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ON THE LEBESGUE AREA OF A DOUBLED MAP

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If X is a metric space and A is a non-empty closed subset of X we construct a space Y by doubling X about A in such a way that X is imbedded homeomorphically in Y, the image of A is the boundary of the image of X, and X is also homeomorphic to the closure of the complement of its homeomorphic image in Y. In this way any function fon X may be doubled in a natural way to yield a function F on Y. In 17 it is shown that if X and A satisfy certain triangulability conditions, and f is continuous to Euclidean n space, E_n , with $n \ge k \ge 2$, then $L_k(F) \le 2L_k(f)$, with L_k denoting k-dimensional Lebesgue area. In 18, 21 and 22 the restrictions of 2-dimensionality are used to show that, when k = 2, we have in fact $L_2(F) = 2L_2(f)$.

In particular if (X, A) is a 2-dimensional manifold with boundary, then Y is a compact 2-dimensional manifold. Furthermore, if X is finitely triangulable, then X and A satisfy the required triangulability conditions and $L_2(F) = 2L_2(f)$. Thus to compute the Lebesgue area of f, we need only to know the Lebesgue area of F, whose domain is a compact 2-dimensional manifold.

Our terminology is consistent with [1]; however, some additional notations are cited below

1. NOTATIONS.

- (i) 0 is the empty set,
- (ii) $\{x\}$ is the set whose sole element is x.
- (iii) $\sigma A = \{x \mid \text{ for some } y, x \in y \in A\}.$
- (iv) R is the set of real numbers.
- $(\mathbf{v}) \quad A^{\cap} = \{x \mid x \subset A\}.$
- (vi) N(f, A, y) is the number of elements, possibly infinite, in the set $\{x \mid x \in A \text{ and } y = f(x)\}$.
- (vii) dmn $f = \{x \mid \text{ for some } y, (x, y) \in f\}.$
- (viii) rng $f = \{y \mid \text{ for some } x, (x, y) \in f\}.$

2. AGREEMENT.

(i) If X is a topological space and i is a positive integer, then $X^i = \{A \mid A \text{ is an } i\text{-cell in } A\}.$

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- (ii) If for some positive integer i, A is an i-cell and f is any homeomorphism of A into E_i note that the set {x | f(x) ∈ bdry rng f } is independent of the homeomorphism f selected. Consequently we agree to denote this unique set by Â.
- 3. NOTATIONS.
- (i) If n is a positive integer, k is an integer and $k \leq n$, then H_n^k is k-dimensional Hausdorff measure over E_n .
- (ii) Let X be a k-dimensional finitely triangulable topological space and let f be a continuous function on X to E_n with $k \leq n$. Then $L_k(f)$ is the k-dimensional Lebesgue area of f. More precisely, $L_k(f)$ is the infimum of the set of all $t \in R$ such that for any $\varepsilon > 0$ there exists a quasi-linear function g on X to E_n such that $|g(x) - f(x)| \leq \varepsilon$ for each $x \in X$ and $\int_{B_n} N(g, X, y) dH_n^k y < t$.

4. DEFINITION. X is a k-dimensional manifold if and only if X is such a connected separable metric space that for any $x \in X$ there exists A such that A is an open k-cell in X with $x \in A$.

5. DEFINITION. (X, A) is a 2-dimensional manifold with boundary if and only if the following conditions are satisfied:

- (i) X is a compact metric space and A is a closed subset of X.
- (ii) X A is a 2-dimensional manifold.
- (iii) If $x \in A$, there exist h and β such that β is an open subset of $X, x \in \beta$, and h is a homeomorphism of β onto $E_2 \cap \{z \mid z_2 \ge 0\}$ such that rng $(h \mid A) = E_2 \cap \{z \mid z_2 = 0\}$.

THEOREM. Let (X, A) be a 2-dimensional manifold with boundary. Then

- (i) A has a finite number of components;
- (ii) each component of A is a simple closed curve.

Proof. A is compact, and finitely many of the open sets β described in 5 (iii) cover A. For each such β the set $A \cap \beta$ is connected. Thus A has a finite number of components.

Let x be a component of A and let $t \in x$. Then $\operatorname{ord}_x t = 2$, [2, §46], and x is a simple closed curve.

