ON THE VAN KAMPEN THEOREM

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1. Introduction. The van Kampen theorem provides a defining set of generators and relations for the fundamental group of the union of two topological spaces X and Y where the fundamental groups of X, Y, and their intersection are given by defining sets of generators and relations. An intrinsic, purely group-theoretic formulation has been given by Fox using his direct limits of systems of groups [4]; however, the corresponding abstract proof had not been worked out. The present paper supplies such a proof (distilled from an earlier proof by Fox of the van Kampen theorem) to a natural generalization of the van Kampen theorem, which includes for example, in addition to the original theorem, the determination of the fundamental group of the union of an increasing nest of open sets each of whose groups is known [2].

In proving the principal result, Theorem (3.1), we depart from the usual development of the fundamental group in that paths and loops are not required to have the fixed unit interval as domains. In particular, a *path* a is a continuous mapping of the interval $[0, \|a\|]$ into the space in question for some $\|a\| \geq 0$. For paths $a : [0, \|a\|] \to X$ and $b : [0, \|b\|] \to X$ which satisfy $a(\|a\|) = b(0)$, we define the *product path* $a \cdot b$ by

$$a \cdot b(t) = \begin{cases} a(t) \text{ for } 0 \leq t \leq \|a\| \\ b(t - \|a\|) \text{ for } \|a\| \leq t \leq \|a\| + \|b\| . \end{cases}$$

Thus, path multiplication is associative. Paths a and b, having the same initial and terminal points, are *equivalent*, denoted by $a \simeq b$, iff there exists a collection of paths $h_s : [0, \|h_s\|] \to X$, $0 \leq s \leq 1$, such that

$h_0 = a$ and $h_1 = b$,
$h_s(0) = a(0) = b(0)$,
$h_s(\|h_s\|) = a(\|a\|) = b(\|b\|)$,
$\|h_s\|$ is a continuous function of s,
$h_s(t)$ is simultaneously continuous in s and t.

We note that, for any path a and positive number t, there is a path b equivalent to a with $\|b\| = t$. Furthermore, $\|h_s\|$ can always be taken as a linear function of s and thus, in view of the preceding sentence, may be arranged to be constant. The induced multiplication of equivalence classes of paths and the definitions of the fundamental groupoid and group of X are made in the usual way.

2. **Systems of groups and direct limits** (cf. [4]). A *system* is any collection $\mathfrak{S}$ of groups and homomorphisms such that if $\theta ; G_\alpha \to G_\beta$ is in $\mathfrak{S}$, then G_α and G_β are in $\mathfrak{S}$. A *homomorphism* $\Phi ; \mathfrak{S} \to G$ of a *system* $\mathfrak{S}$ *into a group* G is a function which assigns to each group G_α in $\mathfrak{S}$ a homomorphism $\varphi_\alpha : G_\alpha \to G$ such that, for every $\theta : G_\alpha \to G_\beta$ in $\mathfrak{S}$, we have $\varphi_\alpha = \varphi_\beta\theta$. The *image* of Φ is the smallest subgroup of G which contains the image of every homomorphism φ_α in Φ, and Φ is *onto* iff its image is G itself.

A homomorphism $\Phi : \mathfrak{S} \to G$ is a *direct limit* iff (i) Φ is onto and (ii) for any group H and homomorphism $\Psi : \mathfrak{S} \to H$, there exists a homomorphism $\lambda : G \to H$ such that $\Psi = \lambda\Phi$, that is, for every group G_α in $\mathfrak{S}$, $\psi_\alpha = \lambda\varphi_\alpha$.

(2.1) THEOREM *Any system $\mathfrak{S}$ has a direct limit unique to within isomorphism.*

The proof is straightforward and is given in [4]. As a result of (2.1), one may relax the above terminology and speak simply of the group G as the direct limit of the system $\mathfrak{S}$.

