ON INTEGRATION OF 1-FORMS

Maurice Sion

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1. **Introduction.** It has been noted by several people that in order to define the integral of some differential 1-form ω along a curve C, the latter need not be of bounded variation. For example, in the extreme (and trivial) case where ω is the differential of some function f, the integral can be defined as the difference of the values assumed by f at the end-points of C. No condition on C is necessary. H. Whithney [4], with J. H. Wolfe, by the introduction of certain norms, has found general abstract spaces of curves along which the integral of 1-forms satisfying certain conditions can be defined. In fact, H. Whitney considers integration of p-forms with $p \geq 1$. In a previous paper [2], we obtained rather awkward conditions for a decent integral to exist that depended on the number of higher derivatives of ω on C.

In this paper, we consider 1-forms ω possessing 'higher derivatives' on C in a sense somewhat different from that due to H. Whitney [3] which we used previously. A Lipschitz type condition on the remainders of the Taylor expansion is imposed (see 4.1.). We define the α-variation of a curve as the supremum of sums of αth powers of chords (see 2.7) and show that the integral of ω along C exists if the α-variation of C is bounded, where α is related to the number of 'higher derivatives' of ω on C. Under somewhat stronger hypotheses on C, we show that this integral is an anti-derivative of ω on C.

2. **Notation and basic definitions.** Throughout this paper, N is a positive integer and we use the following notation.

2.1. E denotes Euclidean $(N + 1)$-space.

2.2. $\|x\| = \left(\sum_{i=0}^{N} x_i^2\right)^{1/2}$ for $x \in E$.

2.3. $\operatorname{diam} U = \sup\{d : d = \|x - y\|$ for some $x \in U$ and $y \in U\}$

2.4. φ is a continuous function on the closed unit enterval to E and $C = \operatorname{range} \varphi$.

2.5. $\mathscr{S}$ is the set of all subdivisions of the unit interval, i.e. functions T on $\{0, 1, \cdots, k\}$ for some positive integer k such that:
$T(0) = 0$, $T(k) = 1$, $T(i - 1) < T(i)$ for $i = 1, \cdots, k$

2.6. $[T/a, b] = \{i : a \leq T(i - 1) < T(i) \leq b\}$

2.7. $V_\alpha(a, b) = \sup_{T \in \mathscr{S}} \sum_{i \in [T/a, b]} \|\varphi(T(i - 1)) - \varphi(T(i))\|^\alpha$

3. Properties of V_α.

3.1. LEMMA. *If* $0 \le a \le b \le c \le 1$, *then*

$$V_\alpha(a, b) + V_\alpha(b, c) \le \alpha(a, c) \le V_\alpha(a, b) + V_\alpha(b, c) + (\text{diam } C)^\alpha$$

3.2. LEMMA. *If* $\alpha < \beta$ *and* $V_\alpha(a, b) < \infty$, *then* $V_\beta(a, b) > \infty$.

Proof. Since $V_\alpha(a, b) < \infty$, there is an integer n such that there are at most n elements $i \in [T/a, b]$ with $\| \varphi(T(i - 1)) - \varphi(T(i)) \| \ge 1$ for any $T \in \mathscr{S}$. For any other $i \in [T/a, b]$ we have

$$\| \varphi(T(i - 1)) - \varphi(T(i)) \|^\beta < \| \varphi(T(i - 1)) - \varphi(T(i)) \|^\alpha .$$

Hence,

$$V_\beta(a, b) < V_\alpha(a, b) + n(\text{diam } C)^\beta < \infty .$$

4. Integration of 1-forms.

In this section, we first define the kind of differential form we shall be dealing with. Our definition is a variant of Whitney's definition of a function m times differentiable on a closed set [3]. Next, we choose a special sequence of subdivisions and proceed to define the integral of the form over the curve C by taking sums of polynomials of degree m and then passing to the limit. Under conditions involving the generalized variation V_α, we show that the integral exists and possesses, in particular, the properties of linearity and 'anti-derivative'.

Throughout this section, m is a positive integer, $\eta \ge 0$, $K > 0$.

