

# Pacific Journal of Mathematics

ON THE RADIUS OF UNIVALENCE OF THE FUNCTION  
 $\exp z^2 \int_0^z \exp(-t^2) dt$

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# ON THE RADIUS OF UNIVALENCE OF THE FUNCTION

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**1. Introduction.** We shall determine the radius of univalence  $\rho_u$  of the function

$$(1.1) \quad E(z) = e^{z^2} \int_0^z e^{-t^2} dt .$$

We shall write  $E(z) = w = u(x, y) + iv(x, y)$ . On the imaginary axis we have  $u = 0$  and  $v$ , regarded as a function of  $y$ , has a single maximum at the solution  $y = \rho$  of

$$2yv(0, y) = 1 .$$

The value of  $\rho$  to eight decimal places has been determined by Lash Miller and Gordon [1] and is

$$(1.2) \quad \rho = 0.92413887 .$$

It is evident that  $\rho_u \leq \rho$ . We shall prove the following theorem.

**THEOREM.** *The number  $\rho$  is the radius of univalence of  $E(z)$ .*  
Recently, the radius of univalence of the error function

$$\operatorname{erf}(z) = \int_0^z e^{-t^2} dt$$

was determined [2]. It is interesting to note that when proceeding from  $\operatorname{erf}(z)$  to  $E(z)$  we meet an entirely different situation. In the case of  $\operatorname{erf}(z)$ , points  $z_1, z_2$  closest to the origin and such that  $\operatorname{erf}(z_1) = \operatorname{erf}(z_2)$  are conjugate complex and lie far apart from each other. In the case of  $E(z)$  points of that nature can be found in an arbitrarily small neighborhood of the point  $z = i\rho$ .

The actual situation is made clear by the diagram and tables given below. In Fig. 1 we show the curves  $R = |E| = \text{constant}$  and  $\gamma = \arg E = \text{constant}$  in the square  $0 \leq x \leq 1.5, 0 \leq y \leq 1.5$  of the  $z$ -plane. The table shows the values of  $E$  for  $z$  on the curve  $C$  (defined below). The values given were obtained by summing an adequate number of terms of the power series on the Datatron 205 at the California Institute of Technology; some were checked by comparison with the tables of Karpov [4, 5] from which values of  $E(z)$  can be obtained.

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2. Idea of proof. Since

$$(2.1) \quad E(z) = \sum_{n=0}^{\infty} \frac{2^n}{1.3.5 \cdots (2n+1)} z^{2n+1}, \quad |z| < \infty,$$

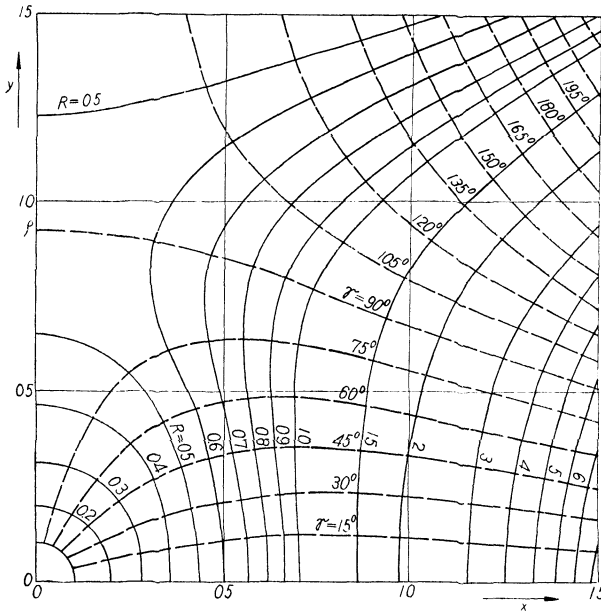


Fig. 1. Curves  $R = |E| = \text{const.}$  and  $\gamma = \text{arg } E = \text{const.}$  in the  $z$ -plane.

$x$	$E(x)$	$\phi$	$E(\rho e^{i\phi})$	$y$	$E(iy)$
0	0	$0^\circ$	1.6837	0	0
0.1	0.1007	$10^\circ$	$1.4957 + 0.6121i$	0.1	0.0993i
0.2	0.2054	$20^\circ$	$1.0573 + 0.9759i$	0.2	0.1948i
0.3	0.3187	$30^\circ$	$0.6079 + 1.0473i$	0.3	0.2826i
0.4	0.4455	$40^\circ$	$0.2919 + 0.9463i$	0.4	0.3599i
0.5	0.5923	$50^\circ$	$0.1189 + 0.8024i$	0.5	0.4244i
0.6	0.7671	$60^\circ$	$0.0401 + 0.6817i$	0.6	0.4748i
0.7	0.9805	$70^\circ$	$0.0099 + 0.6003i$	0.7	0.5105i
0.8	1.2473	$80^\circ$	$0.0011 + 0.5553i$	0.8	0.5321i
0.9	1.5876	$90^\circ$	0.5410i	0.9	0.5407i

we have  $E(\bar{z}) = \overline{E(z)}$  and  $E(-z) = -E(z)$  and may restrict our consideration to the first quadrant  $x \geq 0, y \geq 0$  in the  $z$ -plane.

In the subsequent section we shall prove the following lemma.

LEMMA.

$$(2.2) \quad E(z_1) \neq E(z_2)$$

for any two points on the boundary  $C$  of the open sector  $S$  of the circular disk  $|z| < \rho$  in the first quadrant.

