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1. The object of this paper is to obtain a bound for the orders of the components of a system of algebraic difference equations, *each component of which is of dimension zero.* In the analytic case, this roughly amounts to determining the maximum number of arbitrary functions of period unity which each corresponding manifold can possess.

2. We deal with difference polynomials in n indeterminates $y_1, \cdots, y_n$ having coefficients in an inversive difference field, $\mathscr{F}$, of characteristic zero. Transforms are denoted by means of a second subscript appended to Latin letters having a single subscript. Thus, for example, $A_{3,4}^{(2)}$ denotes the fourth transform of $A_3^{(2)}$. The symbol $\mathscr{F}\{y_1, \cdots, y_n\}$ denotes the ring of difference polynomials in the indeterminates $y_1, \cdots, y_n$. The perfect difference ideal generated by a system Φ of difference polynomials is designated $\{\Phi\}$. Unless there is a possibility for confusion, the term "ideal" is used for the longer "reflexive difference ideal". It is well known that every perfect ideal is the intersection of a finite number of prime ideals, none of which contain any other, [4]. As in ordinary or in differential algebra, these prime ideals are termed *components* of the decomposition of the perfect ideal.

If Λ is a prime ideal in $\mathscr{F}\{u_1, \cdots, u_q; y_1, \cdots, y_p\}$, then the u_i are said to constitute a *parametric set of indeterminates*, or briefly *parameters*, of Λ if

(1) Λ contains no nonzero difference polynomial in the u_i alone;

(2) for each k, $1 \leqq k \leqq p$, there exists in Λ a nonzero difference polynomial in y_k and $u_1, \cdots, u_q$.

It is shown in [1, p. 141] that all parametric sets of a given reflexive prime difference ideal Λ contain the same number of parameters. This number is known as the *dimension* of Λ, and is briefly denoted *dim* Λ. If the prime ideal has no parameters, we say its dimension is zero.

By the *order* of a prime ideal Λ in $\mathscr{F}\{y_1, \cdots, y_n\}$, we mean the algebraic dimension of Λ, that is $\partial^0\mathscr{F}(\eta_1, \cdots, \eta_n; \eta_{11}, \cdots, \eta_{n1}; \eta_{12}, \cdots, \eta_{n2}; \cdots)/\mathscr{F}$ or $\partial^0\mathscr{F} < \eta_1, \cdots, \eta_n > /\mathscr{F}$, where $\eta_1, \cdots, \eta_n$ is a generic zero of Λ.

A system of difference (differential) polynomials in $\mathscr{F}\{y_1, \cdots, y_n\}$ is said to be of *type* $(r_1, \cdots, r_n)$ if $r_1, \cdots, r_n$ are the maximum orders of the transforms (derivatives) of $y_1, \cdots, y_n$ respectively that appear in the system.

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3. Ritt proved the following theorem in [2].

If Φ is a system of nonzero differential polynomials in $\mathscr{F}\{y_1, \cdots, y_n\}$ of type $(r_1, \cdots, r_n)$ and Σ is a component of $\{\Phi\}$ of dimension zero, then the order of Σ does not exceed $r_1 + \cdots + r_n$.

We shall prove the following analogous, but weaker theorem for a system of difference polynomials.

THEOREM. *Let $\mathscr{F}$ be an inversive difference field of characteristic zero. If Φ is a system of nonzero difference polynomials in $\mathscr{F}\{y_1, \cdots, y_n\}$ of type $(r_1, \cdots, r_n)$ and every component of $\{\Phi\}$ is of dimension zero, then the order of each component is at most $r_1 + \cdots + r_n$.*

4. LEMMA. *Let $A_1, \cdots, A_p$ be a chain in $\mathscr{F}[u_1, \cdots, u_q; y_1, \cdots, y_p]$, A_i being of class $q + i$. Suppose $A_1, \cdots, A_{p-1}$ is a characteristic set of a prime ideal. Then there exist nonzero polynomials $G_1. \cdots, G_r$ with the following properties*

(i) *For each j, $1 \leqq j \leqq r$, the set*

$$A_1, \cdots, A_{p-1};\ G_j$$

is a characteristic set of a prime ideal.

(ii) *There exists a polynomial G in $u_1, \cdots, u_q$ and a product I of powers of initials of $A_1, \cdots, A_{p-1}$ such that $I(GA_p - G_1 \cdots G_r)$ is a linear combination of $A_1, \cdots, A_{p-1}$.*

(iii) *The G_j are of positive degree in y_p and the sum of these degrees is the degree of A_p in y_p.*

Proof.[1] Let $(\gamma) = (\tau_1, \cdots, \tau_q; \eta_1, \cdots, \eta_{p-1})$ be a generic zero of the prime ideal. Let $C_1, \cdots, C_r$ be the irreducible factors of $A_p(\gamma; y_p)$ in $\mathscr{F}(\gamma)[y_p]$. We note that when the coefficients of the C_j are written in the form φ/ψ, each ψ may be chosen to be a polynomial in the τ_i only, and each φ may be chosen of degree in y_i less than that of A_i in y_i, $(1 \leqq i \leqq p-1)$. Now there exist $G_j \in \mathscr{F}[u_1, \cdots, u_q; y_1, \cdots, y_p]$, $(1 \leqq j \leqq r)$, and $B \in \mathscr{F}[u_1, \cdots, u_q; y_1, \cdots, y_{p-1}]$ with $B(\gamma) \neq 0$, $C_j = G_j(\gamma; y_p)/B(\gamma)$, $deg_{y_p} G_j = deg_{y_p} C_j$. In particular, (iii) holds.

Let $G = B^r$. Then $GA_p - G_1 \cdots G_r$ vanishes when $(u_1, \cdots, u_q; y_1, \cdots, y_{p-1})$ is replaced by (γ). For some I as described in the lemma

$$I(GA_p - G_1 \cdots G_r) \equiv C, \qquad [A_1, \cdots, A_{p-1}]\ ,$$

where $C \in \mathscr{F}[u_1, \cdots, u_q; y_1, \cdots, y_p]$, $\deg_{y_p} C < \deg_{y_p} A_i$, $(1 \leqq i \leqq p-1)$.