A given system $\mathfrak{S}$ may always be enlarged to a system $\mathfrak{S}'$ by adjoining all, or any number of, identity homomorphisms and finite compositions of homomorphisms of $\mathfrak{S}$. It is obvious that any homomorphism of $\mathfrak{S}$ is also a homomorphism of $\mathfrak{S}'$, and conversely. Thus,

(2.2) *Any direct limit $\Phi : \mathfrak{S} \to G$ is a direct limit $\Phi : \mathfrak{S}' \to G$, and conversely.*

3. **The generalized van Kampen theorem.** Consider a collection of pathwise-connected, open subsets X_α of a topological space X closed under finite intersections and such that

$$X = \bigcup X_\alpha$$
$$p \in \bigcap X_\alpha, \text{ for some point } p$$

The set $\mathfrak{S}$ of fundamental groups $G_\alpha = \pi(X_\alpha, p)$ and all homomorphisms $\theta : G_\alpha \to G_\beta$ induced by inclusion is a system, and the homomorphisms $\varphi_\alpha : G_\alpha \to G = \pi(X, p)$ induced by inclusion constitute a homomorphism $\Phi : \mathfrak{S} \to G$.

(3.1) VAN KAMPEN THEOREM. $\Phi : \mathfrak{S} \to G$ *is a direct limit.*

Proof. There are two propositions to verify:

I. Φ *is onto.* Consider an arbitrary non-trivial element $A \in G$ and a loop a representing A. Since $A \neq 1$, we know that $\|a\| > 0$. We construct a subdivision.

$$0 = t_0 < t_1 < \cdots < t_n = \|a\|$$

such that each $t_i - t_{i-1}$ is less than the Lebesgue number of the open covering of $[0, \|a\|]$ consisting of all inverse images $a^{-1}X_x$. We then choose X_{α_i}, $i = 1, \cdots, n$, such that

$$a[t_{i-1}, t_i] \subset X_{\alpha_i} \qquad\qquad i = 1, \cdots, n.$$

For each point t_i, $i = 0, \cdots, n$, of the subdivision, we select a path b_i in X subject to the conditions:

(i) $b_i(0) = p$ and $b_i(\|b_i\|) = a(t_i)$
(ii)   If $a(t_i) = p$, then $b_i \equiv p$
(iii) $b_i(t) \in X_{\alpha_i} \cap X_{\alpha_{i+1}}$, $0 \le t \le \|b_i\|$ and $i = 1, \cdots, n-1$.

Note that (iii) uses the fact that the collection of subsets X_α is closed under finite intersections. Next, consider paths $a_i : [0, t_i - t_{i-1}] \to X$, $i = 1, \cdots, n$, defined by $a_i(t) = a(t + t_{i-1})$.
Clearly,

$$a = \prod_{i=1}^{n} a_i$$

and

$$a \simeq \prod_{i=1}^{n} b_{i-1} \cdot a_i \cdot b_i^{-1} .$$

Each path $b_{i-1} \cdot a_i \cdot b_i^{-1}$ is a p-based loop whose image lies entirely in X_{α_i} and which, therefore, is a representative loop of $\varphi_{\alpha_i} A_i$ for some $A_i \in G_{\alpha_i}$. Thus,

$$A = \prod_{i=1}^{n} \varphi_{\alpha_i} A_i$$

and the proof of I is complete.

II. *For any group H and homomorphism $\Psi : \mathfrak{G} \to H$, there exists a homomorphism $\lambda : G \to H$ such that $\Psi = \lambda \Phi$.*
Proving II obviously amounts to proving that, for any $A_i \in G_{\alpha_i}$, $i = 1, \cdots, r$,

$$\prod_{i=1}^{r} \varphi_{\alpha_i} A_i = 1 \text{ implies } \prod_{i=1}^{r} \psi_{\alpha_i} A_i = 1 .$$

We select representative loops $a_i \in A_i$, $i = 1, \cdots, r$. Then the product

$$a = \prod_{i=1}^{r} \varphi_{\alpha_i} a_i$$

is contractible (We denote an inclusion mapping and its induced homo-

morphism of the fundamental groups by the same symbol), and there exists a homotopy $h : R \to X$, where $R = [0, \|a\|] \times [0, 1]$, which satisfies

$$h(t, 0) = a(t)$$
$$h(0, s) = h(t, 1) = h(\|a\|, s) = p$$

The vertical lines $t = \sum_{k=1}^{i} \|a_k\|$, $i = 1, \cdots, r$, provide a decomposition of R, and we consider a refinement

$$0 = t_0 < t_1 < \cdots < t_n = \|a\|$$
$$0 = s_0 < s_1 < \cdots < s_m = 1$$

into rectangles

$$R_{ij} = \{(t, s) \mid t_{i-1} \le t \le t_i \text{ and } s_{j-1} \le s \le s_j\}$$

the maximum of whose diameters is less than the Lebesgue number of the open covering of R consisting of all inverse images $h^{-1} X_\alpha$. Consequently, there exists a function $\alpha(i, j)$ such that

$$h(R_{ij}) \subset X_{\alpha(i,j)} \qquad i = 1, \cdots, n \text{ and } j = 1, \cdots, m.$$

For each lattice point (t_i, s_j), we select a path e_{ij} in X subject to the following conditions.