4.1. The Differential Form.

Let

$$\sigma k = \sum_{i=0}^{N} k_i \text{ for any } (N + 1)\text{-tuple } k .$$

A differential 1-form ω on C is a function on the set of all $(N + 1)$-tuples k, for which k_i is a non-negative integer for $i = 0, \cdots, N$ and $1 \le \sigma k \le m$, to the set of real-valued functions on C such that

$$\omega_k(y) = \sum_{\sigma j=0}^{m-\sigma k} \omega_{k+j}(x) \frac{(y_0 - x_0)^{j_0} \cdots (y_N - x_N)^{j_N}}{j_0 ! \cdots j_N !} + R_k(x, y)$$

where

$$| R_k(x, y) | < K \| x - y \|^{m+\eta-\sigma k} \text{ for } x \in C \text{ and } y \in C .$$

It is important to note that, in case $m = 1$ and $\eta > 0$, ω is a differential form on C satisfying a Hölder condition. If however $m > 1$, then ω is also a closed differential form on C, that is, $d\omega = 0$ on C.

By taking $m = 1$ and $\eta = 1$, we get the sharp forms considered by Whitney. The conditions we impose on C, however, are quite different and we feel, in practice easier to check than those obtained in [4].

4.2. *The sequence of subdivisions.* We define first, for each $(n + 1)$-tuple of non-negative integers $(s_0, \cdots, s_n)$, a point $t(s_0, \cdots, s_n)$ by recursion on n and on s_n. These will be the end-points of the nth subdivision of the unit interval.

4.2.1. DEFINITION. $t(0) = 0, \qquad t(1) = 1$,

$$t(s_0, \cdots, s_n, 0) = t(s_0, \cdots, s_n) ,$$

$$t(s_0, \cdots, s_n, j + 1) = \sup \{u : t(s_0, \cdots, s_n, j) \leq u \leq t(s_0, \cdots, s_n + 1)$$

and $\| \varphi(u') - \varphi(t(s_0, \cdots, s_n, j)) \| \leq \dfrac{1}{2^{n+1}}$ for $t(s_0, \cdots, s_n, j) \leq u' \leq u\}$

for any non-negative integers n and j.

We shall denote by T the sequence of subdivisions of the unit interval such that :

range $T_n = \{u : u = t(s_0, \cdots, s_n)$ for some n-tuple $(s_0, \cdots, s_n)\}$.

4.2.2. LEMMA. *For any non-negative integers n and j, we have*

$$t(s_0, \cdots, s_n) \leq t(s_0, \cdots, s_n, j) \leq t(s_0, \cdots, s_n + 1) .$$

4.2.3. LEMMA. *For any positive integer n, $i \in [T_n/0, 1], j \in [T_{n-1}/0, 1]$ we have* : T_{n+1} *is a refinement of* T_n, *i.e.* range $T_n \subset$ range T_{n+1};

if $$T_n(i - 1) \leq u \leq T_n(i) ,$$

then

$$\| \varphi(T_n(i - 1)) - \varphi(u) \| \leq \frac{1}{2^n} ;$$

if

$$T_{n-1}(j - 1) \leq T_n(i - 1) < T_n(i) < T_{n-1}(j) ,$$

then

$$\| \varphi(T_n(i - 1)) - \varphi(T_n(i)) \| = \frac{1}{2^n} .$$

4.2.4 LEMMA. *If $F(x, y)$ is a real number whenever $0 \leq x \leq y \leq 1$, $a \in$ range T_n, $b \in$ range T_n, and $a \leq b$, then*

$$\sum_{i \in [T_{n+1}/a, b]} F(T_{n+1}(i - 1), T_{n+1}(i)) = \sum_{j \in [T_n/a, b]} \sum_{i \in [T_{n+1}/T_n(j-1), T_n(j)]} F(T_{n+1}(i - 1), T_{n+1}(i)) .$$

4.3. *The integral of* ω. First, we define $\int_b^a \omega d\varphi$ as the limit of certain sums of polynomials.

4.3.1. *Definitions.*

$$P'(x, y) = \sum_{\sigma k=1}^{m} \omega_k(x) \frac{(y_0 - x_0)^{k_0} \cdots (y_N - x_N)^{k_N}}{k_0 ! \cdots k_N !} ,$$

$$P(a, b) \;= P'(\varphi(a), \varphi(b)),$$

$$S_n(a, b) = \sum_{i \in [T_n/a, b]} P(T_n(i-1), T_n(i)) ,$$

$$\int_a^b \omega d\varphi \;= \lim_{n \to \infty} S_n(a, b) .$$

Next, in order to prove the existence of $\int_a^b \omega d\varphi$ and some of its properties under conditions involving $V_\alpha(a, b)$ for some $\alpha < m + \eta$, we introduce the following.