[1] We are indebted to the referee for this proof, which is somewhat shorter than ours which consisted of a modification of an old proof of J. F. Ritt's.

Since $C(\gamma; y_p) = 0$, it follows that $C = 0$, so that (ii) holds.

Letting ζ_j be a root of C_j, we see that $A_1, \cdots, A_{p-1}, G_j$ form a characteristic set of the prime ideal with generic zero $(\gamma; \zeta_j)$. This proves (i).

5.. We now prove the theorem. Let Σ be a component of $\{\Phi\}$, If we treat the transforms of $y_1, \cdots, y_n$ as indeterminates in the algebraic sense, then difference polynomials of Σ can be thought of as ordinary polynomials. Let $\Sigma_{u_1,\cdots,u_n}$ denote the set of difference polynomials of Σ considered as algebraic polynomials in the ring $\mathscr{F}[y_1, \cdots, y_{1u_1}; \cdots; y_n, \cdots, y_{nu_n}]$. It is readily seen that $\Sigma_{u_1,\cdots,u_n}$ is an algebraic ideal, and as Σ is prime, it is a prime ideal.

Denote $\Sigma_{r_1,\cdots,r_n}$ by $\bar{\Sigma}$. Then $\bar{\Sigma} \supseteqq \Phi$. Assume $r_1 \geqq r_2 \geqq \cdots \geqq r_n$. Let $r_i - r_{i+1} = k_i$, $(i = 1, \cdots, n-1)$, and $m_i = \sum_{j=i}^{n-1} k_j$. Consider the following array.

$$
(1) \qquad \begin{array}{lllll}
y_1, \cdots, y_{1k_1}, \cdots, y_{1,k_1+k_2}, & \cdots, y_{1m_1}, & \cdots, y_{1r_1} \\
\qquad\qquad y_2, \cdots, y_{2k_2}, & \cdots, y_{2m_2}, & \cdots, y_{2r_2} \\
\qquad\qquad\qquad\qquad y_3, & \cdots, y_{3m_3}, & \cdots, y_{3r_3} \\
& \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\
& & y_n, \cdots, y_{nr_n}\,.
\end{array}
$$

For the purpose of constructing a characteristic set of Σ, let the indeterminates be ordered by reading the foregoing array columnwise. Thus, we have the ordering

$$
(2) \qquad \begin{array}{l}
y_1, \cdots, y_{1k_1}, y_2, \cdots, y_{1,k_1+k_2}, y_{2k_2}, y_3, \cdots, \\
\qquad y_{1m_1}, y_{2m_2}, y_{3m_3}, \cdots, y_n, \cdots, \\
\qquad\qquad y_{1r_1}, y_{2r_2}, y_{3r_3}, \cdots, y_{nr_n}\,.
\end{array}
$$

Let $\mathfrak{A}$ denote the characteristic set of $\bar{\Sigma}$ which we are going to construct with respect to the ordering (2). Denote the polynomial of $\mathfrak{A}$ which introduces y_{ij} by $A_i^{(j)}$. We shall show

(α1) If y_{ih}, $0 \leqq h < r_i$, is introduced by a polynomial in $\mathfrak{A}$, then $y_{i,h+1}$ is introduced by a polynomial in $\mathfrak{A}$;[2]

(α2) y_{ir_i}, $(i = 1, \cdots, n)$, is introduced by a polynomial of $\mathfrak{A}$.

Let A_j denote the jth polynomial of $\mathfrak{A}$. Take A_1 as irreducible. Assume $h \neq r_i$ and $A_i^{(h)} = A_a$. In the construction of $\mathfrak{A}$, suppose all letters of (2) up to but not including $y_{i,h+1}$ have been considered. Thus, if

$$(3) \qquad A_1, \cdots, A_b$$

[2] $A_i^{(0)}$ will denote the polynomial of $\mathfrak{A}$ which introduces y_i. The symbol "y_{i0}" sometimes will be used to designate y_i.

is the beginning of the characteristic set of $\bar{\Sigma}$ so far constructed, then $1 \leqq a \leqq b$. Consider the difference polynomials of Σ as ordinary algebraic polynomials and let Ω_j, $(j = 1, \cdots, b)$, denote the set of all polynomials of Σ of class not more than the class of A_j relative to the ordering (2). Ω_j will then be a prime algebraic ideal having $A_1, \cdots, A_j$ as its characteristic set.

Let R be the algebraic remainder of A_{a1} with respect to (3). Then there is a relation

$$R = BA_{a1} + K_1A_1 + \cdots + K_bA_b ,$$

where B is a product of powers of the algebraic initials of $A_1, \cdots, A_b$, and $K_1, \cdots, K_b$ are polynomials. Since $A_a \in \Sigma$, $A_{a1} \in \Sigma$; and as $h < r_i$, $A_{a1} \in \bar{\Sigma}$. Therefore, $R \in \bar{\Sigma}$. Let γ be the highest power of $y_{i,h+1}$ that appears in A_{a1} and let $\bar{R}$ be the coefficient of $y_{i,h+1}^{\gamma}$ in R. Then

$$\bar{R} = B\bar{A}_{a1} + \bar{K}_1A_1 + \cdots + \bar{K}_bA_b ,$$

where $\bar{A}_{a1}$ is the transform of the algebraic initial of A_a and $\bar{K}_1, \cdots, \bar{K}_b$ are the coefficients of $y_{i,h+1}^{\gamma}$ in $K_1, \cdots, K_b$ respectively. Now as $B \notin \bar{\Sigma}$, we see that $B, A_{a1} \notin \Sigma$. Thus, as each of $A_1, \cdots, A_b$ belongs to Σ, it follows that $\bar{R} \notin \Sigma$, whence a fortiori is not zero. Therefore, R effectively involves $y_{i,h+1}$, and $\bar{R}$ is its algebraic initial.

