Pacific Journal of Mathematics

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Vol. 9, No. 1 May 1959

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1. Introduction. In 1953 Milnor studied certain positional 2-person games and defined what he called sums of such games [1]. He investigated the optimal strategies for these games and gave some information about them in terms of properties of the individual games.

In this paper we shall consider some other strategies for these sum games. They are in general not optimal. However, the difference between what a player gets when playing one of them instead of playing an optimal strategy can be estimated. For the sum of n copies of the same game this difference is bounded for all n. Hence, in mean this difference is small for large n.

2. Description of the games. Essentially following Milnor [1] we describe the games as follows.

Each game contains a finite set of positions P. There are two players, A_1 and A_2. For each $p \in P$ and each player A_i, $i = 1, 2$, there is a set of possible moves $M_i(p) \subset P$. For each p either both $M_1(p)$ and $M_2(p)$ contain at least one move or they are both vacuous. In the latter case p is called an end position. For any chain $p_0, p_1, \cdots, p_l$ with $p_{j+1} \in M_1(p_j) \cup M_2(p_j)$, we shall have $p_j \neq p_k$ for $j \neq k$. The maximal number l of steps in all such chains starting with $p_0 = p$ will be denoted by $l(p)$. Then

$$p_1 \in M_1(p) \cup M_2(p) \text{ implies } l(p_1) < l(p). \tag{2.1}$$

Note that a pass, $p \in M(p)$, is never possible. The positions with $l(p) = 0$ are just the end positions.

For each end position the payoff functions $k_1(p) = -k_2(p)$ are defined. They shall satisfy a condition given below. The players start with some position and move alternatively until an end position is reached. Then each player collects his payoff.

For each player A_i and position p, let $v_i(p)$ be the value of the game for A_i when it is his turn to move at position p. It is given by

$$\begin{aligned} v_i(p) &= k_i(p) && \text{for } l(p) = 0, \\ v_i(p) &= \max\{-v_{3-i}(p_1) \mid p_1 \in M_i(p)\} && \text{for } l(p) > 0. \end{aligned} \tag{2.2}$$

Because of (2.1) these formulas define $v_i(p)$ by induction on $l(p)$.

The numbers $k_i(p)$ are defined when p is an end position. We require that they shall be given in such a way that

Received August 12, 1958.

$$v_1(p) + v_2(p) \geqq 0 \qquad \text{for every } p \in P. \tag{2.3}$$

Since the value at p for A_i is $v_i(p)$ if he has the move and $-v_{3-i}(p)$ if the other player has the move, the amount $v_1(p) + v_2(p)$ is the gain for a player of having the move. Inequality (2.3) therefore says that it is at least as good to move as to pass (if this would be allowed).

3. Sums of games. We now define the sum of two games G and G'. A position in the sum game $G + G'$ is a pair $(p, p') \in P \times P'$. A move in $G + G'$ is a move in one of the games G and G' and a pass in the other. Thus the moves in position $p + p' = (p, p')$ are

$$M_i(p + p') = M_i(p) \times p' \cup p \times M_i(p') .$$

We notice that

$$l(p + p') = l(p) + l(p') . \tag{3.1}$$

In particular the condition is still satisfied that in a chain of successive positions all positions are different. The position $p + p'$ is an end position if and only if p and p' both are end positions. For the end positions we define $k_i(p + p')$ by

$$k_i(p + p') = k_i(p) + k_i(p') . \tag{3.2}$$

It is not obvious that the sum of two games satisfying condition (2.3) also satisfies this condition. That this is the fact was proved by Milnor [1]. It will also be proved in §8 below as a consequence of Theorem 1.

It is clear that game addition is an associative and commutative operation and that the formulas corresponding to (3.1) and (3.2) hold for the sum of any finite number of games. A move in the sum of several games is a move in one of them and a pass in all the others.

4. The main problem. The problem for us will be to give good strategies for sums of games in terms of properties of the individual games. Then we must decide what kind of strategies we shall consider to be good.

One way to attack this problem is as follows. Consider n copies of a game G and take their sum nG. Let them all be started in the same position p. Then the value of the sum game is $v_i(np)$, where we have written np instead of $p + \cdots + p$. Now, what happens to the mean value $v_i(np)/n$ when n tends to infinity? In fact this number tends to a limit $m_i(p)$ which will be called the mean value of the game G at p. In later sections we shall prove that $m_i(p)$ satisfies

$$m_1(p) + m_2(p) = 0 , \tag{4.1}$$

$$(4.2) \qquad m_i(p) \leqq v_i(p) .$$

If we change i to $3 - i$ in (4.2) and apply (4.1) we get

$$(4.3) \qquad -v_{3-i}(p) \leqq m_i(p) .$$

Thus $m_i(p)$ lies between $v_i(p)$ and $-v_{3-i}(p)$ which represent the values for A_i when the game is started at p by him or by A_{3-i} respectively.

Of a good strategy we now require that it guarantees at least $m_i(p)$. We see from (4.3) that though such a strategy may not guarantee $v_i(p)$, A_i will nevertheless get more by playing it than by passing (if this would be allowed).

That the limit of $v_i(np)/n$ exists can be proved directly by an inequality given by Milnor [1, p. 294]:

$$v_i(p) - v_{3-i}(p') \leqq v_i(p + p') \leqq v_i(p) + v_i(p') .$$

We get

$$v_i((m + n)p) \leqq v_i(mp) + v_i(np) ,$$

and the existence of the limit of $v_i(np)/n$ follows (cf. [2], Erster Abschnitt, Aufgabe 98).

Another way of attacking our problem also leading to the number $m_i(p)$ will be used below. When a player shall move in a sum of games he chooses one game, say G, and there makes a move. Thereby he loses the possibility to make the move in one of the other games. If the value of this possibility is put equal to t it is natural to compare the situation with the case when the player has to move in G and pay the amount t to the other player when moving. This will lead to the games G_t and G_t^* given in the next section. In this approach the value $m_i(p)$ is defined by induction on $l(p)$, thus by a finite procedure and not by a limit process.

Conventions for the figures. When giving examples of games by figures we use the following conventions. The positions are given by points and the moves indicated by segments joining them. A move by A_1 is a segment going down and to the left, a move by A_2 a segment going down and to the right. At an end position we put the value $k_1(p)$ and at any other position we put the two numbers (m, σ), where $m = m_1(p)$ and $\sigma = \sigma(p)$ defined in the next section. Unless anything else is said, the game shall be played with the highest point as starting position.

