Pacific Journal of Mathematics

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Vol. 8, No. 4 June 1958

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Let $\mathfrak{M}$ be the set of all Lebesgue measurable subsets of the closed interval [0, 1], and let $A, B \in \mathfrak{M}$. It is well-known that $\mathfrak{M}$ becomes a pseudo-metric space if distance is defined by

$$d(A, B) = m(A - B) + m(B - A) = m[(A - B) \cup (B - A)] ,$$

m denoting the Lebesgue measure. See [1, pp. 31–32]. It is the purpose of this paper to extend $\mathfrak{M}$ to include the non-measurable sets and to examine some of the properties of the resulting space.

If we remove the restriction that A and B be measurable, and let them be any subsets of [0, 1], then if

$$\rho(A, B) = m^*(A - B) + m^*(B - A), \text{ and } \delta(A, B) = m^*[A - B) \cup (B - A)]$$

(where m^* denotes the exterior Lebesgue measure), it is easily seen that pseudo-metric spaces $\mathfrak{S}$ and $\mathfrak{T}$ are obtained, corresponding to ρ and δ respectively. The properties which we discuss of $\mathfrak{S}$ and $\mathfrak{T}$ are the same and are proved analogously, so we shall state and prove our results for the space $\mathfrak{S}$ only, it being understood that similar theorems and proofs hold for $\mathfrak{T}$.

LEMMA 1. *A necessary and sufficient condition that $\rho(A, B) = 0$ is the existence of sets Z_1 and Z_2, both of Lebesgue measure zero, such that $A \cup Z_1 = B \cup Z_2$.*

Necessity. If $\rho(A, B) = 0$, then $m(A - B) = m(B - A) = 0$. Since $A \cup (B - A) = A \cup B = B \cup A = B \cup (A - B)$, Z_1 and Z_2 may be taken as $B - A$ and $A - B$, respectively.

Sufficiency. If $A \cup Z_1 = B \cup Z_2$, then

$$\rho(A, B) \leqq \rho(A, A \cup Z_1) + \rho(A \cup Z_1, B \cup Z_2) + \rho(B \cup Z_2, B) = 0$$

The relation $\rho(A, B) = 0$ is seen to be an equivalence relation defined on the elements of $\mathfrak{S}$; hence, those elements are partitioned into equivalence classes. Let $[A]$ denote the equivalence class which contains A. It is clear that if $C \in [A]$ and $D \in [B]$, then $\rho(A, B) = \rho(C, D)$. If $\mathfrak{S}^*$ is the set of all equivalence classes defined above, and if $\rho([A], [B]) = \rho(A, B)$, then $\mathfrak{S}^*$ becomes a metric space with the metric $\rho([A], [B])$.

Received June 11, 1958.

LEMMA 2. *If* $B_n \in [A_n]$ *for* $n = 1, 2, \cdots$, *then* $[\bigcup_{n=1}^{\infty} A_n] = [\bigcup_{n=1}^{\infty} B_n]$ and $[\bigcap_{n=1}^{\infty} A_n] = \bigcap_{n=1}^{\infty} B_n]$.

LEMMA 3. *If* A *is measurable and* $B \in [A]$, *then* B *is measurable.*

There exist Z_1 and Z_2 such that $A \cup Z_1 = B \cup Z_2$ with $m(Z_1) = m(Z_2) = 0$. Let $\tilde{B}$ denote $[0, 1] - B$. Then $B \cup Z_2$ is measurable and since $B = (B \cup Z_2) - (\tilde{B} \cap Z_2)$, B is measurable.

It follows from Lemma 3 that the sets in each equivalence class are either all measurable or all non-measurable. Thus the space $\mathfrak{S}^* = \mathfrak{M}^* \cup \mathfrak{N}^*$, where $\mathfrak{M}^*$ is the space of all equivalence classes of measurable sets, and $\mathfrak{N}^*$ is the space of all equivalence classes of non-measurable sets. It should be noted that $\mathfrak{M}^*$ is the metric space corresponding to the well-known pseudo-metric space $\mathfrak{M}$ defined at the beginning of the paper.

In the following we will omit the asterisks and square brackets, and will write $\mathfrak{S}$ for $\mathfrak{S}^*$, etc., and $\rho(A, B)$ for $\rho([A], [B])$. When we write $A \in \mathfrak{S}$, A may be considered either as an equivalence class or as a representative element of that class.

THEOREM 1. *The space* $\mathfrak{S}$ *is complete.*

The proof is similar to that given in [**1**, p. 32].

THEOREM 2. *For every* $A \in \mathfrak{S}$ *and every positive number* $\varepsilon < 1$, *there exists* $B \in \mathfrak{S}$ such that $0 < \rho(A, B) < \varepsilon$.

Proof Case I. $m(A) = 0$.

If $m(A) = 0$, then $A \in [\phi]$, ϕ denoting the empty set. Let $B \in \mathfrak{S}$ be an interval of length $< \varepsilon$. Then $\rho(A, B) = \rho(\phi, B) = m(B) < \varepsilon$.

Case II. $m^*(A) > 0$.

Let $I \in \mathfrak{S}$ be an interval of length $< \varepsilon$, such that $m^*(I \cap A) > 0$. If $B = A - I$, then

$$\rho(A, B) = \rho(A, A - I) = m^*[A - (A - I)] = m^*(I \cap A) \leqq m^*(I) < \varepsilon .$$

COROLLARY 1. *If in Theorem* 2, $A \in \mathfrak{M}$, *then* B (*as constructed*) $\in \mathfrak{M}$.