From this it follows, since  $E(z)$  is entire and thus regular in  $S \cup C$  that  $E(z)$  maps  $S$  conformally and one-to-one onto the interior of the simple closed curve  $C^*$  corresponding to  $C$  in the  $w$ -plane [3, p. 121]. This establishes our theorem.

**3. Proof of the lemma.** Let  $z = re^{i\phi}$ . The curve  $C$  consists of

- the segment  $S_1$ :  $y = 0, 0 < x < \rho$ ,  
 the circular arc  $K$ :  $|z| = \rho, 0 < \phi < \pi/2$ ,  
 the segment  $S_2$ :  $x = 0, 0 < y < \rho$ .

and the three common end points of these three arcs.

- (A) On  $S_1$ ,  $E(z)$  is real and increases steadily with  $x$ .  
 (B) On  $S_2$ ,  $E(z)$  is imaginary, and  $v$  increases steadily with  $y$ .  
 (C)  $v \neq 0$  on  $K$ .  
 (D) On  $K$ ,  $|E(z)|$  decreases steadily with increasing  $\phi$ .

(A) is obvious from (2.1), and (B) follows from the definition of  $\rho$ .

*Proof of (C).* Integrating along segments parallel to the coordinate axes we have

$$v(x, y) = e^{-y^2} \left[ \cos 2xy \int_0^y e^{\tau^2} \cos 2x\tau d\tau \right. \\ \left. + \sin 2xy \left\{ e^{x^2} \int_0^x e^{-t^2} dt + \int_0^y e^{\tau^2} \sin 2x\tau d\tau \right\} \right].$$

In  $\{x > 0, y > 0\} \cap \{|z| \leq \rho\}$  we have  $\cos 2xy > 0, \sin 2xy > 0$ . Therefore  $v > 0$  on  $K$ .

*Proof of (D).* Integrating along a radius  $\phi = \text{constant}$  from 0 to  $\rho$  we have

$$E(z) = e^{i\phi} \int_0^\rho e^{h(r, \phi)} dr$$

where

$$h(r, \phi) = a(r, \phi) + ib(r, \phi), \\ a(r, \phi) = (\rho^2 - r^2) \cos 2\phi, \quad b(r, \phi) = (\rho^2 - r^2) \sin 2\phi.$$

Therefore

$$|E|^2 = \int_0^\rho e^h dr \int_0^\rho e^{\bar{h}} dr.$$

Differentiating with respect to  $\phi$  and setting

$$h^* = a^* + ib^*, \quad a^* = a(r^*, \phi), \quad b^* = b(r^*, \phi), \\ f = \cos(b^* - b) - i \sin(b^* - b)$$

we obtain

$$\begin{aligned} (|E|^2)_\phi &= \int_0^\rho e^h h_\phi dr \int_0^\rho e^{\bar{h}^*} d\bar{r}^* + \int_0^\rho e^{h^*} d\bar{r}^* \int_0^\rho e^{\bar{h}} h_\phi dr \\ &= \int_0^\rho \int_0^\rho e^{a+a^*} \{f h_\phi + \bar{f} \bar{h}_\phi\} dr d\bar{r}^* . \end{aligned}$$

Now

$$a_\phi = -2(\rho^2 - r^2) \sin 2\phi, \quad b_\phi = 2(\rho^2 - r^2) \cos 2\phi$$

and therefore

$$\begin{aligned} f h_\phi + \bar{f} \bar{h}_\phi &= 2\Re f h_\phi = 2[\cos(b^* - b)a_\phi + \sin(b^* - b)b_\phi] \\ &= -4(\rho^2 - r^2) \sin(\alpha(\phi)) \end{aligned}$$

where

$$\alpha(\phi) = 2\phi + b - b^* = (r^{*2} - r^2) \sin 2\phi + 2\phi .$$

This yields

$$(3.1) \quad (|E|^2)_\phi = -4 \int_0^\rho \int_0^\rho e^{a+a^*} (\rho^2 - r^2) \sin(\alpha(\phi)) dr d\bar{r}^* .$$

Since from (1.2) we have  $|r^{*2} - r^2| < 1$ , we obtain

$$\alpha'(\phi) = 2 + 2(r^{*2} - r^2) \cos 2\phi > 0 .$$

Hence  $\alpha(\phi)$ ,  $0 \leq \phi \leq \pi/2$ , has its maximum at  $\phi = \pi/2$ . Therefore  $0 \leq \alpha(\phi) < \pi$  when  $0 \leq \phi < \pi/2$  and  $\sin(\alpha(\phi)) > 0$  when  $0 < \phi < \pi/2$ . This means that the integrand in (3.1) is positive in the region  $0 \leq r \leq \rho$ ,  $0 \leq r^* \leq \rho$  for all  $\phi$  in the interval  $0 < \phi < \pi/2$ . Thus  $(|E|^2)_\phi < 0$  when  $0 < \phi < \pi/2$ . This proves (D).

We note that (D) remains true if  $K$  is replaced by quadrants of circles of radii somewhat larger than  $\rho$ ; this, however, is of no interest here.

For  $z_1 \in K$ ,  $z_2 \in S_2$ , or  $z_1 \in K$ ,  $z_2 \in K$ , equation (2.2) holds, as follows from (D). For  $z_1 \in K$ ,  $z_2 \in S_1$  the same is true because of (C). In the other cases,  $z_1 \in S_1$ ,  $z_2 \in S_1$ , etc., the validity of (2.2) is obvious. This proves the lemma.

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$$\rho = 0.92413 \ 88730 .$$

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