4.3.2. *Definitions.*

$$R(x, y, z) = P'(x, y) + P'(y, z) - P'(x, z) .$$

$$M = K \sum_{\sigma k=1}^{m} \frac{1}{k_0 ! \cdots k_N !} .$$

$$\beta = m + \eta .$$

4.3.3. LEMMA. *If* $x, y, z \in C, \| x - y \| \leq \delta$ *and* $\| y - z \| \leq \delta$, *then*

$$| R(x, y, z) | < M\delta^\beta .$$

Proof. Let $h(v) = P'(x, v)$ for $v \in E$. Then, h is a polynomial of degree m. Let $\boldsymbol{O}_r = \{k : k$ is an $(N + 1)$-tuple of non-negative integers and $1 \leq \sigma k \leq r\}$.
For $k \in \boldsymbol{O}_r$ and $p \in \boldsymbol{O}_r$, let $p \geq k$ iff $p_i \geq k_i$ for $i = 0, \cdots, N,$ and let

$$D_k h(v) = \frac{\partial^{\sigma k} h(v)}{\partial^{k_0} v_0 \cdots \partial^{k_N} v_N} ,$$

then

$$D_k h(v) = \sum_{\substack{p \in \boldsymbol{O}_m \\ p \geq k}} \omega_p(x) \frac{(v_0 - x_0)^{p_0 - k_0} \cdots (v_N - x_N)^{p_N k_N}}{(p_0 - k_0) ! \cdots (p_N - k_N) !} .$$

Hence, by Taylor's formula

$$h(z) = h(y) + \sum_{k \in \boldsymbol{O}_m} D_k h(y) \frac{(z_0 - y_0)^{k_0} \cdots (z_N - y_N)^{k_N}}{k_0 ! \cdots k_N !} = h(y) +$$

$$+ \sum_{k \in O_m} \left\{ \left[\sum_{\substack{p \in O_m \\ p \geq k}} \omega_p(x) \frac{(y_0 - x_0)^{p_0 - k_0} \cdots (y_N - x_N)^{p_N - k_N}}{(p_0 - k_0)! \cdots (p_N - k_N)!} \right] \right.$$
$$\left. \cdot \frac{(z_0 - y_0)^{k_0} \cdots (z_N - y_N)^{k_N}}{k_0! \cdots k_N!} \right\} .$$

On the other hand from 4.3.1 and 4.1 we have

$$P'(y, z) = \sum_{k \in O_m} \left\{ \left[\omega_k(x) + \sum_{j \in O_{m - \sigma k}} \omega_{k+j}(x) \frac{(y_0 - x_0)^{j_0} \cdots (y_N - x_N)^{j_N}}{j_0! \cdots j_N!} + R_k(x, y) \right] \right.$$
$$\left. \cdot \frac{(z_0 - y_0)^{k_0} \cdots (z_N - y_N)^{k_N}}{k_0! \cdots k_N!} \right\}$$

$$= \sum_{k \in O_m} \left\{ \left[\sum_{\substack{k \in O_m \\ p \geq k}} \omega_p(x) \frac{(y_0 - x_0)^{p_0} - {}^{k_0} \cdots (y_N - x_N)^{p_N - k_N}}{(p_0 - k_0)! \cdots (p_N - k_N)!} + R_k(x, y) \right] \right.$$
$$\left. \cdot \frac{(z_0 - x_0)^{k_0} \cdots (z_N - y_N)^{k_N}}{k_0! \cdots k_N!} \right] \right\}$$

$$= h(z) - h(y) + \sum_{k \in O_m} R_k(x, y) \frac{(z_0 - y_0)^{k_0} \cdots (z_N - y_N)^{k_N}}{k_0! \cdots k_N!} .$$