7. DEFINITION. Y is obtained by doubling X about A if and only if:

(i) X is a metric space and A is a closed, non-empty subset of A;

(ii) Y is the topological space $(\operatorname{rng} g^+ \cup \operatorname{rng} g^-) \subset (X \times R)$ where g^+ and g^- are the functions on X to $(X \times R)$ such that for each $x \in X$,

$$g^+(x) = (x, \operatorname{dist}(\{x\}, A)),$$

 $g^-(x) = (x, \operatorname{-dist}(\{x\}, A)).$

8. AGREEMENT. Throughout this paper we fix X, Y and A such that Y is obtained by doubling X about A. In addition we agree to let g^+ and g^- be the functions specified in 7 (ii).

9. THEOREM. g^+ and g^- are both homeomorphisms of X into Y such that

bdry rng g^+ = bdry rng g^- = rng $(g^+ | A)$ = rng $(g^- | A)$.

The proof is trivial.

10. THEOREM. If (X, A) is a 2-dimensional manifold with boundary then Y is a compact 2-dimensional manifold.

The proof is trivial.

11. DEFINITION. The map F is obtained by doubling the map f if and only if f is a function on X and $F \circ g^+ = F \circ g^- = f$.

12. AGREEMENT. Throughout the remainder of this paper we fix and F such that the map F is obtained by doubling the map f.

13. THEOREM.

- (i) F is a function, dmn F = Y, and rng F = rng f.
- (ii) If f is continuous, then F is continuous.
- (iii) If X is compact and f is light and continuous, then F is light and continuous.

Proof. The proofs of (i) and (ii) are trivial. Suppose X is compact and let $z \in \operatorname{rng} F$. Then

$$\{x \mid F(x) = z\} = (\operatorname{rng} g^+ \cap \{x \mid F(x) = z\}) \cup (\operatorname{rng} g^- \cap \{x \mid F(x) = z\}).$$

Both sets on the right are closed in Y and homeomorphic to $\{x | f(x) = z\}$ which is 0-dimensional. Thus $\{x | F(x) = z\}$ is 0-dimensional.

14. DEFINITION. (P, Q) is a *finitely triangulable pair* if and only if P is a topological space, $Q \subset P$, and there exist (K, τ) and K' such that (K, τ) is a finite triangulation of P, $K' \subset K$ and $\operatorname{rng}(\tau | Q) = \sigma K'$.

15. THEOREM. Let (P, Q) be a 2-dimensional manifold with boundary, such that P is finitely triangulable. Then (P, Q) is a finitely triangulable pair.

The theorem is an immediate consequence of 6.

16. LEMMA. Let X be k-dimensional and suppose that (X, A) is a finitely triangulable pair. Let $\operatorname{rng} f \subset E_n$ with $n \geq k$ and let f be continuous. Let $\varepsilon > 0$.

Let (K, τ) be a finite triangulation of X in E_q and let $K' \subset K$ such that $\operatorname{rng}(\tau | A) = \sigma K'$. Let u be a quasi-linear function on X to E_n such that for each $B \in K$, $(u \circ \operatorname{inv} \tau) | B$ is a barycentric map of B, and such that for each $x \in X$, $|u(x) - f(x)| \leq \varepsilon$.

Then there exists a quasi-linear function h on Y to E_n such that $|h(y) - F(y)| \leq \varepsilon$ for each $y \in Y$, and

$$\int_{E_n} N(h, Y, y) dH_n^k y \leq 2 \int_{E_n} N(u, X, y) dH_n^k y .$$

Proof. We may suppose that τ is the identity map, $X \subset E_q$ and $Y \subset E_{q+1}$.

For each $B \in K$ let B^* be the set of vertices of B. Then for each $B \in K$ let φ_B^+ be the function which maps B barycentrically onto the unique Euclidean simplex in E_{q+1} spanned by the affinely independent set rng $(g^* | B^*)$. More precisely, if $x \in B \in K$, and γ_x is that unique function on B^* to $R \cap \{y | 0 \leq y \leq 1\}$, such that $\sum_{t \in B^*} \gamma_x(t) = 1$ and $x = \sum_{t \in B^*} \gamma_x(t)t$, then let $\varphi_B^+(x) = \sum_{t \in B^*} \gamma_x(t)g^+(t)$.

