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AVERAGES OF FOURIER COEFFICIENTS

RICHARD ROBINSON GOLDBERG

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We shall say the sequence $a_n (n = 1, 2, \dots)$ is a p -sequence ($1 \leq p < \infty$) if there is a function $f \in L^p(0, \pi)$ such that

$$a_n = \int_0^\pi f(t) \cos nt \, dt \quad n = 1, 2, \dots ;$$

(i.e. the a_n are Fourier cosine coefficients of an L^p function).

A famous theorem of Hardy [1] states that if a_n is a p -sequence ($1 \leq p < \infty$) and $b_n = \frac{1}{n}(a_1 + a_2 + \dots + a_n)$, then b_n is also a p -sequence.

In this paper we shall prove the following generalization of Hardy's theorem:

THEOREM 1. *Let $\psi(x)$ be of bounded variation on $0 \leq x \leq 1$, and let $1 \leq p < \infty$. Then, if a_n is a p -sequence and*

$$b_n = \frac{1}{n} \sum_{m=1}^n \psi\left(\frac{m}{n}\right) a_m ,$$

b_n is also a p -sequence.

Hardy's theorem is the special case $\psi(x) = 1$ for $0 \leq x \leq 1$.

If the conclusion of Theorem 1 holds for each of two functions ψ it clearly holds for their difference. Hence it is sufficient to prove Theorem 1 in the case where $\psi(x)$ is non-decreasing for $0 \leq x \leq 1$. Further, since any non-decreasing function may be written as the difference of two non-negative non-decreasing functions (the second of which is constant) to prove Theorem 1 it is sufficient to prove

THEOREM 1A. *Let $\psi(x)$ be non-negative and non-decreasing on $0 \leq x \leq 1$ and let $1 \leq p < \infty$. Then, if a_n is a p -sequence and*

$$b_n = \frac{1}{n} \sum_{m=1}^n \psi\left(\frac{m}{n}\right) a_m ,$$

b_n is also a p -sequence.

The proof of Theorem 1A will follow a sequence of lemmas.

LEMMA 1. *Let $B_\epsilon(x) = \int_0^x \cos yt \, d(y - [y])$. Then there is an $M > 0$*

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such that

$$|B_t(x)| \leq M \quad 0 \leq t \leq \pi; 0 \leq x < \infty .$$

The symbol $[y]$ denotes the greatest integer not exceeding y .

Proof. Let n be any non-negative integer. Then for $t > 0$

$$\int_0^n \cos yt \, dy = \frac{\sin nt}{t}$$

and

$$\int_0^n \cos yt \, d[y] = \sum_{m=1}^n \cos mt = \frac{\sin (n + 1/2)t}{2 \sin t/2} - \frac{1}{2} .$$

Hence

$$\begin{aligned} B_t(n) &= \frac{\sin nt}{t} - \frac{\sin (n + 1/2)t}{2 \sin t/2} + \frac{1}{2} \\ &= \sin nt \left(\frac{1}{t} - \frac{1}{2} \cot \frac{t}{2} \right) - \frac{\cos nt}{2} + \frac{1}{2} \end{aligned}$$

and so

$$(1) \quad |B_t(n)| \leq \left| \frac{1}{t} - \frac{1}{2} \cot \frac{t}{2} \right| + 1 \quad n = 0, 1, 2, \dots$$

The right side of (1) is bounded for $0 < t \leq \pi$. Thus for some $M \geq 1$

$$(2) \quad |B_t(n)| \leq M - 1 \quad n = 0, 1, 2, \dots; 0 < t \leq \pi .$$

Now take any $x \geq 0$ and let $n = [x]$. Then

$$B_t(x) = B_t(n) + \int_n^x \cos ytd(y - [y])$$

so that from (2) we have for any $x \geq 0$

$$|B_t(x)| \leq M - 1 + \int_n^x |d(y - [y])| \leq M - 1 + x - n \leq M, 0 < t \leq \pi$$

and the proof is complete since $B_0(x) : x - [x] \leq 1 \leq M$.

(Henceforth we assume $\psi(x) \geq 0$ and $\psi(x)$ non-decreasing for $0 \leq x \leq 1$.)

LEMMA 2. *There is an $M > 0$ such that*

$$\left| \int_0^n \psi \left(\frac{x}{n} \right) \cos xt \, d(x - [x]) \right| \leq M \quad 0 \leq t \leq \pi; n = 1, 2, \dots$$

Proof. With $B_t(x)$ as in Lemma 1 we have

$$\begin{aligned} \int_0^n \psi\left(\frac{x}{n}\right) \cos xt \, d(x - [x]) &= \int_0^n \psi\left(\frac{x}{n}\right) dB_i(x) \\ &= \psi(1)B_i(n) - \int_0^n B_i(x) d\psi\left(\frac{x}{n}\right). \end{aligned}$$

Thus with M as in Lemma 1

$$\left| \int_0^n \psi\left(\frac{x}{n}\right) \cos xt \, d(x - [x]) \right| \leq M\psi(1) + M \int_0^n d\psi\left(\frac{x}{n}\right) \leq 2M\psi(1),$$

and the lemma is proved (with $2M\psi(1)$ instead of M).

