John William Neuberger, <i>Continuous products and nonlinear integral equations</i>	529
Lawrence Edward Payne and Hans F. Weinberger, <i>New bounds for solutions of second order elliptic partial differential equations</i>	551
William T. Reid, <i>A Prüfer transformation for differential systems</i>	575
Howard L. Rolf, <i>The free lattice generated by a set of chains</i>	585
K. M. Saksena, <i>Inversion and representation theorems for a generalized Laplace integral</i>	597
Daniel Shanks, <i>Two theorems of Gauss</i>	609
Paul Slepian, <i>On the Lebesgue area of a doubled map</i>	613
Otto Szász and Nelson Paul Yeadley, Jr., <i>The representation of an analytic function by general Laguerre series</i>	621
Alan C. Woods, <i>On two-dimensional convex bodies</i>	635