Similarly for each $B \in K$ let $\varphi_{\overline{B}}^{-}$ be the function which maps B barycentrically onto the unique Euclidean simplex in E_{q+1} spanned by the set rng $(g^{-} | B^{*})$.

Then let

$$H = \bigcup_{B \in K} (\{\operatorname{rng} \varphi_B^+\} \cup \{\operatorname{rng} \varphi_B^-\}) .$$

Also let λ^+ and λ^- be defined by,

$$\lambda^+ = \bigcup_{B \in K} \varphi_B^+$$
, $\lambda^- = \bigcup_{B \in K} \varphi_B^-$.

Then let

$$\eta = (\lambda^+ \circ \operatorname{inv} g^+) \cup (\lambda^- \circ \operatorname{inv} g^-)$$
.

Since (X, A) is a finitely triangulable pair, η is a function and (H, η) is a finite triangulation of Y in E_{q+1} .

Next let

$$h = (u \circ \operatorname{inv} g^+) \cup (u \circ \operatorname{inv} g^-)$$
.

Then h is a function, and $(h \circ \operatorname{inv} \eta) | B'$ is a barycentric map of B' for each $B' \in H$. Thus h is a quasi-linear map of Y into E_n . Also $|h(y) - F(y)| \leq \varepsilon$ for each $y \in Y$.

Finally,

$$egin{aligned} &\int_{\mathbb{F}_n} N(h,\,Y,\,y) dH_n^k y = \int_{\mathbb{F}_n} N(h,\,\operatorname{rng}\,g^+\,\cup\,\operatorname{rng}\,g^-,\,y) dH_n^k y \ &\leq \int_{\mathbb{F}_n} N(h,\,\operatorname{rng}\,g^+,\,y) dH_n^k y + \int_{\mathbb{F}_n} N(h,\,\operatorname{rng}\,g^-,\,y) dH_n^k y \ &= 2\int_{\mathbb{F}_n} N(u,\,X,\,y) dH_n^k y \;. \end{aligned}$$

17. COROLLARY. Let X be k-dimensional and suppose that (X, A) is a finitely triangulable pair. Let $\operatorname{rng} f \subset E_n$ with $n \geq k$, and let f be continuous. Then Y is finitely triangulable and $L_k(F) \leq 2L_k(f)$.

Proof. The construction of 16 guarantees that Y is finitely triangulable. Now suppose that $2L_k(f) < \delta < L_k(F)$. Let $\varepsilon > 0$. It suffices to establish a quasi-linear function h on Y to E_n such that $|h(y) - F(y)| \leq \varepsilon$ for each $y \in Y$ and $\int_{E_n} N(h, Y, y) dH_n^k y < \delta$.

Let (K, τ) be a finite triangulation of X in E_q and let $V \subset K$ such that $\operatorname{rng}(\tau | A) = \sigma V$. By 6.24 of [1] there exists K' such that K' is a finite simplicial subdivision of K, and there exists a quasi-linear function u on X to E_n such that $|u(x) - f(x)| \leq \varepsilon$ for each $x \in X$, $\int_{E_n} N(u, X, y) dH_n^k y < \delta/2$, and $(u \circ \operatorname{inv} \tau) | B'$ is a barycentric map of B' for each $B' \in K'$.

Now let $V' = K' \cap \bigcup_{B \in V} B^{\cap}$. Clearly (K', τ) is a finite triangulation of X in E_q , and since K' is a subdivision of K, we can state that rng $(\tau \mid A) = \sigma V'$. Thus Lemma 16 applies to produce a quasi-linear function h on Y to E_n such that $|h(y) - F(y)| \leq \varepsilon$ for each $y \in Y$, and

$$\int_{\mathbb{F}_n} N(h, Y, y) dH_n^k y \leq 2 \int_{\mathbb{F}_n} N(u, X, y) dH_n^k y < \delta \ .$$

18. LEMMA. Let B be a 2-cell metrized by ρ and let $V \in B^1 \cap \hat{B}^{\cap}$. Let $M = (V - \hat{V})$ and let $\varepsilon > 0$. Then there exists a function u such that:

- (i) u is a homeomerphism of B into B.
- (ii) u(x) = x for $x \in (\hat{B} M)$.
- (iii) $\operatorname{rng}(u/M) \subset (B \hat{B}).$
- (iv) $M \cap \operatorname{rng} u = 0$.
- (v) $\rho(x, u(x)) < \varepsilon$ for each $x \in B$.