Now

$$A_1, \cdots, A_b; R \tag{4}$$

may be a characteristic set of some prime algebraic ideal. If not, then by the lemma of § 4, there is a polynomial G such that

$$C(GR - G_1 \cdots G_r) \equiv 0, \qquad [A_1, \cdots, A_b] ,$$

where C is a product of nonnegative integral powers of the initials of $A_1, \cdots, A_b$ and the G_j are nonzero polynomials such that the sum of the degrees in $y_{i;h+1}$ is the degree of R in $y_{i,h+1}$. Moreover, for each j, $1 \leqq j \leqq r$,

$$A_1, \cdots, A_b; G_j$$

is a characteristic set of a prime ideal. Since $A_1, \cdots, A_b$, R belong to $\bar{\Sigma}$, while C does not, at least one of $G_1, \cdots, G_r$ is in $\bar{\Sigma}$, say G_1.

If (4) is a characteristic set of a prime algebraic ideal, designate this ideal by Ω_{b+1} and rename R, A_{b+1}. If not, let G_1 be A_{b+1} and Ω_{b+1} be the prime algebraic ideal of which $A_1, \cdots, A_b, G_1$ is the characteristic set. Thus, a polynomial A_{b+1} in $\bar{\Sigma}$ has been obtained such that

$$A_1, \cdots, A_b, A_{b+1} \tag{5}$$

is a characteristic set of some prime algebraic ideal, Ω_{b+1} in $\mathscr{F}[y_1, \cdots, y_{i,h+1}]$.[3] The initial, $\bar{A}_{b+1}$, of A_{b+1} is reduced with respect to $A_1, \cdots, A_b$ and is lower than A_{b+1}, whence is not contained in $\bar{\Sigma}$. Let the set of all polynomials (considered algebraically) of Σ of class not exceeding that of A_{b+1} be designated Σ'. Then

$$(6) \qquad \Omega_{b+1} \subseteqq \Sigma' ,$$

since the polynomials (5) are in Σ', while their initials are not.

In the characteristic set, (5), of Ω_{b+1}, let A_b, A_{b+1} respectively introduce the cth and dth letters of the ordering (2). Now, as is well known, the dimension of an ideal equals the number of indeterminates diminished by the length of a characteristic set. Consequently,

$$(7) \qquad \dim \Omega_{b+1} = d - (b+1) .$$

Since $A_1, \cdots, A_b$ are polynomials at the beginning of a characteristic set of $\bar{\Sigma}$,

$$(8) \qquad \dim \Sigma' \geqq c - b .$$

Combining (7), (6), and (8), we secure

$$d - b - 1 = \dim \Omega_{b+1} \geqq \dim \Sigma' \geqq c - b .$$

If $d - c = 1$, then $\dim \Omega_{b+1} = c - b = \dim \Sigma'$. Now suppose $d - c > 1$. No characteristic set of $\bar{\Sigma}$ contains a polynomial introducing the $d-c-1$ letters between the cth and dth letters of the ordering (2). Therefore, it follows that the length of any characteristic set of Σ' cannot exceed $b+1$. Consequently, it cannot be that $\dim \Sigma' = c-b+j$, $(j = 0, 1, \cdots, d-c-2)$, for then every characteristic set of Σ' would have length greater than $b+1$. Hence,

$$\dim \Sigma' = \dim \Omega_{b+1} .$$

This, together with (6) imply

$$(9) \qquad \Omega_{b+1} = \Sigma' .$$

Thus, (5) is the beginning of a characteristic set of $\bar{\Sigma}$. A_{b+1} effectively involves $y_{i;h+1}$ since R does. Therefore, A_{b+1} introduces $y_{i,h+1}$ and may be considered as $A_i^{(h+1)}$. Consequently, our assertion ($\alpha 1$) is established.

We now turn to proving ($\alpha 2$). By way of contradiction, suppose

$$(10) \qquad A_1, \cdots, A_e$$

is a characteristic set of $\bar{\Sigma}$ and that y_{i,r_i} is introduced for f values of

[3] The dots in "$\mathscr{F}[y_1, \cdots, y_{i,h+1}]$" represent the letters between y_1 and $y_{i,h+1}$ in the ordering (2).

i, where $f < n$. Let these values be designated $\sigma_1, \cdots, \sigma_f$ and suppose $\sigma_1 < \sigma_2 < \cdots < \sigma_f$. Extend the rows of (1) to include all transforms of $y_1, \cdots, y_n$. Then reading this columnwise, we get an infinite extension of the ordering (2). For convenience we make the following definitions. A polynomial in the ring $\mathscr{F}[y_1, \cdots, y_{ij}]$, where the dots represent the letters between y_1 and y_{ij} in the extension of the ordering (2),[4] will be said to be of *type* (i, j); and if y_{ij} effectively appears in the polynomial, it will be said to be of *effective type* (i, j). Let the set of all polynomials (considered algebraically) of Σ of class not exceeding that of A_i be denoted Σ_i, $(i = 1, \cdots, e)$. We have previously obtained a prime algebraic ideal $\Omega_i = \Sigma_i$, $(i = 1, \cdots, e)$ having $A_1, \cdots, A_i$ as its characteristic set. However, although the method used for getting Ω_i cannot be continued beyond $i = e$, the process will be modified slightly so that an infinite set of prime algebraic ideals Ω_{e+j}, $(j = 1, 2, 3, \cdots)$ will be determined.

Before proceeding, let us make a few observations. Let $A_{\sigma_1}^{(r_{\sigma_1})} = A_g$. Suppose $g \neq 1$. Now if $U_1 \in \Omega_j$, where $j < g$, A_j and A_k are respectively of effective types (u, v) and $(u, v+1)$, then $U_{11} \in \Omega_k$. This follows at once since $U_{11} \in \Sigma_k$. On the other hand, if $U_1 \notin \Omega_j$, where $j < g$, and is of class not exceeding A_j, then $U_{11} \notin \Omega_k$, since otherwise U_{11} would belong to Σ_k and U_1 would be in $\Sigma_j = \Omega_j$.