(iv)   The initial and terminal points of e_{ij} are p and $h(t_i, s_j)$, respectively.

(v)   If $h(t_i, s_j) = p$, then $e_{ij} \equiv p$.

(vi)   The image of e_{ij} is contained in
$X_{\alpha(i,j)} \cap X_{\alpha(i+1,j)} \cap X_{\alpha(i,j+1)} \cap X_{\alpha(i+1,j+1)}$.
(Assume $X_{\alpha(i,j)} = X$ if $i = 0, n + 1$ or if $j = 0, m + 1$.)

(vii)   If $\sum_{k=1}^{j-1} \|a_k\| \le t_{i-1} \le t_i \le \sum_{k=1}^{j} \|a_k\|$, then the image of e_{i1} contained in X_{α_j}.

Next, cf. Fig. 1, consider paths

$$c_{ij}(t) = h(t + t_{i-1}, s_j) \qquad\qquad 0 \le t \le t_i - t_{i-1}$$
$$d_{ij}(s) = h(t_i, s + s_{j-1}) \qquad\qquad 0 \le s \le s_j - s_{j-1}$$

and set

$$a_{ij} = e_{i-1,j} \cdot c_{ij} \cdot e_{ij}^{-1} \qquad\qquad i = 1, \cdots, n \text{ and } j = 0, \cdots, m$$
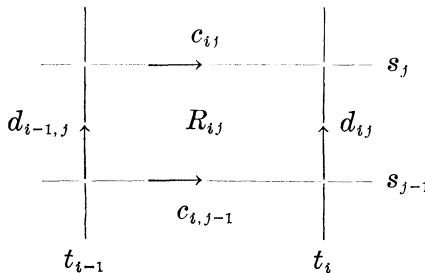$$b_{ij} = e_{i,j-1} \cdot d_{ij} \cdot e_{ij}^{-1} \qquad\qquad i = 0, \cdots, n \text{ and } j = 1, \cdots, m$$

Fig. 1

The image points of the loops a_{ij}, b_{ij}, $a_{i,j-1}$, and $b_{i-1,j}$ all lie in $X_{\alpha(i,j)}$, Consequently, they define group elements A_{ij}, B_{ij}, A'_{ij}, and β'_{ij} respectively. $G_{\alpha(i,j)}$. The product $a_{i,j-1} \cdot b_{ij} \cdot a_{ij}^{-1} \cdot b_{i-1,j}^{-1}$ is obviously contractible in X; moreover, since the image of R_{ij} as well as the images of the four paths lies in $X_{\alpha(i,j)}$, the product is also contractible in $X_{\alpha(i,j)}$. We may conclude that

(1) $$A'_{ij} B_{ij} A_{ij}^{-1} (B'_{ij})^{-1} = 1.$$

The central idea in the proof of II is the fact that *if group elements $A \in G_\alpha$ and $B \in G_\beta$ possess a common representative loop, then $\psi_\alpha A = \psi_\beta B$.*

The proof is easy: By assumption the system $\mathfrak{S}$ contains the fundamental group G_γ of the intersection $X_\gamma = X_\alpha \cap X_\beta$ and the homomorphisms

$$G_\alpha \xleftarrow{\ \theta_1\ } G_\gamma \xrightarrow{\ \theta_2\ } G_\beta$$

induced by inclusion. The assertion that A and B possess a common representative loop states that there exists a p-based loop c in X_γ such that $\theta_1 c \in A$ and $\theta_2 c \in B$. Thus, if c defines $C \in G_\gamma$, we have

$$\theta_1 C = A \text{ and } \theta_2 C = B$$

Since Ψ is consistent with the mappings θ,

$$\psi_\alpha A = \psi_\alpha \theta_1 C = \psi_\gamma C = \psi_\beta \theta_2 C = \psi_\beta B .$$

Applying the central assertion, we obtain

(2) $$\psi_{\alpha(i,j)} A_{ij} = \psi_{\alpha(i,j+1)} A'_{i,j+1}$$
$$\psi_{\alpha(i,j)} B_{ij} = \psi_{\alpha(i+1,j)} B'_{i+1,j}$$