EXAMPLE 1. Let G be the game in Figure 1, and consider the sum of n copies of G. First let us start at p_2 in all games. Then of course in about half of the games A_1 will get 7 and in the rest of them 3. Hence the mean value $m_1(p_2)$ is $(7 + 3)/2 = 5$. Analogously we get

$m_1(p_3) = -1$. If all games are started from p_1, it can be proved that an optimal play by both players is to choose the moves from p_1, p_2, and p_3 in this order of preference. Thus when both play optimally one move will first be made in all games. After these n moves the players start attacking the positions p_2 in the games where A_1 made the move from p_1. At last the remaining games with positions p_3 are played. About 1/4 of the games will end in each of the four end positions. Hence the mean value $m_1(p_1)$ is $(7 + 3 + 0 - 2)/4 = 2$. The order of preference between p_1, p_2, p_3 is to be compared with the numbers $\sigma(p_1)$, $\sigma(p_2)$, $\sigma(p_3)$ which are defined in the next section. As given in the figure, $\sigma(p_1) = 3$, $\sigma(p_2) = 2$, $\sigma(p_3) = 1$. The number $\sigma(p)$ is in a sense the value of the move from position p.

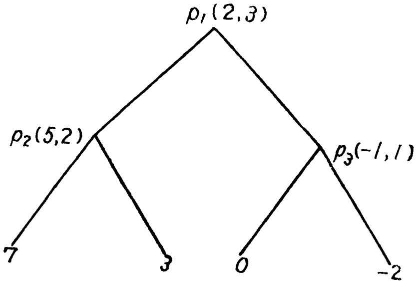


Figure 1

EXAMPLE 2. We change one of the payoff numbers in Figure 1 and get the game in Figure 2. Let us again consider the play of the sum of n copies of the game. If all the games are started from p_1, the optimal play is now to choose the moves from p_1, p_2, p_3 in the order of preference: p_2, p_1, p_3, in accordance with the fact that $\sigma(p_2) = 5$, $\sigma(p_1) = 4$, and $\sigma(p_3) = 1$. Thus if A_1 moves from p_1 to p_2 in a game, A_2 will immediately move in the same game. Thus all games with only one possible exception will end in the position with payoff $k_1(p) = 3$. Thus $m_1(p_1) = 3$. Only if A_2 has the first move one game will end in another end position, the one with $k_1(p) = 0$.

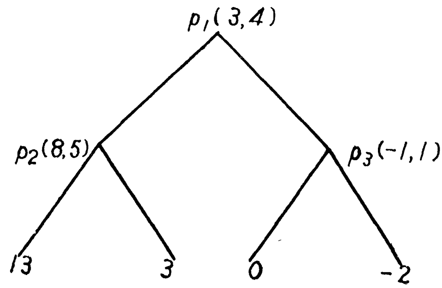


Figure 2

5. The games G_t and G_t^*. Let G be a game satisfying as usual the condition (2.3). Let t be a real number $\geqq 0$. When $l(p) = 0$ put for $i = 1, 2$,

$$v_i(p\,; t) = k_i(p)\ ,$$

$$m_i(p) = k_i(p)\ ,$$

$$\sigma(p) = 0\ .$$

For each p with $l(p) > 0$, we define four functions in $t: u_1(p\,; t)$, $u_2(p\,; t)$,

$v_1(p\,;t)$, $v_2(p\,;t)$ and three numbers $m_1(p)$, $m_2(p)$, and $\sigma(p)$. They shall satisfy (5.1)—(5.7).

(5.1) Each function $u_i(p\,;t)$ and $v_i(p\,;t)$ is a continuous function for $t \geqq 0$ with a derivative for all but a finite number of t-values. In each interval between these exception values the function is linear with derivative 0 or -1. For t greater than the exception values the function $u_i(p\,;t)$ has derivative -1 and $v_i(p\,;t)$ has derivative 0,

$$v_i(p\,;0) = v_i(p)\ , \tag{5.2}$$

$$u_i(p\,;t) = \max\{-v_{3-i}(p_1\,;t) \mid p_1 \in M_i(p)\} - t\ , \tag{5.3}$$

$$u_i(p\,;0) = v_i(p)\ , \tag{5.4}$$

$$\sigma(p) = \min\{t \mid t \geqq 0, u_1(p\,;t) + u_2(p\,;t) = 0\}\ , \tag{5.5}$$

$$m_i(p) = u_i(p\,;\sigma(p))\ , \tag{5.6}$$

$$\begin{aligned} v_i(p\,;t) &= u_i(p\,;t) && \text{for } 0 \leqq t \leqq \sigma(p), \\ &= m_i(p) && \text{for } \quad t > \sigma(p). \end{aligned} \tag{5.7}$$

We shall see below that these conditions are related to two games G_t and G_t^*. Let us first show, however, that they define our functions and numbers by induction on $l(p)$.

For $l(p) = 0$ the function $v_i(p\,;t)$ is constant and equal to $v_i(p)$, hence it satisfies (5.1) and (5.2). Let $l(p) > 0$ and suppose that for each p_1 with $l(p_1) < l(p)$ and in particular for each $p_1 \in M_i(p)$ we have $v_i(p_1\,;t)$ defined satisfying (5.1) and (5.2). Then $u_i(p\,;t)$ can be defined by (5.3). By (5.1) for each $v_{3-i}(p_1\,;t)$ we get immediately (5.1) for $u_i(p\,;t)$ and by (5.2) for each $v_{3-i}(p_1\,;t)$ and by (2.2) we get (5.4). By (5.4) and (2.3) we have $u_1(p\,;0) + u_2(p\,;0) \geqq 0$ and by (5.1) for $u_i(p\,;t)$ we have $u_1(p\,;t) + u_2(p\,;t) \to -\infty$ when $t \to \infty$. Hence, since $u_i(p\,;t)$ is continuous, the set in (5.5) is not vacuous and $\sigma(p)$ is defined and $\geqq 0$. Then (5.6) and (5.7) will define $m_i(p)$ and $v_i(p\,;t)$. That $v_i(p\,;t)$ satisfies (5.1) and (5.2) follows from the corresponding facts for $u_i(p\,;t)$. Hence the induction will work.