Making use of the condition on $R_k(x, y)$ stated in 4.1, we get

$$| P'(x, y) + P'(y, z) - P'(x, z) | < \sum_{k \in O_m} \frac{K \| y - x \|^{\beta - \sigma k} \| z - y \|^{\sigma k}}{k_0! \cdots k_N!} \leq M \delta^\beta .$$

4.3.4 LEMMA. *Suppose* $\| x(0) - x(i) \| \leq A$ *and* $\| x(i - 1) - x(i) \| \leq A$ *for* $i = 1, \cdots, p$, *whereas* $\| x(i - 1) - x(i) \| = A/r$ *for* $i = 1, \cdots, p - 1$, *where all* $x(i) \in C$. *Then*

$$\left| \sum_{i=1}^p P'(x(i - 1), x(i)) - P'(x(0), x(p)) \right| < M r^\alpha A^{\beta - \alpha} \sum_{i=1}^p \| x(i - 1) - x(i) \|^\alpha .$$

Proof. $\left| \sum_{i=1}^p P'(x(i - 1), x(i)) - P'(x(0), x(p)) \right|$

$$\leq \sum_{i=2}^p | P'(x(0), x(i - 1)) + P'(x(i - 1), x(i)) - P'(x(0), x(i)) |$$

$$= \sum_{i=2}^{p-1} | R(x(0), x(i-1), x(i)) | < (p-1) M A^\beta = (p-1) M r^x A^{\beta - \alpha} \left(\frac{A}{r} \right)^\alpha$$

$$= M r^\alpha A^{\beta - \alpha} \sum_{i=1}^{p-1} \| x(i-1) - x(i) \|^\alpha \leq M r^\alpha A^{\beta - \alpha} \sum_{i=1}^p \| x(i-1) - x(i) \|^\alpha .$$

4.3.5 LEMMA. *Let* $n > 1$, $a \in \text{range } T_n$, $b \in \text{range } T_n$, $a \leq b$,
$[T_{n-1}/a, b] = 0$. *Then*
$| S_n(a, b) - P(a, b) | < M 5^\beta V_\beta(a, b)$.

Proof. Let

$$a' = \sup \{ u : u \in \text{range } T_{n-1} \text{ and } u \leq a \}$$
$$b' = \sup \{ u : u \in \text{range } T_{n-1} \text{ and } u \leq b \} .$$

First, suppose $a \leq b' \leq b$. Then $a' < a$ and, by 4.2.3

$$\| \varphi(u) - \varphi(a') \| \leq \frac{1}{2^{n-1}} \qquad \text{for } a' \leq u \leq b'$$

$$\| \varphi(u) - \varphi(b') \| \leq \frac{1}{2^{n-1}} \qquad \text{for } b' \leq u \leq b .$$

Hence

$$\| \varphi(T_n(i)) - \varphi(a) \| \leq \frac{2}{2^{n-1}} \qquad \text{for } i \in [T_n/a, b] ,$$

$$\| \varphi(T_n(i)) - \varphi(b') \| \leq \frac{1}{2^{n-1}} \qquad \text{for } i \in [T_n/b', b] ,$$

$$\| \varphi(T_n(i-1)) - \varphi(T_n(i)) \| = \frac{1}{2^n} \qquad \text{for } i \in [T_n/a, b], T_n(i) \neq b', T_n(i) \neq b .$$

Replacing α by β in 4.3.4 and using 4.3.3 and 3.1, we see that

$$| S_n(a, b) - P(a, b) | = | S_n(a, b') + S_n(b', b) - P(a, b) |$$
$$\leq | S_n(a, b') - P(a, b') | + | S_n(b', b) - P(b', b) | + | P(a, b') + P(b', b) - P(a, b) |$$
$$< M4^\beta V_\beta(a, b') + M2^\beta V_\beta(b', b) + M V_\beta(a, b) \leq M5^\beta V_\beta(a, b) .$$

Next suppose $b' < a$. Then, for $i \in [T_n/a, b]$,

$$\| \varphi(T_n(i)) - \varphi(a) \| \leq \frac{2}{2^{n-1}} ,$$

$$\| \varphi(T_n(i-1)) - \varphi(T_n(i)) \| = \frac{1}{2^n} .$$

Hence, by 4.3.4,

$$| S_n(a, b) - P(a, b) | < M4^\beta V_\beta(a, b) .$$

4.3.6 LEMMA. Let $a \in \operatorname{range} T_n$, $b \in \operatorname{range} T_n$, $a < b$. Then,

$$| S_{n+1}(a, b) - S_n(a, b) | < M2^\alpha V_\alpha(a, b)\left(\frac{1}{2^{\beta-\alpha}}\right)^n .$$

