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ALMOST LOCALLY PURE ABELIAN GROUPS

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0. **Introduction.** It is the purpose of this paper to introduce and to give a preliminary investigation of almost locally pure Abelian groups [see definition 1]. For primary groups the concept of almost locally pure Abelian group coincides with that of no elements of infinite height [Theorem 9].

1. **DEFINITION.** A group (= Abelian group), G , is *almost locally pure* (hereafter abbreviated a.l.p.) if for every finite set of elements g_1, \dots, g_n of G there exists a finitely generated pure subgroup, P , of G which contains g_1, \dots, g_n .

2. **EXAMPLES.** Direct sums of cyclic groups are clearly a.l.p. The complete direct sum of copies of the integers is a.l.p. since by [1] every finite subset is contained in a completely decomposable direct summand and each such summand is free of finite rank.

3. **REMARK.** If one defines a group G to be locally pure if every finite subset generates a pure subgroup, then it is easy to see that G is a direct sum of cyclic groups of prime order, for various primes.

4. **THEOREM.** *A direct sum of a. l. p. groups is a.l.p.*

Proof. Let $G = \sum_{\alpha} \oplus H_{\alpha}$, where \oplus denotes the weak direct sum, and where H_{α} is a.l.p. for all α . Let g_1, \dots, g_n be in G . Now let H_{β} be a summand in which some g_i has a non-zero component, and consider the components $g_{\beta_1}, \dots, g_{\beta_n}$ of g_1, \dots, g_n in H_{β} . In each such H_{β} (there are only a finite number) there exists a finitely generated pure subgroup P_{β} containing $g_{\beta_1}, \dots, g_{\beta_n}$. Then $\sum_{\beta} \oplus P_{\beta}$ is a finitely generated pure subgroup containing g_1, \dots, g_n .

5. **THEOREM.** *If G is a.l.p., if K is a subgroup of G , and if for every finite set of elements g_1, \dots, g_n of G , there exists a pure subgroup, P , of G such that the group generated by K and g_1, \dots, g_n is a subgroup of P and $P|K$ is finitely generated, then $G|K$ is a.l.p.*

If G and $G|K$ are a.l.p., where K is pure in G , then for every finite set of elements g_1, \dots, g_n of G , there exists a pure subgroup, P , of G such that the group generated by K and g_1, \dots, g_n is a subgroup of P , and $P|K$ is finitely generated.

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Proof. For the proof of the first statement, assume there exists such a P in G for each finite set of elements of G . Then if $g_1 + K, \dots, g_n + K$ are elements of G/K , there exists a pure subgroup, P , of G such that the group generated by K and g_1, \dots, g_n is a subgroup of P and P/K is finitely generated. Now P/K is pure in G/K and G/K is a.l.p.

If G/K is a.l.p. and g_1, \dots, g_n are elements of G , then there exists a finitely generated pure subgroup, P/K , of G/K which contains $g_1 + K, \dots, g_n + K$. The inverse image, P , of P/K has the desired properties.

6. COROLLARY. *If G is a.l.p. and H is a finitely generated subgroup, then G/H is a.l.p.*

7. COROLLARY. *If T is the torsion subgroup of an a.l.p. group, then G/T is a.l.p.*

Proof. For g_1, \dots, g_n in G , let H be a finitely generated pure subgroup containing g_1, \dots, g_n . Then by [2], P , the subgroup generated by H and T is pure. Clearly the subgroup generated by T and g_1, \dots, g_n is a subgroup of P and P/T is finitely generated. Hence by the theorem G/T is a.l.p.

8. EXAMPLE. A strong direct sum of a.l.p. groups is not necessarily a.l.p. Consider $G = \sum_n \odot C(p^n)$, where \odot denotes the strong direct sum and $C(p^n)$ is cyclic of order p^n . Then if G were a.l.p., G/T would be a.l.p., where T is the torsion subgroup. But a torsion-free a.l.p. group, F , is only finitely divisible (i.e. for each $x \neq 0$ in F there exists a maximum positive integer, m_x , such that $m_x y = x$ has a solution in F), whereas the element $(0, 1/p, 0, 0, 1/p^2, 0, 0, 0, 1/p^3, 0, 0, 0, 0, 1/p^4, \dots) + T$ is not zero and is divisible by all powers of p .

9. THEOREM. *A torsion group is a.l.p. if and only if its p -components have no elements of infinite height.*

Proof. This follows from the footnote on page 79 of [1] and from 4.

10. LEMMA. *Every subgroup of a torsion-free a.l.p. group is a.l.p.*

Proof. Let H be a subgroup of the torsion-free group, G , and let h_1, \dots, h_n be elements of H . Then there exists a finitely generated pure subgroup, P , of G which contains h_1, \dots, h_n . Since $P \cap H$ is a finitely generated pure subgroup of H , H is a.l.p.

11. LEMMA. *The torsion subgroup, T , of an a.l.p. group, G , is a.l.p.*

Proof. The proof is similar to the proof of Lemma 10.