EXAMPLE 3. We give in the diagram in Figure 4 the functions $u_1(p\,;t)$, $v_1(p\,;t)$, $-u_2(p\,;t)$, $-v_2(p\,;t)$ for the game in Figure 3 and also the values $m_1(p)$ and $\sigma(p)$ for the same game.

Properties (5.1)—(5.7) give some further formulas. Since (5.1)—(5.7) are only known to be true for $l(p) > 0$, we have to verify separately the case $l(p) = 0$ each time we get a formula which has a meaning even in this case. Note that $u_i(p\,;t)$ is not defined when $l(p) = 0$.

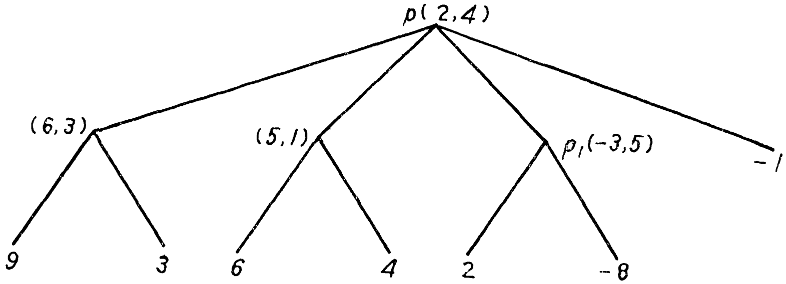
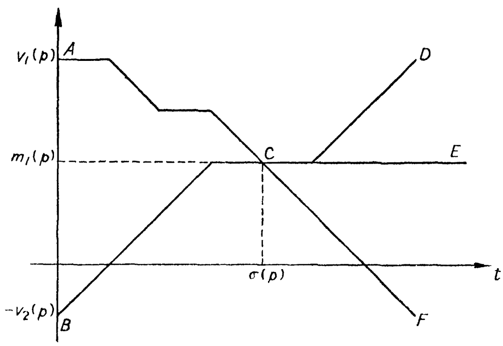


Figure 3



$u_1(p\,;t) : ACF, \quad -u_2(p\,;t) : BCD,$
$v_1(p\,;t) : ACE, \quad -v_2(p\,;t) : BCE.$

Figure 4

Since by (5.1) $u_i(p\,;t)$ is a decreasing function, (5.6) and (5.7) give

(5.8) $$v_i(p\,;t) = \max\,\{u_i(p\,;t),\, m_i(p)\}\ .$$

Hence, in particular

(5.9) $$v_i(p\,;t) \geqq m_i(p)\ .$$

By (5.5) and (5.6)

(5.10) $$m_1(p) + m_2(p) = 0\ .$$

Both (5.9) and (5.10) are true also when $l(p) = 0$ as is easily verified. For any p they imply

(5.11) $$v_1(p\,;t) + v_2(p\,;t) \geqq 0\ .$$

By (5.2) and (5.9) we obtain

(5.12) $$v_i(p) \geqq m_i(p)\ .$$

Since $v_i(p\,;t)$ has derivative 0 or -1, we have for $t_1 < t_2$

$$0 \leqq v_i(p\,;t_1) - v_i(p\,;t_2) \leqq t_2 - t_1\ .$$

Apply this for $t_1 = 0$ and $t_2 = \sigma(p)$. Then by (5.2), (5.6), and (5.7)

$$v_i(p) \leqq m_i(p) + \sigma(p) \ . \tag{5.13}$$

Both (5.12) and (5.13) are also true when $l(p) = 0$. For any p they give a lower and an upper bound for $v_i(p)$.

We are now ready to define the two games G_t and G_t^* mentioned above. Both are defined for each $t \geqq 0$. They are played with the positions in G. The players play alternatively. But each time a player makes a move into a new position in G he has to pay t to the other player. Thus for large t it will be expensive to make a move. Therefore we introduce a new possibility. When A_i has the move in G_t he is allowed to stop the game instead of moving. In G_t^* the same possibility is open except in the starting position, where the player who begins really must move (and pay t). When A_i stops at p he collects $m_i(p)$. Then A_{3-i} gets $m_{3-i}(p)$ by (5.10). The value of G_t at p is $v_i(p\,;t)$ and the value of G_t^* started at p is $u_i(p\,;t)$. This is seen by induction from (5.3) and (5.8).

For large t it is a disadvantage to have to start in G_t^*. The starting player will make a move and pay t and the other player will then immediately stop the game. Thus if t is great enough the starting player will always lose. Thus G_t^* does not satisfy (2.3). The game G_t, however, satisfies (2.3) as is seen from (5.11). In fact we have introduced the number $m_i(p)$ and the possibility to stop just in order to save this property. The number $m_i(p)$ is defined by (5.5) and (5.6) as the value of G_t^* with starting position p, when t has become so large that it is no more an advantage to have the first move in G_t^*. The lowest t-value of this kind is $\sigma(p)$.

6. **The t-optimal moves.** We will call a move in G a t-optimal move if it is optimal in G_t. Thus $p_1 \in M_i(p)$ is t-optimal if

$$v_i(p\,;t) = -v_{3-i}(p_1\,;t) - t \ . \tag{6.1}$$

There is a t-optimal move at p for A_i if $v_i(p\,;t) = u_i(p\,;t)$. Thus we get from (5.7) the following important fact: If $\sigma(p) \geqq t$ and if p is not an end position there always exist t-optimal moves for both players.

If $\sigma(p) \leqq t$ we have $v_i(p\,;t) = m_i(p)$, and an optimal play of G_t is to stop the game at p and collect $m_i(p)$.

Now study a sequence $p_0, p_1, p_2, \cdots, p_l$ of positions that develop when the players play alternatively and make t-optimal moves. If $\sigma(p_l) > t$, there are t-optimal moves at p_l. Therefore the sequence can be continued and we can go on in this way until we reach a position p with $\sigma(p) \leqq t$. We suppose this already done, so that $\sigma(p_l) \leqq t$.