First we determine Ω_{e+1}; the other Ω_{e+j} will be obtained inductively. Let A_g and A_e respectively introduce $y_{u_1, v_1 - 1}$ and y_{u_2, v_2}. In the extension of the ordering (2), let $y_{u_3 v_3}$ be the letter that immediately follows $y_{u_2 v_2}$, and $y_{u_4 v_4}$ the one that immediately precedes $y_{u_1 v_1}$. If R_g is the algebraic remainder of A_{g1} with respect to (10), then there is a relation

$$I_e A_{g1} - R_g \equiv 0, \qquad [A_1, \cdots, A_e], \tag{11}$$

where I_e is a product of nonnegative integral powers of the initials of $A_1, \cdots, A_e$. Therefore,

$$I_e \bar{A}_{g1} - \bar{R}_g \equiv 0, \qquad [A_1, \cdots, A_e],$$

where $\bar{A}_{g1}$ and $\bar{R}_g$ denote the coefficients in A_{g1} and R_g respectively of the highest power of $y_{u_1 v_1}$ in A_{g1}.

Let Ω'_e be the prime ideal in $\mathscr{F}[y_1, \cdots, y_{u_1 v_1}]$ generated by Ω_e. The polynomials of Ω'_e are those polynomials in $y_{u_3 v_3}, \cdots, y_{u_1 v_1}$ having coefficients in Ω_e. It may, of course, happen that u_3, v_3 are respectively equal to u_1, v_1, in which case "$y_{u_3 v_3}, \cdots, y_{u_1 v_1}$" is to be regarded as simply "$y_{u_1 v_1}$". At any rate, we have

$$I_e \bar{A}_{g1} - \bar{R}_g \in \Omega'_e . \tag{12}$$

[4] Here and elsewhere, where no confusion can result, the dots represent the letters of the extension of the ordering (2) between the given letters.

Now, as I_e is free of $y_{u_3v_3}, \cdots, y_{u_1v_1}$ and $\notin \Omega_e$, it follows that $I_e \notin \Omega'_e$. We claim that $\bar{A}_{g1} \notin \Omega'_e$. To prove this, we write $\bar{A}_{g1}$ as a polynomial in the letters $y_{u_3v_3}, \cdots, y_{u_1v_1}$. Its coefficients are of type (u_2, v_2). Regarding $\bar{A}_g$ as a polynomial in $y_{u_3,v_3-1}, \cdots, y_{u_1,v_1-1}$, its coefficients are of type $(u_2, v_2 - 1)$ unless $v_2 = 0$. If $v_2 = 0$, then the coefficients are of type (u_5, v_5), where in the array (1), $y_{u_5v_5}$ is the last letter which appears in the column headed by y_{1,r_1-1}. In any case, since these coefficients are reduced with respect to $A_1, \cdots, A_{g-1}$, they do not belong to Ω_{g-1}. Therefore, their transforms are not in Ω_e. (If $g = 1$, we still see that the transforms of the coefficients of $\bar{A}_g$ are not in Ω_e. For, if they were in Ω_e, then they would belong to Σ, whence $\bar{A}_1$ would belong to Σ, a contradiction.) Consequently, $\bar{A}_{g1} \notin \Omega'_e$, as was asserted. It now follows from (12) that $\bar{R}_g \notin \Omega'_e$. This means $\bar{R}_g \neq 0$, and so that R_g effectively involves $y_{u_1v_1}$, that is $y_{\sigma_1,r\sigma_1+1}$. Hence, $\bar{R}_g$ is the algebraic initial of R_g. From (11) we see $R_g \in \Sigma$.

If

$$A_1, \cdots, A_e; R_g \tag{13}$$

is a characteristic set of a prime algebraic ideal, we denote this ideal by Ω_{e+1} and R_g by A_{e+1}. If (13) is not a characteristic set of any prime algebraic ideal, then by the lemma of §4, there is a polynomial H such that

$$J_e(HR_g - H_1 \cdots H_q) \equiv 0, \qquad [A_1, \cdots, A_e]\,, \tag{14}$$

where J_e is a product of powers of the initials of $A_1, \cdots, A_e$ and the H_j are polynomials of positive degree in $y_{u_1v_1}$ such that the sum of these degrees is the degree of R_g in $y_{u_1v_1}$. Moreover, for each j, $1 \leqq j \leqq q$,

$$A_1, \cdots, A_e; H_j$$

is a characteristic set of a prime ideal. From (14) it is seen that some H_j, say H_1, belongs to Σ. Let H_1 be A_{e+1} and Ω_{e+1} be the prime algebraic ideal of which $A_1, \cdots, A_e, H_1$ is the characteristic set. Thus, a polynomial A_{e+1} in Σ has been obtained such that

$$A_1, \cdots, A_{e+1} \tag{15}$$

is a characteristic set of some prime algebraic ideal Ω_{e+1} in $\mathscr{F}[y_1, \cdots, y_{u_1v_1}]$.

Now let us assume as inductive hypotheses:

(β1) If $U_1 \in \Omega_h$, $h = g - 2 + j$, and A_h and A_k are respectively of effective types (u, v) and $(u, v + 1)$, then $U_{11} \in \Omega_k$.

(β2) If $U_1 \notin \Omega_h$, $h = g - 2 + j$, U_1 is of type (u, v), A_h and A_k are of effective type (u, v) and $(u, v + 1)$, respectively, then $U_{11} \notin \Omega_k$.