12. THEOREM. *Every subgroup of an a.l.p. group is a.l.p.*

Proof. Let G be a.l.p., let S be an arbitrary subgroup of G and let T be the torsion subgroup of G . By 7 G/T is a.l.p. and by 10 $(S \cup T)/T$ is a.l.p. Thus $S/(S \cap T)$ is a.l.p. Now let s_1, \dots, s_n be elements of S . Since $S/(S \cap T)$ is a.l.p. there exists a finitely generated pure subgroup, $P/(S \cap T)$, of $S/(S \cap T)$ such that $s_1 + (S \cap T), \dots, s_n + (S \cap T)$ are elements of $P/(S \cap T)$. Since $P/(S \cap T)$ is finitely generated and torsion-free, $P = (S \cap T) \oplus K$, where K is finitely generated and torsion-free. Since K is finitely generated, it is clearly a.l.p., and it follows from 11 and 9 that $S \cap T$ is a.l.p. Hence by 4, P is a.l.p. and s_1, \dots, s_n are elements of P . Thus there exists a finitely generated pure subgroup, P_1 , of P containing s_1, \dots, s_n . Since $S \cap T$ is a pure subgroup of S , P is a pure subgroup of S . Hence P_1 is pure in S .

13. LEMMA. *A countable torsion-free a.l.p. group, G , is free.*

Proof. Let $H_1 \subset H_2 \subset \dots \subset H_n \subset \dots$ be an ascending chain of subgroups of G , each having finite rank r . Let h_1, \dots, h_r be a maximal linearly independent subset of H_1 , and hence of all the H_i 's. Since G is a.l.p., there exists a finitely generated pure subgroup P containing h_1, \dots, h_r . Hence each H_i is contained in P . Since P is free of finite rank, it satisfies the ascending chain condition, so that by Theorem E, page 168, of [3], G is free.

14. THEOREM. *If the torsion subgroup T , of an a.l.p. group, G , has countable index, then T is a direct summand (and the complementary summand is free).*

Proof. By 7 G/T is a.l.p., countable and torsion-free. Thus by 13 G/T is free. Hence $G = T \oplus K$.

15. LEMMA. *A countable a.l.p. p -group, G , is a direct of sum cyclic groups.*

Proof. By Theorem 9 this is a restatement of a theorem of Prüfer [4].

Now we prove a generalization of Prüfer's theorem.

16. THEOREM. *A countable a.l.p. group, G , is a direct sum of cyclic groups.*

Proof. Let T be the torsion subgroup of G . Then by 14, $G = T \oplus K = T_{p_1} \oplus \cdots \oplus K$, where T_{p_i} is the p_i -component of T . Since G is countable it follows from 4 and 15 that G is a direct sum of cyclic groups.

17. REMARKS. From 12 and 16 it follows that every countable subgroup of an a.l.p. group is a direct sum of cyclic groups.

If one represents the group of rational numbers as a quotient group of a free group, one obtains a pure subgroup (the kernel of the mapping) of an a.l.p. group which is not a direct summand.

From 16 it follows that if H is a pure subgroup of G and G/H is both a.l.p. and countable, then H is a direct summand of G and the complementary summand is a direct sum of cyclic groups.

It follows from Corollary 6 that if T is the torsion subgroup of an a.l.p. group, G , and if H/T is finitely generated then $G/H \cong (G/T)/(H/T)$ is also a.l.p.

18. THEOREM. *If H is pure in G and if H and G/H are a.l.p., the G is a.l.p.*

Proof. Let g_1, \dots, g_n be elements of G . Since G/H is a.l.p. there exists a finitely generated pure subgroup, L/H , of G/H which contains $g_1 + H, \dots, g_n + H$. Since H is pure and L/H is finitely generated, $L = H \oplus K$, K finitely generated. Since g_i is in L for $i = 1, \dots, n$, let $g_i = h_i + k_i$. Since H is a.l.p. let P be a finitely generated pure subgroup of H which contains the h_i . Now g_i is in $P \oplus K$ and $P \oplus K$ is pure in L , which is pure in G . Hence $P \oplus K$ is a finitely generated pure subgroup of G which contains the g_i . Hence G is a.l.p.

19. THEOREM. *Every group, G , has a maximal pure a.l.p. subgroup, M , (which may be 0) and 0 is the only pure a. l. p. subgroup of G/M .*

Proof. The existence of M is easily proved by applying Zorn's lemma. If P/M were a non-zero pure a.l.p. subgroup of G/M , then P would be a pure subgroup of G and by Theorem 18 P would be a.l.p., contradicting the maximality of M .

20. COROLLARY. *If G is a p -group and M is a maximal pure a.l.p. subgroup of G , then G/M is divisible.*

Proof. Otherwise $G/M = D \oplus R$, with D divisible and R reduced and R has a finite cyclic direct summand, P , which is a pure a.l.p. subgroup of G/M .

21. COROLLARY. *If G is a p -group and M is a countable maximal pure a.l.p. subgroup, then M is a basic subgroup of G .*

Proof. By Theorem 16 M is a direct sum of cyclic groups and by 20 G/M is divisible. Hence M is a basic subgroup of G .

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