We want to get some formulas for $m_i(p_k)$, $0 \leqq k \leqq l$. Since all moves in the sequence are t-optimal we know that a player cannot get more when playing G_t by stopping at a position p_k, $k < l$, than by moving into p_{k+1}. Thus if A_i makes the first move and if we put $v_i(p_0\,; t) = v$, we get

$$m_i(p_{2k}) \leqq v \qquad \text{if } 0 \leqq 2k < l, \tag{6.2}$$

$$m_i(p_{2k+1}) \geqq v + t \qquad \text{if } 1 \leqq 2k+1 < l, \tag{6.3}$$

where the term $+t$ in (6.3) is the amount A_i shall have when the game is stopped after an odd number of moves as a compensation for the fact that he has made one more move than A_{3-i}, each player paying t when moving in G_t. Since $\sigma(p_l) \leqq t$, an optimal play at p_l in G_t is to stop the game. Hence

$$m_i(p_l) = v \qquad \text{if } l \text{ is even}\,, \tag{6.4}$$

$$m_i(p_l) = v + t \qquad \text{if } l \text{ is odd}. \tag{6.5}$$

Formulas (6.2)—(6.5) could also have been deduced from (6.1). Since all moves are t-optimal we get $v_i(p_0\,; t) = -v_{3-i}(p_1\,; t) - t = v_i(p_2\,; t) = -v_{3-i}(p_3\,; t) - t = \cdots$ and (6.2)—(6.5) follow if we apply (5.9) and (5.10) and the fact that since $\sigma(p_l) \leqq t$, we have by (5.7), $m_j(p_l) = v_j(p_l\,; t)$ for $j = 1, 2$.

EXAMPLE 4. The game in Figure 5 shows that strong inequality may hold in (6.2) and (6.3). All the moves which lead from p_0 to p_5 are 1-optimal and $v = v(p_0\,; 1) = 1$.

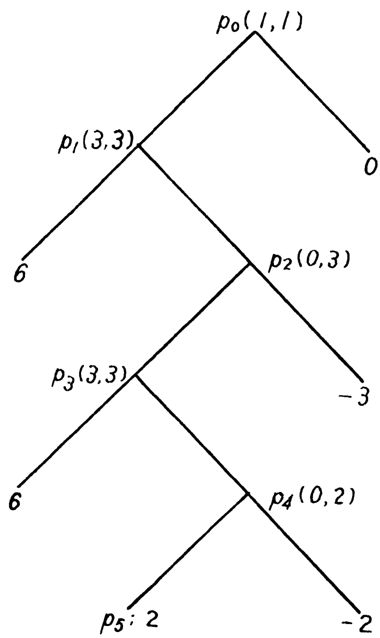


Figure 5

Let now only one player make t-optimal moves when playing G_t. He will get at least as much as when also the other player makes t-optimal moves. Thus we can get some formulas corresponding to (6.2)—(6.5). We put them together into two lemmas.

LEMMA 1. *Let $p_0, p_1, \cdots, p_l$ be a sequence of positions in G such that $p_{2k+1} \in M_i(p_{2k})$, where p_{2k+1} is a t-optimal move at p_{2k}, and such that $p_{2k+2} \in M_{3-i}(p_{2k+1})$. Then if $v_i(p_0\,; t) = v$, we have*

$$m_i(p_{2k+1}) \geqq v + t\,, \tag{6.6}$$

$$m_i(p_l) \geqq v \qquad \textit{if } \sigma(p_l) \leqq t \textit{ and } l \textit{ is even.} \tag{6.7}$$

LEMMA 2. *Let $p_0, p_1, \cdots, p_l$ be a sequence of positions in G such that $p_{2k+1} \in M_i(p_{2k})$ and $p_{2k+2} \in M_{3-i}(p_{2k+1})$, where p_{2k+2} is a t-optimal move at p_{2k+1}. Then if $v_i(p_0\,; t) = v$ we have*

$$m_i(p_{2k}) \leqq v\ , \tag{6.8}$$

$$m_i(p_l) \leqq v + t \qquad \text{if } \sigma(p_l) \leqq t \text{ and } l \text{ is odd.} \tag{6.9}$$

7. The mean strategies for sum games. We now go to our main subject, sums of games.

THEOREM 1. *Let us start the games $G_1, \cdots, G_n$ in positions $q_1, \cdots, q_n$. Put*

$$m_i = m_i(q_1) + \cdots + m_i(q_n)\ ,$$

$$\sigma = \max\,\{\sigma(q_r) \mid 1 \leqq r \leqq n\}\ .$$

Then the value $v_i(q_1 + \cdots + q_n)$ for A_i when he starts at $q_1 + \cdots + q_n$ in $G_1 + \cdots + G_n$ satisfies

$$m_i \leqq v_i(q_1 + \cdots + q_n) \leqq m_i + \sigma\ .$$

Proof. We proceed by induction on $l(q_1 + \cdots + q_n)$. When $l(q_1 + \cdots + q_n) = 0$, all q_r are end positions and our theorem follows directly from $m_i(q_r) = k_i(q_r)$ and $\sigma(q_r) = 0$. By (2.1) we know that if one or several moves are made from $q_1 + \cdots + q_n$ we come to a position, say $p_1 + \cdots + p_n$, with

$$l(p_1 + \cdots + p_n) < l(q_1 + \cdots + q_n)\ .$$

Hence when proving our theorem we may assume that it is true for all positions obtainable from $q_1 + \cdots + q_n$ by one or several moves.

By symmetry we may specialize in the proof so that $i = 1$, i.e. A_1 makes the first move. We then want a strategy for him that secures the amount m_1 and a strategy for A_2 such that A_1 cannot get more than $m_1 + \sigma$. These strategies can be formulated together.

(α) Always make a σ-optimal move in one of the games $G_1, \cdots, G_n$.

(β) Except for the first move, play in the game, in which the other player has just played.

In general it will not be possible to follow this strategy through the whole play of the game, since there are not σ-optimal moves in all positions. The strategy shall therefore be used during a period in the beginning of the play. In the position at the end of this period the induction hypothesis will be used. The length of the period depends upon the moves made. We give two possibilities to end the period.

(γ_1) The other player plays in a game G_r and there leaves a position p_r with $\sigma(p_r) \leqq \sigma$.

(γ_2) Positions p_r, $1 \leqq r \leqq n$, are reached for which $\sigma(p_r) \leqq \sigma$.