LEMMA 3. Let $f \in L(0, \pi)$ and let

$$d_n = \frac{1}{n} \int_0^\pi f(t) dt \int_0^n \psi\left(\frac{x}{n}\right) \cos xt \, d(x - [x]) \quad n = 1, 2, \dots$$

Then

$$(3) \quad d_n = O\left(\frac{1}{n}\right) \quad n \rightarrow \infty$$

and hence d_n is a p -sequence for every $p \geq 1$.

Proof. By Lemma 2 there is an $M > 0$ such that $|d_n| \leq \frac{M}{n} \int_0^\pi |f(t)| dt$ from which (3) follows. From (3) it follows that $\sum_{n=1}^\infty |d_n|^q < \infty$, for every $q > 1$. By the Hausdorff-Young theorem and the fact that $L^p \subseteq L^{p'}$ if $1 \leq p' \leq p$, this implies that d_n is a p -sequence for every $p \geq 1$. (See [2].)

From now on we shall write $f \sim a_n$ as an abbreviation for $a_n = \int_0^\pi f(t) \cos nt \, dt, n = 1, 2, \dots$

LEMMA 4. Let $1 \leq p < \infty, f \in L^p(0, \pi)$ and $a(x) = \int_0^\pi f(t) \cos xt \, dt$ so that

$$f \sim a_n = a(n).$$

Let

$$g(x) = \int_x^\pi \frac{1}{t} \psi\left(\frac{x}{t}\right) f(t) dt \quad c_n = \frac{1}{n} \int_0^n \psi\left(\frac{x}{n}\right) a(x) dx.$$

Then $g \in L^p(0, \pi)$ and

$$g \sim c_n.$$

Proof. Since $|g(x)| \leq \psi(1) \int_x^\pi \frac{|f(t)|}{t} dt$ it follows from the proof in [1] that $g \in L^p$. Also

$$\begin{aligned}
\int_0^\pi g(x) \cos nx \, dx &= \int_0^\pi \cos nx \, dx \int_x^\pi \frac{1}{t} \psi\left(\frac{x}{t}\right) f(t) \, dt \\
&= \int_0^\pi \frac{1}{t} f(t) \, dt \int_0^t \psi\left(\frac{x}{t}\right) \cos nx \, dx = \int_0^\pi f(t) \, dt \int_0^1 \psi(x) \cos nxt \, dt \\
&= \frac{1}{n} \int_0^\pi f(t) \, dt \int_0^n \psi\left(\frac{x}{n}\right) \cos xt \, dt = \frac{1}{n} \int_0^n \psi\left(\frac{x}{n}\right) \int_0^\pi f(t) \cos xt \, dt = c_n .
\end{aligned}$$

The changes in order of integration are valid since

$$\int_0^\pi |f(t)| \, dt \int_0^1 |\psi(x) \cos nxt| \, dx \leq \psi(1) \int_0^\pi |f(t)| \, dt < \infty .$$

(Note $f \in L'(0, \pi)$ since $f \in L^p(0, \pi)$.) Thus $g \sim c_n$, which is what we wished to show.

We can now establish our principal result.

Proof of Theorem 1A. Let $f \in L^p(0, \pi)$ be such that $f \sim a_n$ and let $a(x), g(x), c_n$ be as in Lemma 4. Then

$$b_n = \frac{1}{n} \sum_{m=1}^n \psi\left(\frac{m}{n}\right) a_m = \frac{1}{n} \int_0^n \psi\left(\frac{x}{n}\right) a(x) \, d[x]$$

so that

$$\begin{aligned}
c_n - b_n &= \frac{1}{n} \int_0^n \psi\left(\frac{x}{n}\right) a(x) \, d(x - [x]) = \frac{1}{n} \int_0^n \psi\left(\frac{x}{n}\right) \, d(x - [x]) \int_0^\pi f(t) \cos xt \, dt \\
&= \frac{1}{n} \int_0^\pi f(t) \, dt \int_0^n \psi\left(\frac{x}{n}\right) \cos xt \, d(x - [x]) .
\end{aligned}$$

The last iterated integral clearly converges absolutely, justifying the change in order of integration. By Lemma 3 $c_n - b_n$ is a p -sequence. Also c_n is a p -sequence since, by Lemma 4, $g \in L^p(0, \pi)$ and $g \sim c_n$. Hence $b_n = c_n - (c_n - b_n)$ is a p -sequence and the theorem is proved.