Equation (1) says that the result of reading around each R_{ij} under the homorphism $\psi_{\alpha(i,j)}$ is the identity. Equations (2) show that edges of adjacent rectangles will cancel. It follows (by induction) that the result of reading around the circumference of the large rectangle R is the identity. Furthermore, only the elements along the bottom edge, $s = 0$, are non-trivial. We conclude, therefore, that

$$\prod_{i=1}^{n} \psi_{\alpha(i,0)} A_{ij} = 1 .$$

Since each of the numbers $\sum_{k=1}^{j} \| a_k \|$, $j = 1, \cdots, r$, is a member of $\{t_1, \cdots, t_n\}$, there exists an index function $i(j)$ such that $i(0) = 0$, and

$$t_{i(j)} = \sum_{k=1}^{j} \| a_k \| \qquad\qquad j = 1, \cdots, r.$$

Then,

$$\prod_{i=i(j-1)+1}^{i(j)} a_{i0} \simeq \varphi_{\alpha_j} a_j \qquad\qquad j = 1, \cdots, r.$$

However, by virtue of (vii), we may assume that the equivalence is in X_{α_j}. Thus, each loop a_{i0}, $i = i(j-1)+1, \cdots, i(j)$, determines a group element $A_i' \in G_{\alpha_j}$ and

$$\prod_{i=i(j-1)+1}^{i(j)} A_i' = A_j \,.$$

Since A_{i0} and A_i' possess a common representative loop a_{i0}, it follows from our central assertion that

$$\psi_{\alpha(i,0)} A_{ij} = \psi_{\alpha_j} A_i' \quad i = i(j-1)+1, \cdots, i(j) \,,$$

Finally, therefore,

$$1 = \prod_{j=1}^{r} \prod_{i=i(j-1)+1}^{i(j)} \psi_{\alpha(i,0)} A_{i0} = \prod_{j=1}^{r} \prod_{i=i(j-1)+1}^{i(j)} \psi_{\alpha_j} A_i'$$

$$= \prod_{j=1}^{r} \psi_{\alpha_j} A_j$$

and the proof of the generalized van Kampen theorem is complete.

4. Generators and relations. Since generators and relations describe a group only to within isomorphism, we shall speak of the image group of any direct limit of a system as the direct limit of the system. To obtain a presentation of the direct limit of a system of groups which are given by generators and relations is a simple matter of setting up the proper homomorphisms and chasing around a batch of consistent diagrams. Consider a system $\mathfrak{S}$, each group G_α of which has a presentation (cf [3])

$$G_\alpha = (x_\alpha^1, x_\alpha^2, \cdots : r_\alpha^1, r_\alpha^2, \cdots) \,.$$

Each mapping $\theta : G_\alpha \to G_\beta$ in $\mathfrak{S}$ is described by giving the assignment $\theta x_\alpha^i \in G_\beta$, $i = 1, 2, \cdots$. Then, the direct limit of $\mathfrak{S}$ has the presentation

$$(1) \qquad\qquad G = (\{x_\alpha^i\} : \{r_\alpha^i\}, \{x_\alpha^i (\theta\, x_\alpha^i)^{-1}\})$$

i.e., all generators x_α^i, all relators r_α^i, and all elements $x_\alpha^i (\theta x_\alpha^i)^{-1}$ (a proof is given in [4]). The presentation (1) can be simplified in that, for each homomorphism $\theta : G_\alpha \to G_\beta$, the relators r_α^i, $i = 1, 2, \cdots$, may be dropped. The reason is that, in the free group F generated by all the generators in (1) and of which G is the homomorphic image, the relators r_α^i are a

consequence of the relators r_β^i and the elements $x_\alpha^i(\theta x_\alpha^i)^{-1}$. To prove this assertion consider the diagram

$$
\begin{array}{ccc}
G_\alpha & \xrightarrow{\ \theta\ } & G_\beta \\
\zeta_\alpha \big\uparrow & & \big\uparrow \zeta_\beta \\
F_\alpha & \xrightarrow{\ \bar\theta\ } & F_\beta
\end{array}
$$