THEOREM 3. *If* $A \in \mathfrak{M}$ *and* $\varepsilon > 0$, *then there exists* $C \in \mathfrak{N}$ *such that* $0 < \rho(A, C) < \varepsilon$.

Proof Case I. $m(A) = 0$.

Let M be a set of real numbers such that for every measurable set E, $m^*(M \cap E) = m(E)$ and $m_*(M \cap E) = 0$, m_* denoting the interior Lebesgue measure. (See [2], Theorem E, p. 70.) In Case I of Theorem 2, let $C = B \cap M$. Then

$$\rho(A, C) = \rho(\phi, C) = m^*(C) = m(B) < \varepsilon \text{ and } m_*(C) = 0 .$$

Case II. $m(A) > 0$.

In Case II of Theorem 2, let $C = A - (I \cap M)$, M described above. Then $\rho(A, C) = m^*(A - C) = m^*(A \cap I \cap M) \leqq m(I) < \varepsilon$, and $m^*(A \cap I \cap M) = m(A \cap I) > 0$, $m_*(A \cap I \cap M) = 0$. Since $(A \cap I \cap M) \in \mathfrak{N}$, $C \in \mathfrak{N}$.

THEOREM 4. *$\mathfrak{N}$ is open in $\mathfrak{S}$.*

Proof. Assume Theorem 4 is false. Then there exists $N \in \mathfrak{N}$ and sets $M_m \in \mathfrak{M}$, $m = 1, 2, \cdots$, such that $\lim_{m\to\infty} \rho(N, M_m) = 0$. The sequence M_m, $m = 1, 2, \cdots$, is, therefore, a Cauchy sequence in $\mathfrak{S}$ and so by Theorem 1 has a subsequence M_{m_n}, $n = 1, 2, \cdots$, such that $\lim_{m\to\infty} \rho(\limsup_n M_{m_n}, M_m) = 0$. Since $\limsup_n M_{m_n}$ is measurable, this means that N is measurable by Lemma 3, a contradiction.

The last few results can be summarized as follows.

THEOREM 5. *$\mathfrak{M}$ is perfect and nowhere dense in $\mathfrak{S}$; $\mathfrak{N}$ is open and dense in $\mathfrak{S}$.*

The remainder of the work is valid for both spaces, as only the equivalence classes are dealt with (these being the same for $\mathfrak{S}$ and $\mathfrak{T}$).

After having proven completeness for $\mathfrak{S}$ in Theorem 1, a natural question to ask is "Is the space separable ?". The theorem proved here which demonstrates the existence of $2^{\mathfrak{c}} (= \mathfrak{f})$, where $2^{\aleph_0} = \mathfrak{c}$,equivalence classes in $\mathfrak{S}$ answers this question (and a similar one about a countable basis) in the negative. It is also interesting to note that the space $\mathfrak{M}$ has exactly $\mathfrak{c}$ equivalence classes. (In the following work Ω is the first ordinal belonging to $\mathfrak{c}$.)

THEOREM 6. *There exist $\mathfrak{f}$ equivalence classes in the space $\mathfrak{S}$.*

Proof. It will be sufficient to construct a well-ordered family $\{A_\alpha \mid 0 \leqq \alpha < \Omega\}$ of mutually disjoint subsets of [0, 1], each of which has $m^*(A_\alpha) = 1$.

Consider $\{B_\beta \mid 0 \leqq \beta < \Omega\}$ as a well-ordering of all closed subsets B_β of [0, 1] which have a positive Lebesgue measure. For each β, $0 \leqq \beta < \Omega$, let $\{x_\alpha^\beta \mid 0 \leqq \alpha \leqq \beta\}$ be a well-ordered subset of B_β such that $x_\alpha^\beta \neq x_{\alpha'}^{\beta'}$, if $\beta \neq \beta'$ or $\alpha \neq \alpha'$. This selection is possible since, for each β, the set of all $x_{\alpha'}^{\beta'}$ with $0 \leqq \alpha' \leqq \beta' < \beta$ has a cardinal number $< \mathfrak{c}$. Set $A_\alpha = \{x_\alpha^\beta \mid \alpha \leqq \beta < \Omega\}$, for each α, $0 \leqq \alpha < \Omega$. By a simple argument $A_\alpha \cap A_{\alpha'} = \phi$, for $\alpha \neq \alpha'$. Now consider any A_α; if $m^*(A_\alpha) \neq 1$, then A_α is contained in some open set Y such that $m(Y) < 1$. The complement of Y is closed and has $m([0, 1] - Y) > 0$. But $m\{[0, x] \cap ([0, 1] - Y)\}$

is a continuous function of x for $0 \leqq x \leqq 1$; therefore, this function takes on all values between 0 and $m([0, 1] - Y)$, inclusive. This means that there are non-denumerably many closed sets whose measures are greater than 0 and which do not intersect A_α. This is, of course, impossible by the construction of A_α. Therefore, $m^*(A_\alpha) = 1$.

Form the set of all subsets of the set of A_x's, and take the sum of each element of this power set. Any two such sums belong to two different equivalence classes since they disagree in a set of exterior measure 1. This set of sums has cardinal $\mathfrak{f}$. There are, therefore, at least $\mathfrak{f}$ equivalence classes, at most $\mathfrak{f}$ such classes; hence, exactly $\mathfrak{f}$.

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2. P. R. Halmos, *Measure theory*, van Nostrand, 1950.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

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Pacific Journal of Mathematics

Vol. 8, No. 4 June, 1958