Proof. Using 4.2.4, 4.2.3 and 4.3.4, we see that

$$| S_{n+1}(a, b) - S_n(a, b) |$$
$$= \left| \sum_{j \in [T_n/a, b]} \left[\sum_{i \in [T_{n+1}/T_n(j-1), T_n(j)]} P(T_{n+1}(i-1), T_{n+1}(i)) - P(T_n(j-1), T_n(j)) \right] \right|$$
$$< \sum_{j \in [T_n/a, b]} \left[M2^\alpha \left(\frac{1}{2^n}\right)^{\beta-\alpha} \sum_{t \in [T_{n+1}/T_n(j-1), T_n(j)]} \| \varphi(T_{n+1}(i-)) - \varphi(T_{n+1}(i)) \|^\alpha \right]$$
$$= M2^\alpha \left(\frac{1}{2^n}\right)^{\beta-\alpha} \sum_{i \in [T_{n+1}/a, b]} \| \varphi(T_{n+1}(i-1)) - \varphi(T_{n+1}(i)) \|^\alpha \leq M2^\alpha V_\alpha(a, b)\left(\frac{1}{2^{\beta-\alpha}}\right)^n .$$

4.3.7. THEOREM. *If $0 \leq a \leq b \leq 1$, $\alpha < \beta$, $V_\alpha(a, b) < \infty$, then*

$$\left| \int_n^b \omega d\varphi \right| < \infty \ .$$

Proof.   Let

$$a_n' = \inf\{u : u \in \text{range } T_n \text{ and } a \le u\} \ ,$$
$$b_n' = \sup\{u : u \in \text{range } T_n \text{ and } u \le b\} \ .$$

If $a = b$, the theorem is trivial.   If $a < b$, for n sufficiently large, we have

$$a \le a_{n+1}' \le a_n' \le b_n' \le b_{n+1}' \le b \ ,$$
$$[T_n/a, a_n'] = 0 \qquad \text{and} \quad [T_n/b_n', b] = 0 \ ,$$
$$\| \varphi(a_{n+1}')\varphi - (a_n')\| \le \frac{2}{2^n} \qquad \text{and} \quad \| \varphi(b_n') - \varphi(b_{n+1}')\| \le \frac{1}{2^n} \ .$$

Hence

$$|S_{n+1}(a, b) - S_n(a, b)| = |S_{n+1}(a_{n+1}', b_{n+1}') - S_n(a_n', b_n')|$$
$$= |S_{n+1}(a_{n+1}', a_n') + S_{n+1}(a_n', b_n') + S_{n+1}(b_n', b_{n+1}') - S_n(a_n', b_n')|$$
$$\le |S_{n+1}(a_{n+1}', a_n') - P(a_{n+1}', a_n')| + |S_{n+1}(a_n', b_n') - S_n(a_n', b_n')|$$
$$+ |S_{n+1}(b_n', b_{n+1}')| - P(b_n', b_{n+1}')| + |P(a_{n+1}', a_n')| + 1P(b_n', b_{n+1}'| < \text{(by 4.3.5, 4.3.6)}$$
$$< M5^\beta V_\beta(a_{n+1}', a_n') + M2^*V_\alpha(a_n', b_n')\left(\frac{1}{2^{\beta-\alpha}}\right)^n + M5^\beta V_\beta(b_n', b_{n+1}') + M'\frac{2}{2^n} + M'\frac{1}{2^n} \ ,$$

where

$$M' = \sup_{\substack{x \in O \\ 1 \le \sigma k < m}} |\omega_k(x)| \sum_{\sigma k = 1}^m \frac{1}{k_0! \cdots k_N!} \ .$$