Proof. We may suppose that $B \subset E_2$. In fact, letting

$$egin{aligned} a &= (1/2,\,1) \in E_2 ext{ ,} \ eta &= (-1/2,\,1) \in E_2 ext{ ,} \ \gamma &= (0,\,0) \in E_2 ext{ ,} \end{aligned}$$

we may assume that B is the convex hull of the set $\{a, \beta, \gamma\} \subset E_2$. Furthermore we may suppose that

$$V = \{at + (1 - t)\beta \,|\, t \in \{z \,|\, 0 \leq z \leq 1\}\}.$$

Now let v be the function on $B - \{\gamma\}$ such that for each $x \in (B - \{\gamma\})$ we have $v(x) = [\sigma(V \cap \{tx \mid t \in R\})]_1$.

Then let w be the function on R such that $w(x) = (\varepsilon/4) - \varepsilon x^2$. For each $x \in R$.

Finally let u be the function on B such that

$$u(x) = [1 - w(v(x))]x$$
, if $x \in (B - \{r\})$,
 $u(r) = r$.

It is easy to check that u satisfies the required conditions.

19. REMARK. Let (K, τ) be a finite triangulation of a topological space P and let $\varepsilon > 0$. Then by barycentrically subdividing each element of K, we obtain K' such that (K', τ) is a finite triangulation of P, K' is a finite simplicial subdivision of K and each element of K' is less than ε in diameter.

20. DEFINITION. A subset V of E_n is k-removable ([1, 6.26]) if and only if V is a closed set with the following property.

If u is a continuous function on a k-dimensional finitely triangulable space T, to E_n , and

 $G = \{P | P \text{ is a finitely triangulable subset of } T \text{ and } \operatorname{rng}(u | P) \cap V = 0\}$, then $L_k(u) = \sup_{P \in G} L_k(u | P)$.

In the following lemma we make use of the fact that any finite subset of E_n is k removable.

21. LEMMA. Let M be a metric space. Let K be a finite 2-dimensional cell-complex in M such that $M = \sigma K$. Suppose there exists a finite non-empty set $P \subset (K \cap M^1)$ and a function γ on P such that for each $x \in P$

(i)
$$\{\gamma(x)\} = \{B \mid (B \in (K \cap M^2)) \text{ and } (x \subset \hat{B})\},$$

and ¹

¹ Geometrically the conditions (i) and (ii) state that each 1-cell of P is a subset of the boundary of exactly one 2-cell of K, and furthermore, this 2-cell of K meets no other element of P.

(ii) $\sigma P \cap \gamma(x) = x$.

Let J be that set of all M' such that M' is a finitely triangulable subset of M and $M' \cap \sigma P = 0$. Let u be a continuous function on M to E_n with $n \ge 2$. Then $L_2(u) = \sup_{M \in J} L_2(u \mid M')$.

Proof. Let ρ metrize M. It suffices to show that $L_2(u) \leq \sup_{M' \in J} L_2(u \mid M')$.

The remainder of the proof is divided into 2 parts.

Part 1. Let $\varepsilon > 0$. There exists a function φ such that: (i) φ is a homeomorphism of M into M. (ii) For each $x \in M$, $\rho(x, \varphi(x)) < \varepsilon$. (iii) $\bigcup_{B \in P} \hat{B} = \sigma P \cap \operatorname{rng} \varphi = \sigma P \cap \operatorname{rng} (\varphi | \sigma P)$.

Proof of Part 1. For each $x \in P$ we apply Lemma 18 to produce a function d_x is satisfying the following conditions:

(i') d_x is a homeomorphism of $\gamma(x)$ into $\gamma(x)$.

(ii')
$$d_x(t) = t$$
, for $t \in [\dot{\gamma}(x) - (x - \hat{x})]$

- (iii') rng $(d_x | (x x)) \subset (\gamma(x) \gamma(x))$.
- (iv') $(x \hat{x}) \cap \operatorname{rng} d_x = 0$.