($\beta 3$) $\Omega_{e+1}, \cdots, \Omega_{e+j}$ have been constructed by a process similar to the one described on the preceding pages. That is, if

$$A_1, \cdots, A_{e+j-1} \tag{16}$$

is a characteristic set of Ω_{e+j-1}, the characteristic set of Ω_{e+j} will be

$$A_1, \cdots, A_{e+j-1}, A_{e+j} ,$$

where A_{e+j} is either the algebraic remainder, R_{g+j-1}, of $A_{g+j-1,1}$ with respect to (16) or else is one of the F_i obtained from a factorization equation of the type

$$J_{e+j-1}(FR_{g+j-1} - F_1 \cdots F_r) \equiv 0, \qquad [A_1, \cdots, A_{e+j-1}] , \tag{17}$$

where J_{e+j-1} is a product of nonnegative integral powers of the initials of $A_1, \cdots, A_{e+j-1}$ and the F_i and F are polynomials having properties analogous to those of the G_i and G, respectively, of the lemma of §4.

Our hypotheses have been shown to hold when $j = 1$ if $g > 1$; and, in fact, it has been proven that ($\beta 3$) is true even if $g = 1$. We now verify ($\beta 1$) and ($\beta 2$) for $h = 1$, $g = 1$; that is, for $j = 2$, $g = 1$. Thus, we must prove:

($\varphi 1$) If $g = 1$ and $U_1 \in \Omega_1$, then $U_{11} \in \Omega_{e+1}$.

($\varphi 2$) If $g = 1$ and $U_1 \notin \Omega_1$, where U_1 is of type (σ_1, r_{σ_1}), then $U_{11} \notin \Omega_{e+1}$.

If $U_1 \in \Omega_1$, then for a suitable power N_1, of the initial $\bar{A}_1$ of A_1, we have

$$N_1 U_1 \equiv 0, \qquad [A_1] .$$

Consequently,

$$N_{11} U_{11} \equiv 0, \qquad [A_{11}] .$$

If $N_{11} \in \Omega_{e+1}$, it would then follow that $N_1 \in \Sigma$, which is false. Therefore, $N_{11} \notin \Omega_{e+1}$. Now either A_{e+1} equals the algebraic remainder, R_1, of A_{11} with respect to (10), or else A_{e+1} is an H_i resulting from a factorization equation of the type (14). In either case, we see that $R_1 \in \Omega_{e+1}$, whence, from (11), $A_{11} \in \Omega_{e+1}$. Thus, $U_{11} \in \Omega_{e+1}$ and ($\varphi 1$) is proven.

On the other hand, if U_1 is of type (σ_1, r_{σ_1}) and $\notin \Omega_1$, then $U_1 \notin \Sigma$, whence $U_{11} \notin \Sigma$. Therefore, $U_{11} \notin \Omega_{e+1}$ and ($\varphi 2$) is proven.

We are now ready for our induction. We shall prove ($\gamma 1$), ($\gamma 2$), ($\gamma 3$), where these respectively are like ($\beta 1$), ($\beta 2$), ($\beta 3$) with $j + 1$ replacing j,

Let $U_1 \in \Omega_{h+1}$ and A_{h+1} be of effective type (u, v). Since $h = g - 2 + j$ and $e = g - 1 + f$, it follows that $h + 1 + f = e + j$. Therefore, by hypothesis ($\beta 3$), Ω_{h+1+f} and A_{h+1+f} have been determined. Obviously, A_{h+1+f} is of effective type $(u, v + 1)$. Now for a suitable product of powers, M_1, of the initials of $A_1, \cdots, A_{h+1}$, we have

$$M_1U_1 \equiv 0, \qquad [A_1, \cdots, A_{h+1}] .$$

Consequently,

$$M_{11}U_{11} \equiv 0, \qquad [A_{11}, \cdots, A_{h+1,1}] . \tag{18}$$

By inductive hypothesis (β1), $A_{11}, \cdots, A_{h1} \in \Omega_{h+f} \subset \Omega_{h+1+f}$. Let Ω'_{h+f} be the prime algebraic ideal in $\mathscr{F}[y_1, \cdots, y_{u,v+1}]$ generated by Ω_{h+f}. We know that A_h introduces some indeterminate, say the letter immediately preceding $y_{\bar{u}\bar{v}}$ in the extension of the ordering (2). Also, we know that A_{h+1} introduces y_{uv}. We assert that $M_{11} \notin \Omega_{h+1+f}$. To prove this, suppose otherwise. Then M_{11} has zero remainder with respect to $A_1, \cdots, A_{h+1+f}$, and since M_{11} is free of $y_{u,v+1}$, it has in fact zero remainder with respect to $A_1, \cdots, A_{h+f}$. Thus, $M_{11} \in \Omega'_{h+f}$. Hence, if we consider M_{11} as a polynomial in $y_{\bar{u},\bar{v}+1}, \cdots, y_{u,v+1}$, its coefficients belong to Ω_{h+f}. By the induction hypothesis (β2), therefore, the coefficients of M_1, considered as a polynomial in $y_{\bar{u}\bar{v}}, \cdots, y_{uv}$, are contained in $\Omega_h \subset \Omega_{h+1}$. But then $M_1 \in \Omega_{h+1}$, a contradiction. Hence, our assertion that $M_{11} \notin \Omega_{h+1+f}$ is proved.

If we show that $A_{h+1,1} \in \Omega_{h+1+f}$, then by (18) we shall have $U_{11} \in \Omega_{h+1+f}$ and so (γ1). Now A_{h+1+f} either equals the algebraic remainder, R_{h+1}, of $A_{h+1,1}$ with respect to $A_1, \cdots, A_{h+f}$ or else is some polynomial F_i resulting from a factorization equation of the type (17). That is, we have either

$$I_{h+f}A_{h+1,1} - A_{h+1+f} \equiv 0, \qquad [A_1, \cdots, A_{h+f}] \tag{19}$$

or taking A_{h+1+f} to be F_1,

$$J_{h+f}(FR_{h+1} - A_{h+1+f} \cdot F_2 \cdots F_s) \equiv 0, \qquad [A_1, \cdots, A_{h+f}] , \tag{20}$$

where I_{h+f} and J_{+f} are each products of powers of the initials of $A_1, \cdots, A_{h+f}$ and F is a polynomial having properties analogous to G of the lemma of §4.