Therefore, for any positive integer p we have

$$|S_{n+p}(a, b) - S_n(a, b)| \le \sum_{q=0}^{p-1} |S_{n+q+1}(a, b) - S_{n+q}(a, b)|$$
$$< M5^\beta \sum_{q=0}^\infty [V_\beta(a_{n+q+1}', a_{n+q}') + V_\beta(b_{n+q}', b_{n+n+q+1}')] + M2^*V_\alpha(a, b)\sum_{q=0}^\infty \left(\frac{1}{2^{\beta-\alpha}}\right)^{n+q}$$
$$+ 3M' \sum_{q=0}^\infty \frac{1}{2^{n+q}} < M5^\beta(V_\beta(a, a_n') + V_\beta(b_n', b)) + M\frac{2^3}{2^{3-\alpha}-1}V_\alpha(a, b)\left(\frac{1}{2^{3-\alpha}}\right)^n + \frac{6M'}{2^n} \ .$$

Since, by 3.2, $V_\beta(a, b) < \infty$, with the help of 3.1 we see that $V_\beta(a, a_n') \to 0$ and $V_\beta(b_n', b) \to 0$ as $n \to \infty$.   Thus, the $S_n(a, b)$ form a Cauchy sequence and $\left| \int_a^b \omega d\varphi \right| < \infty$.

4.3.8.   Theorem.   *Suppose $\delta > 0$, $\alpha < \beta$, $L < \infty$, $\| \varphi(a) - \varphi(b) \| < 1$, and*

$$V_\alpha(a, b) < L \| \varphi(a) - \varphi(b) \|^\alpha$$

whenever $0 \le a \le b \le 1$ and $b - a < \delta$.   Then, for some $M' < \infty$,

$$\left| \int_a^b \omega d\varphi - P(a, b) \right| < M' \| \varphi(a) - \varphi(b) \|^\alpha$$

whenever $0 \leq a \leq b \leq 1$ and $b - a < \delta$.

Proof. Given $0 \leq a \leq b \leq 1$ and $b - a < \delta$, let

$$a'_q = \inf\{u : u \in \text{range } T_q \text{ and } a \leq u\} \ ,$$
$$b'_q = \sup\{u : u \in \text{range } T_q \text{ and } u \leq b\} \ ;$$

and let n be the integer such that $[T_{n-1}/a, b] = 0$, $[T_n/a, b] \neq 0$.
   Given $\varepsilon > 0$, we can choose p so that

$$\left| \int_a^b \omega d\varphi - S_{n+p}(a'_{n+p}, b'_{n+p}) \right| < \varepsilon$$

and

$$| P(a, b) - P('_{n+p}, b'_{n+p}) | < \varepsilon$$

and

$$| \, \| \varphi(a) - \varphi(b) \| - \| \varphi(a'_{n+p}) - \varphi(b'_{n+p}) \| \, | < \varepsilon \ .$$

Hence we need only to show that

$$| S_{n+p}(a'_{n+p}, b'_{n+p}) - P(a'_{n+p}, b'_{n+p}) | < M' \| \varphi(a'_{n+p}) - \varphi(b'_{n+p}) \|^\alpha$$

for some $M' < \infty$ and all positive integers p.
   We can check that

$$| S_{n+p}(a'_{n+p}, b'_{n+p}) - P(a'_{n+p}, b'_{n+p}) |$$
$$\leq | S_n(a'_n, b'_n) - P(a'_n, b'_n) | + | P(a'_{n+p}, a'_n) + P(a'_n, b'_n) - P(a'_{n+p}, b'_n) |$$
$$+ | P(a'_{n+p}, b'_n) + P(b'_n, b'_{n+p}) - P(a'_{n+p}, b'_{n+p}) |$$
$$+ \sum_{k=0}^{p-1} \{ | P(a'_{n+p}, a'_{n+k+1}) + P((a'_{n+k+1}, a'_{n+k}) - P(a'_{n+p}, a'_{n+k}) |$$
$$+ | P(b'_{n+k}, b'_{n+k+1}) + P(b'_{n+k+1}, b'_{n+p}) - P(b'_{n+k}, b'_{n+p}) |$$
$$+ | S_{n+k+1}(a'_{n+k+1}, a'_{n+k}) - P(a'_{n+k+1}, a'_{n+k}) |$$
$$+ | S_{n+k+1}(b'_{n+k}, b'_{n+k+1}) - P(b'_{n+k+1}, b'_{n+k}) |$$
$$+ | S'_{n+k+1}(a'_{n+k}, b'_{n+k}) - S_{n+k}(a'_{n+k}, b'_{n+k}) | \} \ .$$