(v') For each $t \in \gamma(x)$, $\rho(d_x(t), t) < \varepsilon$.

Let Ψ be the identity map of $(M - \sigma \operatorname{rng} \gamma)$ onto itself and let

$$arphi = igcup_{x\in P} d_x \cup arPmu$$
 .

Part 2. $L_2(u) \leq \sup_{M' \in J} L_2(u \mid M')$.

Proof of Part 2. Let $\varepsilon > 0$ and produce a function φ satisfying the conditions (i)-(iii) of part 1.

The finite set $\operatorname{rng}(u \mid [\sigma P \cap \operatorname{rng} \varphi])$ is 2-removable. Thus if we let W be the set of all Q such that Q is a finitely triangulable subset of rng φ and

$$\operatorname{rng}(u \mid Q) \cap \operatorname{rng}(u \mid [\sigma P \cap \operatorname{rng} \varphi]) = 0$$

we can state that

$$L_{\mathbf{z}}(u \circ \varphi) = L_{\mathbf{z}}(u \mid \operatorname{rng} \varphi) = \sup_{\mathbf{Q} \in W} L_{\mathbf{z}}(u \mid Q) \leq \sup_{\mathbf{M}' \in J} L_{\mathbf{z}}(u \mid M') \ .$$

Due to the arbitrary nature of ε we have

$$L_2(u) \leq L_2(u \circ arphi) \leq \sup_{M' \in J} L_2(u \mid M') \;.$$

22. COROLLARY. Let K be a finite 2-dimensional cell complex in X such that $X = \sigma K$. Suppose there exists a finite non-empty set $P \subset (K \cap X^1)$

such that $A \subset \sigma P$, and there exists a function γ on P such that for each $x \in P$,

$$\{\gamma(x)\} = \{B \mid (B \in (K \cap X^2)) \text{ and } (x \subset \hat{B})\}$$
,

and

$$\sigma P \cap \gamma(x) = x.$$

Let $\operatorname{rng} f \subset E_n$ with $n \geq 2$ and let f be continuous. Then $2L_2(f) \leq L_2(F)$.

Proof. Let J be the set of all X' such that X' is a finitely triangulable subset of X and $X' \cap \sigma P = 0$. Let $V \in J$. Then since

 $\operatorname{rng}(g^+ \mid V) \cap \operatorname{rng}(g^- \mid V) = 0$,

we infer that,

$$L_2(F|\operatorname{rng}(g^+|V)) + L_2(F|\operatorname{rng}(g^-|V)) = L_2(F|(\operatorname{rng}(g^+|V) \cup \operatorname{rng}(g^-|V)))$$

 $\leq L_2(F)$.

Since $F \circ g^- = F \circ g^- = f$,

$$2L_2(f \mid V) \leqq L_2(F)$$
 ,

and

$$2L_2(f) = 2 \sup_{X' \in J} L_2(f \,|\, X') \leq L_2(F) \;.$$

23. COROLLARY. Suppose that (X, A) is a 2-dimensional manifold with boundary and X is finitely triangulable. Let $\operatorname{rng} f \subset E_n$ with $n \geq 2$ and let f be continuous. Then $2L_2(f) = L_2(F)$.

Proof. From 15 and 17 we infer that $L_2(F) \leq 2L_2(f)$.

Let (K, τ) be a finite triangulation of X. By appropriately subdividing each 2-cell of K we can easily produce H such that H is a finite 2-dimensional cell-complex in X, $\sigma H = X$, and such that $B \cap A \in H$ for each $B \in H$ with $B \cap A \neq 0$. Let $P = A^{\cap} \cap H \cap X^{1}$. Note that if $x \in P$, then the set

 $\{B \mid (B \in (H \cap X^2)) \text{ and } (x \subset \hat{B})\}$

has precisely one element.

Thus let γ be the function on P such that for each $x \in P$

$$\gamma(x) = \sigma\{B \mid (B \in (H \cap X^2)) \text{ and } (x \subset B)\}$$
.

The construction of H guarantees that $\sigma P \cap \gamma(x) = x$ for each $x \in P$. Thus 22 applies and $2L_2(f) = L_2(F)$.

Refernces

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