If (19) is the case, it is immediate that $A_{h+1,1} \in \Omega_{h+1+f}$. On the other hand, if we have (20), then $R_{h+1} \in \Omega_{h+1+f}$, in which case once again we have $A_{h+1,1} \in \Omega_{h+1+f}$. The proof of ($\gamma$1) is therefore complete.

We turn now to (γ2). Let $U_1 \notin \Omega_{h+1}$, U_1 be of type (u, v), and A_{h+1} be of effective type (u, v). Then A_{h+1+f} is of effective type $(u, v+1)$. Since every component of the ideal $[U_1, \Omega_{h+1}]$ is of lower dimension than $\dim \Omega_{h+1}$, a polynomial, V_1, in the parameters of Ω_{h+1} can be found such that

$$V_1 = W_1U_1 + X_1 , \tag{21}$$

where $X_1 \in \Omega_{h+1}$ and W_1 is of type (u, v). From (21) we secure

$$V_{11} = W_{11}U_{11} + X_{11} .$$

Since we just proved ($\gamma 1$), we see that $X_{11} \in \Omega_{h+1+f}$. If we prove that $V_{11} \notin \Omega_{h+1+f}$, we would have $W_{11}U_{11} \notin \Omega_{h+1+f}$. Then $U_{11} \notin \Omega_{h+1+f}$ and ($\gamma 2$) would immediately follow.

By way of contradiction, suppose $V_{11} \in \Omega_{h+1+f}$. Then V_{11} has zero remainder with respect to $A_1, \cdots, A_{h+1+f}$. Let $A_{\pi_1}, \cdots, A_{\pi_f}$ be those A's of the characteristic set, (10), of $\bar{\Sigma}$ which respectively introduce the least transforms of $y_{\sigma_1}, \cdots, y_{\sigma_f}$. Suppose $A_{\pi_i} = A_{\sigma_i}^{(t_i)}$, $(i = 1, \cdots, f)$. Note that it is not necessarily the case that $\pi_1 < \pi_2 < \cdots < \pi_f$. Let $\lambda_1, \cdots, \lambda_f$ be $\pi_1, \cdots, \pi_f$ arranged in order of increasing magnitude. Then $\lambda_1 = 1$. Now the class of V_{11} in each of the letters $y_{\sigma_1}, \cdots, y_{\sigma_f}$ does not exceed the class of $A_{\pi_1}, \cdots, A_{\pi_f}$ in each of these letters respectively. Thus, V_{11} has zero remainder with respect to $A_1, \cdots, A_{\lambda_f}$. Suppose no transform of y_i, $(i = \rho_1, \cdots, \rho_{n-f})$, is introduced by any of the polynomials of (10). Write V_{11} as a polynomial in the y_{ij} following $y_{\sigma_f t_f}$ in the extension of the ordering (2). These y_{ij} will all be transforms of $y_{\rho_1}, \cdots, y_{\rho_{n-f}}$, hence parameters of Ω_{h+1+f}. Therefore, the coefficients will have zero remainder with respect to $A_1, \cdots, A_{\lambda_f}$, and so belong to $\bar{\Sigma}$. But the inverse transforms are of order less than t_i in y_{σ_i}, $(i = 1, \cdots, f)$. As $\bar{\Sigma}$ contains no polynomials of this sort, a contradiction has been obtained. Consequently, $V_{11} \notin \Omega_{h+1+f}$. This proves ($\gamma 2$).

To establish ($\gamma 3$), that is to construct Ω_{e+j+1}, one need only to proceed in a manner analogous to the way in which Ω_{e+1} was determined, except for the specification "$H_1 \in \Sigma$".

Thus, we have demonstrated for all i:

($\delta 3$) There exist prime algebraic ideals $\Omega_1, \Omega_2, \Omega_3, \cdots$ having the properties

(i) $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots$;

(ii) a characteristic set of Ω_i is $A_1, \cdots, A_i$;

(iii) $\Omega_j = \Sigma_j$, $(j = 1, \cdots, e)$;

(iv) if Ω_{g-1+i} is an ideal in $\mathscr{F}[y_1, \cdots, y_{uv}]$, then $\Omega_{g-1+i+f}$ is an ideal in $\mathscr{F}[y_1, \cdots, y_{u\,v+1}]$.

($\delta 1$) If $U_1 \in \Omega_i$ and A_i and A_k are respectively of effective types (u, v) and $(u, v + 1)$, then $U_{11} \in \Omega_k$.

($\delta 2$) If $U_1 \notin \Omega_i$, U_1 is of type (u, v), A_i and A_k are of effiective types (u, v) and $(u, v + 1)$ respectively, then $U_{11} \notin \Omega_k$.

Let Ω be the union of the Ω_i of ($\delta 3$). Ω is obviously a prime algebraic ideal, and indeed, as we shall see, a reflexive prime difference ideal. If $U_1 \in \Omega$, then there is some i such that $U_1 \in \Omega_i$, whence by ($\delta 1$), for a suitable k, $U_{11} \in \Omega_k \subset \Omega$. Conversely, now suppose $U_{11} \in \Omega$. Then there are positive integers k, i, u, v such that A_i and A_k are of effective types (u, v) and $(u, v + 1)$ respectively, and $U_{11} \in \Omega_k$. Therefore, by ($\delta 2$), $U_1 \in \Omega_i \subset \Omega$. This proves our assertion that Ω is reflexive.

Since the indeterminates $y_{\rho_1}, \cdots, y_{\rho_{n-f}}$ are the parameters of Ω, it follows that $\dim \Omega = n - f$. Now Ω is a divisor of a component of $\{\Phi\}$, say Λ, Therefore, $\dim \Lambda \geqq \dim \Omega = n - f \neq 0$. Hence we have a contradiction of the hypothesis of the theorem that every component of $\{\Phi\}$ is of dimension zero.

Our assertion, ($\alpha 2$), thus has been established.

Let

$$A_1, \cdots, A_t \tag{22}$$

be a characteristic set of $\bar{\Sigma}$. Before proceeding, several consequences of ($\alpha 1$) and ($\alpha 2$) should be noted.