REMARK. Note that except for the result of Lemma 1 the only properties of the cosine function used were its boundedness and the fact that $O\left(\frac{1}{n}\right)$ is a p -sequence for all $p \geq 1$.

LEMMA 5. Let $C_t(x) = \int_0^x \sin yt \, d(y - [y])$. Then there is an $M > 0$ such that

$$|C_t(x)| \leq M \quad 0 \leq t \leq \pi; 0 \leq x < \infty .$$

Proof. Let n be any non-negative integer. Then for $t > 0$

$$\int_0^n \sin yt \, dy = \frac{1}{t} - \frac{\cos nt}{t}$$

and

$$\int_0^n \sin yt \, d[y] = \sum_{k=1}^n \sin kt = \frac{\cos t/2 - \cos (n + 1/2)t}{2 \sin t/2}.$$

Hence

$$\begin{aligned} C_i(n) &= \frac{1}{t} - \frac{\cos nt}{t} - \frac{\cos t/2 - \cos (n + 1/2)t}{2 \sin t/2} \\ &= (1 - \cos nt) \left(\frac{1}{t} - \frac{1}{2} \cot \frac{t}{2} \right) - \frac{\sin nt}{2}. \end{aligned}$$

The remainder of the proof follows as in Lemma 1.

In view of Lemma 5 and the remark preceding it the exact analogue of Theorem 1 for sine coefficients must hold. This we now state:

THEOREM 2. *Fix $p \geq 1$. If, for some $f \in L^p$,*

$$a_n = \int_0^\pi f(t) \sin ntdt \qquad n = 1, 2, \dots,$$

and if $b_n = \frac{1}{n} \sum_{m=1}^n \psi\left(\frac{m}{n}\right) a_m$ where $\psi(x)$ is of bounded variation on $0 \leq x \leq 1$ then there exists $g \in L^p$ such that

$$b_n = \int_0^\pi g(t) \sin ntdt \qquad n = 1, 2, \dots$$

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2. A. Zygmund, *Trigonometrical Series*, Warsaw 1935 p. 190.

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Pacific Journal of Mathematics

Vol. 9, No. 3

July, 1959

Errett Albert Bishop, <i>A minimal boundary for function algebras</i>	629
John W. Brace, <i>The topology of almost uniform convergence</i>	643
Cecil Edmund Burgess, <i>Chainable continua and indecomposability</i>	653
L. Carlitz, <i>Multiplication formulas for products of Bernoulli and Euler polynomials</i>	661
Eckford Cohen, <i>A class of residue systems (mod r) and related arithmetical functions. II. Higher dimensional analogues</i>	667
Shaul Foguel, <i>Boolean algebras of projections of finite multiplicity</i>	681
Richard Robinson Goldberg, <i>Averages of Fourier coefficients</i>	695
Seymour Goldberg, <i>Ranges and inverses of perturbed linear operators</i>	701
Philip Hartman, <i>On functions representable as a difference of convex functions</i>	707
Milton Vernon Johns, Jr. and Ronald Pyke, <i>On conditional expectation and quasi-rings</i>	715
Robert Jacob Koch, <i>Arcs in partially ordered spaces</i>	723
Gregers Louis Krabbe, <i>A space of multipliers of type $L^p(-\infty, \infty)$</i>	729
John W. Lamperti and Patrick Colonel Suppes, <i>Chains of infinite order and their application to learning theory</i>	739
Edith Hirsch Luchins, <i>On radicals and continuity of homomorphisms into Banach algebras</i>	755
T. M. MacRobert, <i>Multiplication formulae for the E-functions regarded as functions of their parameters</i>	759
Michael Bahir Maschler, <i>Classes of minimal and representative domains and their kernel functions</i>	763
William Schumacher Massey, <i>On the imbeddability of the real projective spaces in Euclidean space</i>	783
Thomas Wilson Mullikin, <i>Semi-groups of class (C_0) in L_p determined by parabolic differential equations</i>	791
Steven Orey, <i>Recurrent Markov chains</i>	805
Ernest Tilden Parker, <i>On quadruply transitive groups</i>	829
Calvin R. Putnam, <i>On Toeplitz matrices, absolute continuity, and unitary equivalence</i>	837
Helmut Heinrich Schaefer, <i>On nonlinear positive operators</i>	847
Robert Seall and Marion Wetzel, <i>Some connections between continued fractions and convex sets</i>	861
Robert Steinberg, <i>Variations on a theme of Chevalley</i>	875
Olga Taussky and Hans Zassenhaus, <i>On the similarity transformation between a matrix and its transpose</i>	893
Emery Thomas, <i>The suspension of the generalized Pontrjagin cohomology operations</i>	897
Joseph L. Ullman, <i>On Tchebycheff polynomials</i>	913
Richard Steven Varga, <i>Orderings of the successive overrelaxation scheme</i>	925
Orlando Eugenio Villamayor, Sr., <i>On weak dimension of algebras</i>	941