We have to show that a player can follow (α) and (β) until (γ_1) or (γ_2) occurs. We first see that A_1 always can make his first move. In fact, by the definition of σ there is a q_r with $\sigma(q_r) = \sigma$. Thus there is a σ-optimal move in G_r. For all later moves the player following the strategy shall play in the position p_r which the other player has just left. Then if (γ_1) does not occur, $\sigma(p_r) > \sigma$ and there is a σ-optimal move at p_r. Hence the game can be continued until (γ_1) occurs or until the player following the strategy ends the whole sum game by playing into an end position. Then $\sigma(p_r) = 0$ for all games G_r and (γ_2) is satisfied. Hence it is possible to follow (α) and (β) until (γ_1) or (γ_2) occurs.

In order to be able to use the induction hypothesis we have to compare m_1 with

$$m_1(p_1) + \cdots + m_1(p_n) ,$$

where p_r is the position in G_r at the end of the period. Therefore we first compare $m_1(q_r)$ with $m_1(p_r)$ for each r. Hence we are interested in those moves in the period that are made in G_r. Note that when at least one player follows the strategy, (β) implies that these moves are played alternatively by the players. Thus for each G_r we are able to apply Lemmas 1 and 2 of the preceding section with $t = \sigma$. Since $\sigma \geqq \sigma(q_r)$, the number $v = v_i(q_r ; \sigma)$ in these lemmas is $= m_i(q_r)$.

Let first A_1 follow the strategy. Denote by p_r the position in G_r at the end of the period. Then if the move into p_r is made by A_2, we know, since A_1 follows (β), that this move is the last move in the period, and whether the period ends with (γ_1) or (γ_2) we get $\sigma(p_r) \leqq \sigma$ in this game G_r. Using the fact (5.10) for q_r and p_r, $1 \leqq r \leqq n$, we apply Lemma 1 with $i = 1$ and Lemma 2 with $i = 2$. Then (6.6), (6.7), (6.8), and (6.9) imply respectively the following four formulas, depending upon who makes the first move and the last move in G_r.

(7.1) $\qquad m_1(p_r) \geqq m_1(q_r) + \sigma \qquad A_1$ first and last move,

(7.2) $\qquad m_1(p_r) \geqq m_1(q_r) \qquad A_1$ first move, A_2 last move,

(7.3) $\qquad m_1(p_r) \geqq m_1(q_r) \qquad A_2$ first move, A_1 last move,

(7.4) $\qquad m_1(p_r) \geqq m_1(q_r) - \sigma \qquad A_2$ first and last move.

We add the trivial fact

(7.5) $\qquad m_1(p_r) = m_1(q_r) \qquad$ if no move is made in G_r.

Formulas (7.1)—(7.5) can be taken together in one formula

$$m_1(p_r) \geqq m_1(q_r) + l_{1r}\sigma - l_{2r}\sigma \ , \tag{7.6}$$

where l_{ir} is the number of moves made by A_i in G_r during the period. Let us take the sum of the inequalities (7.6) for all r. Then

$$m_1(p_1) + \cdots + m_1(p_n) \geqq m_1 + l_1\sigma - l_2\sigma \ , \tag{7.7}$$

where l_i is the number of moves made by A_i during the period.

If the number of moves in the period is even we have $l_1 = l_2$. A_1 who makes the first move in the period shall also make the first move after the period (if there is any move to be made). A_1 can play so after the period that he secures $v_1(p_1 + \cdots + p_n)$. By the induction hypothesis this is $\geqq m_1(p_1) + \cdots + m_1(p_n)$ which by (7.7) is $\geqq m_1$. Hence we have shown that A_1 has been able to play from $q_1 + \cdots + q_n$ so as to secure m_1, and the left-hand inequality of our theorem is proved in this case.

We also have to consider the case that the period contains an odd number of moves. Then since A_1 makes the first move he also makes the last move and the period is not ended by (γ_1), hence by (γ_2). Thus $\sigma(p_r) \leqq \sigma$ for each G_r. We have now $l_1 = l_2 + 1$. A_1 can play so after the period that he secures $-v_2(p_1 + \cdots +$ By$p_n)$. the induction hypothesis, by $\sigma(p_r) \leqq \sigma$, and by (7.7) we get

$$\begin{aligned} -v_2(p_1 + \cdots + p_n) &\geqq -m_2(p_1) - \cdots - m_2(p_n) - \max\{\sigma(p_r)\} \\ &\geqq -m_2(p_1) - \cdots - m_2(p_n) - \sigma \\ &= m_1(p_1) + \cdots + m_1(p_n) - \sigma \\ &\geqq m_1 \ . \end{aligned}$$

Hence the left-hand inequality of the theorem is proved even in this case.

In order to prove the right-hand inequality of the theorem we let A_2 follow the strategy. Then by (β) A_1 makes the first move in each G_r (if there is any move in G_r during the period). Lemma 2 with $i = 1$ gives now depending upon who makes the last move in G_r

$$m_1(p_r) \leqq m_1(q_r) \qquad A_2 \text{ last move,} \tag{7.8}$$

$$m_1(p_r) \leqq m_1(q_r) + \sigma \qquad A_1 \text{ last move.} \tag{7.9}$$

Proceeding as above we get a formula like (7.7), namely

$$m_1(p_1) + \cdots + m_1(p_n) \leqq m_1 + l_1\sigma - l_2\sigma \ . \tag{7.10}$$

If the period contains an odd number of moves, $l_1 = l_2 + 1$. A_2 makes then the first move after the period (if there is any move to be made). He can therefore play so that A_1 gets at most $-v_2(p_1 + \cdots + p_n)$. By the induction hypothesis and by (7.10)

$$\begin{aligned}-v_2(p_1 + \cdots + p_n) &\leqq -m_2(p_1) - \cdots - m_2(p_n)\\ &= m_1(p_1) + \cdots + m_1(p_n)\\ &\leqq m_1 + \sigma\ ,\end{aligned}$$

so that the right-hand inequality is proved in this case.

Finally if the period contains an even number of moves, $l_1 = l_2$, and the period ends by (γ_2), so that $\sigma(p_r) \leqq \sigma$. Then A_1 gets at most $v_1(p_1 + \cdots + p_n)$ and by the induction hypothesis and by (7.10)

$$\begin{aligned}v_1(p_1 + \cdots + p_n) &\leqq m_1(p_1) + \cdots + m_1(p_n) + \max\{\sigma(p_r)\}\\ &\leqq m_1(p_1) + \cdots + m_1(p_n) + \sigma\\ &\leqq m_1 + \sigma\ ,\end{aligned}$$

and the right-hand inequality is proved even in this case.