($\varepsilon 1$). When we reach the point in the construction of (22) where transforms of each of the letters $y_1, \cdots, y_n$ have been introduced, all succeeding polynomials of (22) introduce y_1's, $\cdots$, y_n's in order, no further transforms from then on being omitted.

($\varepsilon 2$). $y_{1r_1}, \cdots, y_{nr_n}$ respectively are introduced by the last n polynomials in (22).

($\varepsilon 3$). In forming (22), certain letters are not introduced by any polynomial of (22). The indeterminates represented by these letters constitute a parametric set of $\bar{\Sigma}$.

If we continue the construction which yielded (22), new polynomials, $A_i^{(j)}$,[5] can be formed such that for any positive integer m

$$A_1, \cdots, A_t; A_1^{(r_1+1)}, \cdots, A_n^{(r_n+1)}; \cdots ; A_1^{(r_1+m)}, \cdots, A_n^{(r_n+m)}$$

is a characteristic set of the prime algebraic ideal $\Sigma r_{1+m,\cdots,r_n+m}$ consisting of all polynomials of Σ of type $(r_n, r_n + m)$ with respect to the extension of the ordering (2). Let this ideal be denoted $\Sigma^{(m)}$. By ($\varepsilon 2$), ($\varepsilon 3$), ($\varepsilon 1$), it follows that the maximum number of parameters in $\Sigma^{(m)}$ for any nonnegative integer m is $r_1 + \cdots + r_n$. Consequently,

$$\dim \Sigma^{(m)} \leqq r_1 + \cdots + r_n \, . \tag{23}$$

We prove by way of contradiction that the order of Σ is at most $r_1 + \cdots + r_n$. Suppose the order of Σ is more than $r_1 + \cdots + r_n$. Then for all sufficiently large $a_1, \cdots, a_n$, the dimension of $\Sigma_{a_1,\cdots,a_n}$ is greater than $r_1 + \cdots + r_n$, since by definition the order of Σ is the algebraic dimension of Σ. However, this is a contradiction of (23). Hence, the theorem.

6. The bound

$$r_1 + \cdots + r_n \tag{24}$$

[5] We are extending the meaning of $A_i^{(j)}$, which previously was defined as a polynomial of $\mathfrak{U}$, that is of (22).

which was obtained in the previous section will be denoted $\mathscr{R}$ and called the *Ritt bound*. Let

$$A_1^{(s_1)}, \cdots, A_n^{(s_n)} \tag{25}$$

be those polynomials of the characteristic set, $\mathfrak{U}$, of $\bar{\Sigma}$ which respectively introduce the least transforms of $y_1, \cdots, y_n$. Then $s_i \leqq r_i$, $(i = 1, \cdots, n)$. Then by (ε3) it is clear that the order of Σ will be given by

$$\mathscr{S} = s_1 + \cdots + s_n \,.$$

In the case of differential equations, Jacobi investigated the problem of determining the number of arbitrary constants in the solution of a system of n equations in the variable x and n dependent variables $y_1, \cdots, y_n$. If these equations are denoted

$$B_i = 0, \qquad (i = 1, \cdots, n)\,, \tag{26}$$

and α_{ij} stands for the greatest order of the derivatives of y_i in B_j, then Jacobi asserted, [5] that the number of arbitrary constants in the solution of (26) is no greater than

$$\max\,(\alpha_{1j_1} + \cdots + \alpha_{nj_n}) \tag{27}$$

where $j_1, \cdots, j_n$ is a permutation of $1, \cdots, n$. However, Jacobi's work was largely heuristic and lacked logical rigor.

Ritt in [2, p. 136] has shown that in the case of two algebraic differential equations in two unknowns, Jacobi is essentially correct. That is, Ritt proved:

If Σ, of dimension zero, is a component of the system B_1, B_2, then the order of Σ is at most $\max\,(\alpha_{11} + \alpha_{22}, \alpha_{12} + \alpha_{21})$.

We shall be interested, in the case of n difference equations in n indeterminates, in obtaining an improvement on the Ritt Bound, and in seeing how it compares with the Jacobi number, (27), where that number now applies to difference polynomials. The number, (27), will be denoted $\mathscr{J}$.

7. Let $F_1, \cdots, F_n$ be a system of n nonzero difference polynomials of type $(r_1, \cdots, r_n)$ in the n indeterminates $y_1, \cdots, y_n$, where every component of $\{F_1, \cdots, F_n\}$ is of dimension zero. Suppose among the F_i, there is at least one, say F_k, which does not effectively involve any y_{jr_j} for $j = 1, \cdots, n$. If F_k is of effective type (θ, φ), then the characteristic set of $\{F_1, \cdots, F_n\}$ certainly must contain a polynomial A of the ring $\mathscr{F}[y_1, \cdots, y_{\theta\varphi}]$. Suppose A is of effective type (σ, τ). By (ε3) and (α1), we are sure, therefore, that $y_{\sigma\tau}, \cdots, y_{\sigma r_\sigma}$ are not parameters of $\bar{\Sigma}$, that is of $\Sigma_{r_1,\cdots,r_n}$. Since $\tau < r_\sigma$, we have an improvement on the Ritt bound; r_σ in (24) is to be replaced by τ. However, we have no simple

way of determining σ and τ. But since $r_\sigma - \tau \geqq r_\theta - \varphi > 0$, if we replace r_θ in (24) by φ, we shall still have a bound which is an improvement on $\mathscr{R}$. As θ and φ are given, the new bound is easily found. Should it happen that several of the F_i are devoid of the y_{jr_j}, then possibly (although not necessarily) we may get a further refinement.