$$\text{inclusion}\!-\!-\!-\!\searrow\ \ \swarrow\!-\!-\!\text{inclusion}$$
$$F$$
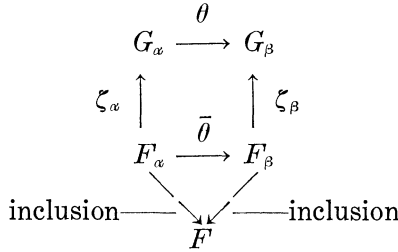
F_α is the free group generated by x_α^i, $i = 1, 2, \cdots$, and ζ_α is the con-onical homomorphism whose kernel is the consequence of r_α^i, $i = 1, 2, \cdots$. The mapping $\bar\theta$, which strictly speaking should be used in (1), is simply θ lifted to the free groups. Consider an arbitrary homomorphism η of F which maps r_β^i, $x_\alpha^i(\bar\theta\, x_\alpha^i)^{-1}$, $i = 1, 2, \cdots$, onto 1. Then, for any $u \in F_\alpha$,

$$\eta\, u = \eta\, \bar\theta\, u \ .$$

Since

$$\zeta_\beta\, \bar\theta\, r_\alpha^i = \theta\, \zeta_\alpha\, r_\alpha^i = 1 \ ,$$

each $\bar\theta\, r_\alpha^i$ is a consequence of the elements r_β^i. Hence,

$$\eta\, \bar\theta\, r_\alpha^i = \eta\, r_\alpha^i = 1$$

and the assertion is proved.

Consider a topogical space X which is the union of two pathwise-connected open subsets X_1 and X_2 whose intersection $X_0 = X_1 \cap X_2$ is also pathwise-connected and contains a point p. Suppose we are given presentations of the fundamental groups $G_i = \pi(X_i, p)$, $i = 0, 1, 2$,

$$G_1 = (x_1, x_2, \cdots : r_1, r_2, \cdots)$$
$$G_2 = (y_1, y_2, \cdots : s_1, s_2, \cdots)$$
$$G_0 = (z_1, z_2, \cdots : t_1, t_2, \cdots)$$

and the mapping $\theta_i : G_0 \to G_i$, $i = 1, 2$, induced by inclusion are described by assignments $\theta_i z_j \in G_i$, $i = 1, 2, j = 1, 2, \cdots$. By our principal Theorem (3.1) and the results of the preceding paragraph, the fundamental group $G = \pi(X, p)$ has the presentation

$$G = (\{x_j\}, \{y_j\}, \{z_j\} : \{r_j\}, \{s_j\}, \{z_j(\theta_i z_j)^{-1}\})$$

This presentation is equivalent to (cf. [3])

$$G = (\{x_j\}, \{y_j\} : \{r_j\}, \{s_j\}, \{\theta_1 z_j\, (\theta_2 z_j)^{-1}\})$$

which is the assertion of the usnal formulation of the van Kampen theorem.

Consider a system $\mathfrak{S}$ of groups and mappings

$$G_1 \xrightarrow{\theta_1} G_2 \xrightarrow{\theta_2} G_3 \xrightarrow{\theta_3} \cdots \text{ with presentations}$$

(2) $$G_i = (x_i^1, x_i^2, \cdots : r_i^1, r_i^2, \cdots)$$

such that

(3) $$\theta_i x_i^j = x_{i+1}^j \text{ and } \theta_i r_i^j = r_{i+1}^j$$

(G_{i+1} may have more generators and relators than G_i). We may define a group

(4) $$G = (y_1, y_2, \cdots : s_1, s_2, \cdots)$$

and a homomorphism $\varPhi : \mathfrak{S} \to G$ such that

$$\varphi_i x_i^j = y_j \text{ and } \varphi_i r_i^j = s_j .$$

It is easy to check that G (more precisely, $\varPhi : \mathfrak{S} \to G$) is the direct limit of $\mathfrak{S}$.

Finally, we consider an ascending chain of non-empty, open subsets $X_1 \subset X_2 \subset \cdots$ of some topological space. We have by (3.1) and (2.2) that the fundamental group G of the union is the direct limit of the system

$$G_1 \xrightarrow{\theta_1} G_2 \xrightarrow{\theta_2} \cdots,$$

where $G_i = \pi(X_i, p)$ is the fundamental group and θ_1 is induced by inclusion. Using the results of the preceding paragraph, we obtain a presentation (4) for G, if presentations (2) satisfying conditions (3), are given. This procedure is used in [1] to obtain (among other examples) a presentation of the group of the exterior of the Alexander Horned Sphere.

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