This completes the proof of Theorem 1.

In the proof just completed the strategy given by (α) and (β) is used only in a period in the beginning of the play. When this period is ended we have used the induction hypothesis in the proof of the theorem. This means, however, that we shall start counting a new period and then again apply (α) and (β). Continuing in this way we get the following consequence of the proof of Theorem 1.

THEOREM 2. *Make the same assumptions as in Theorem* 1. *Suppose one player, A_k, follows a strategy satisfying* (a)—(d) *below. Then A_i, the player making the first move, will get at least m_i when $k = i$ and at most $m_i + \sigma$ when $k = 3-i$.*

(a) *Divide the moves made by the two players into periods.*

(b) *For each period let τ be the maximum of $\sigma(p_r)$ for the positions p_r at the beginning of the period. With this τ defined for a period, always make τ-optimal moves in the period.*

(c) *Except for the first move in a period play in the game in which the other player has just played.*

(d) *Start counting a new period when one of the following two situations occurs,*

(d_1) *the other player plays in G_r into a position p_r with $\sigma(p_r) \leqq \tau$,*

(d_2) *positions p_r with $\sigma(p_r) \leqq \tau$ are reached in all G_r, $1 \leqq r \leqq n$.*

We call the strategies that satisfies (a)—(d) of this theorem mean strategies.

8. Properties of $m_i(p)$ and $\sigma(p)$. By Theorem 1 we easily prove the fact that the sum of games satisfying (2.3) also satisfies (2.3) (proved by Milnor [1, p. 294]). In fact by Theorem 1

$$v_i(q_1 + \cdots + q_n) \geqq m_i\ .$$

Since $m_1(q_r) + m_2(q_r) = 0$ for each r, we have $m_1 + m_2 = 0$. Hence

$$v_1(q_1 + \cdots + q_n) + v_2(q_1 + \cdots + q_n) \geqq 0 ,$$

which is (2.3) for $G_1 + \cdots + G_n$.

Thus $G_1 + \cdots + G_n$ is a game of the kind described in §2. We can therefore apply §5 and define e.g. $u_i(q_1 + \cdots + q_n\,; t)$, $m_1(q_1 + \cdots + q_n)$, and $\sigma(q_1 + \cdots + q_n)$.

THEOREM 3. *Let us start the games* $G_1, \cdots, G_n$ *in positions* $q_1, \cdots, q_n$. *Then*

$$m_i(q_1 + \cdots + q_n) = m_i(q_1) + \cdots + m_i(q_n) , \tag{8.1}$$

$$\sigma(q_1 + \cdots + q_n) \leqq \max \{\sigma(q_r) \mid 1 \leqq r \leqq n\} . \tag{8.2}$$

The right-hand side of these formulas is just m_i and σ respectively defined in Theorem 1.

Proof. We need the following lemma.

LEMMA 3.

$$u_i(q_1 + \cdots + q_n\,; \sigma) = m_i \quad \textit{when } l(q_1 + \cdots + q_n) > 0.$$

Before proving the lemma let us see that Theorem 3 follows from it. If $l(q_1 + \cdots + q_n) = 0$, (8.1) and (8.2) are certainly true. If $l(q_1 + \cdots + q_n) > 0$ we get from Lemma 3, since $m_1 + m_2 = 0$,

$$u_1(q_1 + \cdots + q_n\,; \sigma) + u_2(q_1 + \cdots + q_n\,; \sigma) = 0 .$$

Then (8.2) follows from (5.5). We also see from (5.5) and the fact that $u_i(q_1 + \cdots + q_n\,; t)$, $i = 1, 2$, are decreasing functions in t, that they are constant in the interval $(\sigma(q_1 + \cdots + q_n), \sigma)$. Then (8.1) follows from (5.6) and Lemma 3.

Proof of Lemma 3. The proof will be somewhat similar to that of Theorem 1. Without losing generality we put $i = 1$. We make the induction hypothesis that Theorem 3 is true for all $p_1 + \cdots + p_n$ obtainable from $q_1 + \cdots + q_n$ by one or several moves. We will prove

$$u_1(q_1 + \cdots + q_n\,; \sigma) \geqq m_1 , \tag{8.3}$$

$$u_1(q_1 + \cdots + q_n\,; \sigma) \leqq m_1 . \tag{8.4}$$

Of course they together will give Lemma 3. The number $u_1(q_1 + \cdots + q_n\,; \sigma)$ is the value for A_1 in the game $(G_1 + \cdots + G_n)^*_\sigma$. To prove (8.3) and (8.4) we define strategies for A_1 and A_2 in this game: Follow (α) and (β) of the proof of Theorem 1. Unless the other player stops the game in some position, continue until (γ_1) occurs and then stop the game. When the game is stopped at $p_1 + \cdots + p_n$, A_1 collects $m_1(p_1 + \cdots + p_n)$. If then A_1 has made l_1 and A_2 l_2 moves ($l_1 = l_2$ or $l_1 = l_2 + 1$), A_1 has paid $l_1\sigma$ to A_2 and got $l_2\sigma$ from him. Hence the result will be that A_1 gets

$$m_1(p_1 + \cdots + p_n) - l_1\sigma + l_2\sigma .$$

Since by the induction we may apply Theorem 3, this is equal to

$$m_1(p_1) + \cdots + m_n(p_n) - l_1\sigma + l_2\sigma .$$

Thus in order to prove (8.3) and (8.4) we only need to verify that (7.7) and (7.10) are true when A_1 and A_2 respectively use the strategy described above.

Let A_1 follow the strategy, and let p_r be the position in G_r when the game is stopped. Then if the move into p_r is made by A_2, we know since A_1 follows (β), that this is the last move made before the game is stopped by A_1. Hence (γ_1) is true, and we have $\sigma(p_r) \leqq \sigma$ for this game G_r. The proof of the formulas (7.1)—(7.4) now follows as in the proof of Theorem 1, and (7.7) will again be a consequence of these formulas. Hence we have given a strategy for A_1 in $(G_1 + \cdots + G_n)^*_\sigma$ which secures m_1. Thus (8.3) is proved.