If a transform[6] of y_j appears in F_i, let α_{ij} stand for the greatest order of the transforms of y_j in F_i, $(i, j = 1, \cdots, n)$. For a fixed i, consider the set of numbers

$$r_k - \alpha_{ik}, \qquad (k = 1, \cdots, n)\,. \tag{28}$$

(If some α_{ik} are undefined, then (28) will consist of fewer than n numbers). Let $\mathfrak{S}_i$ be the set of values of k among $1, \cdots, n$ which will yield the minimum of the numbers (28). If b_i denotes the greatest member of $\mathfrak{S}_i$, then it will follow that F_i is of effective type (b_i, α_{ib_i}). Hence, if we replace r_{b_i} in (24) by α_{ib_i}, the result will be an improvement on $\mathscr{R}$ if F_i does not effectively involve any y_{jr_j}.

Let $w = \max(r_{b_i} - \alpha_{ib_i})$, $(i = 1, \cdots, n)$, and $\mathscr{G} = \mathscr{R} - w$. Then $\mathscr{G} \leqq \mathscr{R}$, and we have the following

THEOREM. *Let $\mathscr{F}$ be an inversive difference field of characteristic zero. If $F_1, \cdots, F_n$ is a system of n nonzero difference polynomials in $\mathscr{F}\{y_1. \cdots, y_n\}$ of type $(r_1, \cdots, r_n)$ and every component of $\{F_1, \cdots, F_n\}$ is of dimension zero, then the order of each component is at most $\mathscr{G}$.*

8. Although $\mathscr{G}$ is an improvement on $\mathscr{R}$, still in many situations it is larger than $\mathscr{J}$, and of course, under no circumstances[7] is it less than $\mathscr{J}$. However, we shall show in the case of two nonzero difference polynomials F_1, F_2 in y_1, y_2, that $\mathscr{J} = \mathscr{G}$, whence in such a situation *Jacobi's number is a bound.*[8]

To prove that $\mathscr{J} = \mathscr{G}$ in the case of two difference polynomials F_1, F_2 in y_1, y_2, first note that we may assume without loss of generality that $\alpha_{11} = \max(\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22})$. Then $r_1 = \alpha_{11}$ and $r_2 = \max(\alpha_{12}\, \alpha_{22})$. It is easily seen that $\mathscr{J} < \mathscr{R}$ if and only if

$$\alpha_{11} > \alpha_{21} \text{ and } \alpha_{12} > \alpha_{22}\,. \tag{29}$$

Now, since $\mathscr{J} \leqq \mathscr{G} \leqq \mathscr{R}$, it follows that if (29) is not satisfied that $\mathscr{J} = \mathscr{G} = \mathscr{R}$. Therefore, suppose the condition (29) holds. In such

[6] Recall y_j itself is considered as the zero-th transform of y_j.

[7] The Jacobi number, $\mathscr{J}$, has been defined only in the case where no y_j is missing from each F_i. If a polynomial does not involve one of the indeterminates, we shall define its order in that letter to be -1, in which case $\mathscr{J}$ would always have a meaning. In such a situation, $\mathscr{G}$ may be less than $\mathscr{J}$.

[8] If one of y_1, y_2 is missing from one of F_1, F_2, and $\mathscr{J}$ is defined as in footnote 7), then $\mathscr{G}$ is a better bound than $\mathscr{J}$, since in this case $\mathscr{G} \leqq \mathscr{J}$.

an event, $r_1 - \alpha_{11} = 0$, $r_2 - \alpha_{12} = 0$, $r_1 - \alpha_{21} > 0$, $r_2 - \alpha_{22} > 0$. Hence, $w = \min(r_1 - \alpha_{21}, r_2 - \alpha_{22})$ and $\mathscr{G} = r_1 + r_2 - \min(r_1 - \alpha_{21}, r_2 - \alpha_{22}) = \max(r_1 + \alpha_{22}, r_2 + \alpha_{21}) = \mathscr{J}$.

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Pacific Journal of Mathematics
Vol. 9, No. 2 June, 1959

Lee William Anderson, *On the breadth and co-dimension of a topological lattice* . . . 327
Frank W. Anderson and Robert L. Blair, *Characterizations of certain lattices of functions* . . . 335
Donald Charles Benson, *Extensions of a theorem of Loewner on integral operators* . . . 365
Errett Albert Bishop, *A duality theorem for an arbitrary operator* . . . 379
Robert McCallum Blumenthal and Ronald Kay Getoor, *The asymptotic distribution of the eigenvalues for a class of Markov operators* . . . 399
Delmar L. Boyer and Elbert A. Walker, *Almost locally pure Abelian groups* . . . 409
Paul Civin and Bertram Yood, *Involutions on Banach algebras* . . . 415
Lincoln Kearney Durst, *Exceptional real Lehmer sequences* . . . 437
Eldon Dyer and Allen Lowell Shields, *Connectivity of topological lattices* . . . 443
Ronald Kay Getoor, *Markov operators and their associated semi-groups* . . . 449
Bernard Greenspan, *A bound for the orders of the components of a system of algebraic difference equations* . . . 473
Branko Grünbaum, *On some covering and intersection properties in Minkowski spaces* . . . 487
Bruno Harris, *Derivations of Jordan algebras* . . . 495
Henry Berge Helson, *Conjugate series in several variables* . . . 513
Isidore Isaac Hirschman, Jr., *A maximal problem in harmonic analysis. II* . . . 525
Alfred Horn and Robert Steinberg, *Eigenvalues of the unitary part of a matrix* . . . 541
Edith Hirsch Luchins, *On strictly semi-simple Banach algebras* . . . 551
William D. Munro, *Some iterative methods for determining zeros of functions of a complex variable* . . . 555
John Rainwater, *Spaces whose finest uniformity is metric* . . . 567
William T. Reid, *Variational aspects of generalized convex functions* . . . 571
A. Sade, *Isomorphisme d'hypergroupoï des isotopes* . . . 583
Isadore Manual Singer, *The geometric interpretation of a special connection* . . . 585
Charles Andrew Swanson, *Asymptotic perturbation series for characteristic value problems* . . . 591
Jack Phillip Tull, *Dirichlet multiplication in lattice point problems. II* . . . 609
Richard Steven Varga, *p-cyclic matrices: A generalization of the Young-Frankel successive overrelaxation scheme* . . . 617