Now, we observe that

$$\| \varphi(u) - \varphi(v) \| \leq \frac{2}{2^{n+k}} \qquad \text{for } a'_{n+p} \leq u \leq v \leq a'_{n+k} \ ,$$

$$\| \varphi(u) - \varphi(v) \| \leq \frac{1}{2^{n+k}} \qquad \text{for } b'_{n+k} \leq u \leq v \leq b'_{n+p} \ ,$$

$$[T_{n+k}/a'_{n+k+1}, a'_{n+k}] = 0 \ ,$$
$$[T_{n+k}/b'_{n+k}, b'_{n+k+1}] = 0 \ .$$

Hence by 4.3.5, 4.3.3, 4.3.6 we have

$$| S'_{n+p}(a'_{n+p}, b'_{n+p}) - P(a'_{n+p}, b'_{n+p}) |$$

$$< M5^\beta V_\beta(a'_n, b'_n) + MV_\beta(a'_{n+p}, b'_n) + MV_\beta(a'_{n+p}, b'_{n+p})$$

$$+ M \sum_{k=0}^{p-1} \left\{ V_\alpha(a'_{n+p}, a'_{n+k}) \left(\frac{2}{2^{n+k}} \right)^{\beta-\alpha} + V_\alpha(b'_{n+k}, b'_{n+p}) \left(\frac{1}{2^{n+k}} \right)^{\beta-\alpha} \right.$$

$$\left. + 5^\beta V_\beta(a'_{n+k+1}, a'_{n+k}) + 5^\beta V_\beta(b'_{n+k}, b'_{n+k+1}) + 2^\alpha V_\alpha(a'_{n+k}, b'_{n+k}) \left(\frac{1}{2^{\beta-\alpha}} \right)^{n+k} \right\}$$

$$< M5^\beta V_\beta(a'_{n+p}, b'_{n+p}) + 2MV_\beta(a'_{n+p}, b'_{n+p})$$

$$+ MV_\alpha(a'_{n+p}, b'_{n+p})(2^{\beta-\alpha} + 1 + 2^\alpha) \sum_{k=0}^{\infty} \left(\frac{1}{2^{\beta-\alpha}} \right)^{n+k}$$

$$< MV_\alpha(a'_{n+p}, b'_{n+p}) \left[5^\beta + 2 + (2^{3-\alpha} + 1 + 2^\alpha) \sum_{k=0}^{\infty} \left(\frac{1}{2^{\beta-\alpha}} \right)^{n+k} \right]$$

$$< M' \| \varphi(a'_{n+p}) - \varphi(b'_{n+p}) \|^\alpha$$

where

$$M' = ML \left[5^\beta + 2 + (2^{3-\alpha} + 1 + 2^\alpha) \sum_{k=0}^{\infty} \left(\frac{1}{2^{\beta-\alpha}} \right)^{n+k} \right] < \infty .$$

4.3.9. THEOREM. *If* $0 \leq a \leq b \leq c \leq 1$, $\left| \int_a^b \omega d\varphi + \int_b^c \omega d\varphi \right| < \infty$, *then*

$$\int_a^c \omega d\varphi = \int_a^b \omega d\varphi + \int_b^c \omega d\varphi .$$

Proof. Let

$$a'_n = \sup\{u : u \in \text{range } T_n \text{ and } u \leq b\}$$

$$b'_n = \inf\{u : u \in \text{range } T_n \text{ and } b \leq u\} .$$

We have $\lim_{n \to \infty} P(a'_n, b'_n) = 0$ and for sufficiently large n

$$S_n(a, c) = S_n(a, b) + P(a'_n, b'_n) + S_n(b, c) .$$

Taking the limit on both sides we get the desired result.

4.3.10. REMARK. If ω and ω' are both 1-forms in the sense of 4.1, then so is $(\omega + \omega')$ and

$$\int_a^b (\omega + \omega') d\varphi = \int_a^b \omega d\varphi + \int_a^b \omega' d\varphi$$

provided the right hand side is bounded. This is an immediate consequence of the definitions.

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UNIVERSITY OF CALIFORNIA

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