Similarly if A_2 follows the strategy, we verify (7.8) and (7.9) thereby proving (7.10). Thus we have given a strategy for A_2 in $(G_1 + \cdots + G_n)^*_\sigma$ such that A_1 gets $\leqq m_1$. This proves (8.4). Thus Lemma 3 is proved and also Theorem 3.

Theorem 3 can be looked upon as a sharper form of Theorem 1. In fact we get Theorem 1 from Theorem 3 simply by applying (5.12) and (5.13) for $p = q_1 + \cdots + q_n$.

Let now the games $G_1, \cdots, G_n$ be n copies of one and the same game G and let $p_1, \cdots, p_n$ correspond to p in G. We write np for $p_1 + \cdots + p_n$. By Theorem 1

$$nm_i(p) \leqq v_i(np) \leqq nm_i(p) + \sigma(p) .$$

Divide by n and let $n \to \infty$. Then, because of (5.10), we get the following result.

THEOREM 4. *The two expressions*

$$\frac{1}{n} v_i(np) \qquad \textit{and} \qquad \frac{1}{n}(-v_{3-i}(np))$$

which represent the mean value for A_i in the sum of n equal games when he or the other player has the first move, both tend to the same limit $m_i(p)$ when $n \to \infty$.

This theorem justifies the name mean value for the number $m_i(p)$.

The name mean strategies for the strategies described in Theorem 2 is chosen, since it secures the mean value for the player who makes the first move.

We know by Theorem 3 that

$$m_i(p_1 + \cdots + p_n) = m_i(p_1) + \cdots + m_i(p_n) \tag{8.5}$$

and get from (5.10) and (5.12)

$$-v_{3-i}(p) \leqq m_i(p) \leqq v_i(p) . \tag{8.6}$$

Let us show that the two properties (8.5) and (8.6) determine $m_i(p)$ uniquely. Let $m(p)$ be given for all p satisfying (8.5) and (8.6). We get

$$-v_{3-i}(np) \leqq nm(p) \leqq v_i(np) .$$

Divide by n and let $n \to \infty$. Then, by Theorem 4 we get $m(p) = m_i(p)$, showing the uniqueness of $m_i(p)$.

9. Both players use mean strategies.

THEOREM 5. *Let in a sum $G_1 + \cdots + G_n$ both players follow a mean strategy, such as described by* (a)—(d) *in Theorem* 2. *Then*

(1) *the players will count the same periods,*

(2) *in each period both players will make all their moves in only one of the games G_r,*

(3) *the number τ defined by* (b) *of Theorem* 2 *is a decreasing function of the period,*

(4) *if to $m_i(q_1) + \cdots + m_i(q_n)$, where q_r is the starting position of the game G_r, $1 \leqq r \leqq n$, we add τ for each move A_i makes and $-\tau$ for each move A_{3-i} makes, where τ is defined by* (b) *for the period containing the move, then the result will be A_i's payoff.*

Proof. Here (1) will follow by induction if we show that the first period ends at the same moment for both players. When both players play in their first periods (c) implies that they both move in the same game, say in G_s. Then for $r \neq s$, $p_r = q_r$ for all positions $p_1 + \cdots + p_n$ that are reached in the period and therefore since $\sigma(q_r) \leqq \sigma$ by the definition of σ (see Theorem 1), we get $\sigma(p_r) \leqq \sigma$, $r \neq s$. Thus when (d_1) occurs for one player (d_2) also occurs and since (d_2) is symmetric with respect to the two players the first period will be the same for both players. This proves (1).

When we know that the players count the same periods, (2) is a simple consequence of (c). (3) follows from the fact that each period ends with (d_2).

To prove (4) it will be sufficient to show that if in G_r, q_r is the position at the beginning of a period and p_r is the position at the end of the same period then whether A_i or A_{3-i} starts the period,

$$(9.1) \quad m_i(p_1) + \cdots + m_i(p_n) = m_i(q_1) + \cdots + m_i(q_n) + l_i\tau - l_{3-i}\tau ,$$

where l_i is the number of moves by A_i in the period. Since $p_r = q_r$ for $r \neq s$, where G_s is the game in which all moves are made during the period, (9.1) reduces to

$$(9.2) \qquad m_i(p_s) = m_i(q_s) + l_i\tau - l_{3-i}\tau .$$

If A_i makes the first move in the period, (9.2) follows from (6.4) and (6.5). In fact these two formulas are proved for the case when both players make t-optimal moves until a position p_l is reached with $\sigma(p_l) \leqq t$. But putting $t = \tau$ we get in our case by (d_2) that for the final position p_s of the period, $\sigma(p_s) \leqq \tau$.

If A_{3-i} makes the first move in the period, (9.2) is just proved with $3 - i$ substituted for i. However, the formula thus obtained reduces to (9.2) by the use of $m_i(p) + m_{3-i}(p) = 0$.

Thus Theorem 5 is proved.

Since τ is decreasing we see by (4) of Theorem 5 that A_i's payoff is the sum of $m_i = m_i(q_1) + \cdots + m_i(q_n)$ and a sequence of terms with alternating signs and decreasing modules. If A_i starts playing, the first term is positive and equal to $\sigma = \max\{\sigma(q_r)\}$ and the sum of the terms in the sequence is therefore $\geqq 0$ and $\leqq \sigma$, and A_i will get at least m_i and at most $m_i + \sigma$. This last result is of course contained in Theorem 2. Theorem 2 says even more, since it says that a mean strategy always guarantees a certain amount even if used against a player which plays any strategy, e.g. an optimal strategy.

10. Some examples. Conditions (a)—(d) of Theorem 2 do not in general determine a unique strategy. There are still some choices which the player may use to get as good result as possible. Thus there may be different τ-optimal moves in the same game and, when the first move of a period shall be made, there may be several games in which there are τ-optimal moves. In this connection it may be worth while to notice that there may be a τ-optimal move even in a position p with $\sigma(p) < \tau$. The number τ is determined as the maximum of $\sigma(p_r)$, $1 \leqq r \leqq n$ when the period starts, but it is not necessary to start the period in one of the games for which $\sigma(p_r)$ reaches this maximum. There may be τ-optimal moves even in other games.

EXAMPLE 5. Let us study the game given in Figure 3. The move $p_1 \in M_2(p)$ is t-optimal for A_2 even when $4 < t \leqq 5$. In fact for these t-values $u_2(p\,; t) = v_2(p\,; t) = m_2(p)$ so that there must be a t-optimal move for A_2.

If a position has to be played in optimal way it is unimportant if this position is the starting position of the game or if it is a position which has developed during the play. This is not the case when mean strategies are used.

EXAMPLE 6. Compare the game in Figure 6 started by A_2 and the game in Figure 7 started by A_1. When A_2 has moved into p_1 in Figure 6 the situation for A_1 will be the same as when he starts in p_1 in Figure

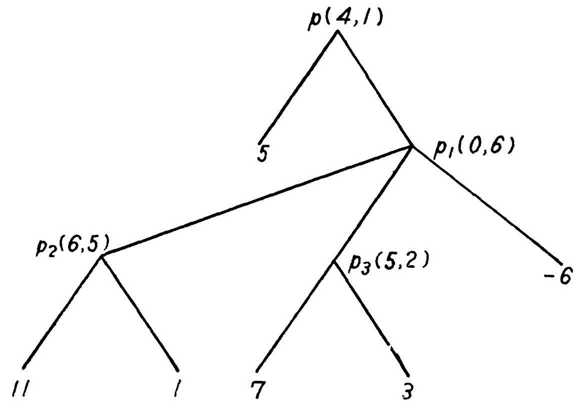


Figure 6

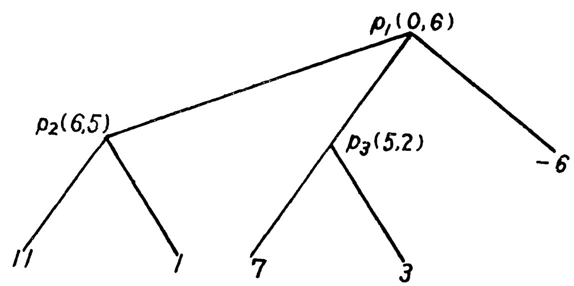


Figure 7

7. However, playing a mean strategy he will handle the two cases in different way. In Figure 6 A_1 plays in a period with $\tau = 1$. He will therefore make a 1-optimal move, the one into p_3. In Figure 7 he just starts a period with $\tau = 6$ and moves into p_2.

This difference may be explained thus. The move recommended by a mean strategy shall be a good move when played in the sum of n copies of the game. We see readily that in n copies of the game in Figure 6 the move into p_3 is the correct answer to A_2's move into p_1. If A_1 always moves into p_2 he gets only about $5\frac{n}{2} + 1\frac{n}{2} = 3n$ though $m_1(p) = 4$. In n copies of the game in Figure 7 the move into p_2 is correct. If A_1 always moves into p_3 he gets about $(-6)\frac{n}{2} + 7\frac{n}{4} + 3\frac{n}{4} = -\frac{n}{2}$ though $m_1(p_1) = 0$.

In a sense (4) of Theorem 5 means that the value of making a move is equal to the number τ for the period containing the move, where τ is max $\{\sigma(p_r)\}$ at the beginning of the period. One may try to change the rules for a mean strategy by requiring each move to be played at the position p where $\sigma(p)$ is highest. The following example shows, however, that such a play does not guarantee the mean value.

EXAMPLE 7. Consider the sum game given in Figure 8. Suppose that A_1 starts and plays in the left game and that A_2 answers in the right game. Then $\sigma(p) = 7$ in the left game and $\sigma(p) = 6$ in the right

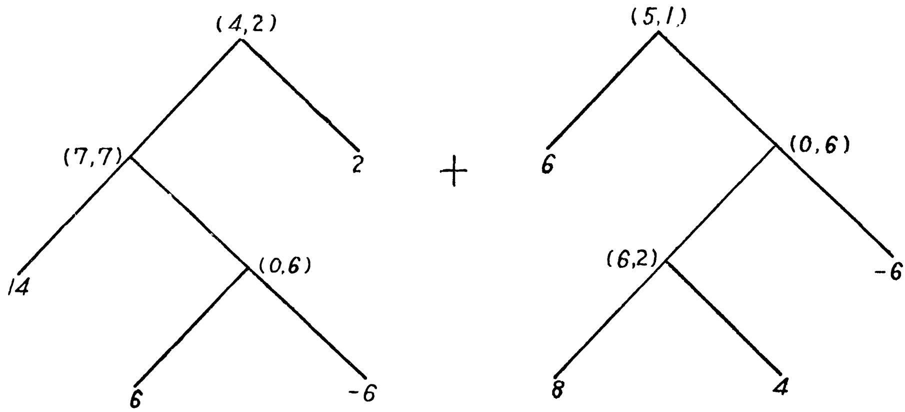


Figure 8

game. But if A_1 plays in the left game, where $\sigma(p)$ is highest he will get only $14 + (-6) = 8$ which is less than the mean value $4 + 5 = 9$. In fact A_1's second move is made in a period with $\tau = 2$. Hence if he follows (a)—(d) he shall play a 2-optimal move in the game where the other player has just played, i.e. he shall play in the right game. Then he will get at least $6 + 4 = 10$ which is more than the mean value 9.

Let us in a final example show that an optimal move in a sum game need not be t-optimal for any t in the summand G in which it is made. Hence this move can never be recommended by a mean strategy.

EXAMPLE 8. The optimal move for A_1 in the sum game in Figure 9 is the move into p_1 in the left game, the mean strategy move is the move in the right game. The move into p_1 is never t-optimal in the left game for any t.

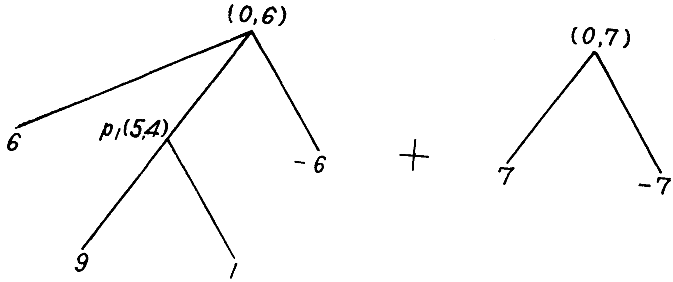


Figure 9

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Printed in Japan by Kokusai Bunken Insatsusha
(International Academic Printing Co., Ltd.), Tokyo, Japan

Pacific Journal of Mathematics

Vol. 9, No. 1 May, 1959