ON FUNCTIONS REPRESENTABLE AS A DIFFERENCE OF CONVEX FUNCTIONS

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1. Introduction. A function $f(x)$ defined on a convex x-set D will be called a d.c. function on D if there exists a pair of convex functions $F_1(x)$, $F_2(x)$ on D such that $f(x)$ is the difference

$$(1) \qquad f(x) = F_1(x) - F_2(x) .$$

In this note, "convex" function means "continuous and convex" function. D.c. functions have been considered, for example, by Alexandroff [1]. E. G. Straus mentioned them in a lecture in Professor Beckenbach's seminar (and used the abbreviation "d.c.").

When x is a real variable, so that D is a (bounded or unbounded) interval, then $f(x)$ is a d.c. function if and only if f has left and right derivatives (where these are meaningful) and these derivatives are of bounded variation on every closed bounded interval interior to D. Straus remarked that this fact implies that if $f_1(x), f_2(x)$ are d.c. functions of a real variable, then so are the product $f_1(x)f_2(x)$, the quotient $f_1(x)/f_2(x)$ when $f_2(x) \neq 0$, and the composite $f_1(f_2(x))$ under suitable conditions on f_2. He raised the question whether or not this remark can be extended to cases where x is a variable on a more general space. The object of this note is to give an affirmative answer to this question if x is a point in a finite dimensional (Euclidean) space.

2. Local d.c. functions. Let $f(x)$ be defined on a convex x-set D. The function $f(x)$ will be said to be d.c. at a point x_0 of D if there exists a convex neighborhood U of x_0 such that $f(x)$ is d.c. on $U \cap D$. When $f(x)$ is d.c. at every point x of D, it will be said to be locally d.c. on D.

(I) *Let D be a convex set in an m-dimensional Euclidean x-space and let D be either open or closed. Let $f(x)$ be locally d.c. on D. Then $f(x)$ is* d.c. *on D.*

While the proof of (I) cannot be generalized to the case where the m-dimensional x-space is replaced by a more general linear space, it will be clear that (II), below remains valid if the Euclidean x-space (but not the y-space) is replaced by a more general space.

(II) *Let* $x = (x^1, \cdots, x^m)$ *and* $y = (y^1, \cdots, y^n)$. *Let* D *and* E *be convex sets in the* x- *and* y-*spaces, respectively; let* D *be either open or closed and let* E *be open. Let* $g(y)$ *be a* d.c. *function on* E *and let* $y^j = y^j(x)$ *where* $j = 1, \cdots, n$, *be* d.c. *functions on* D *such that* $y = y(x) \in E$ *for* $x \in D$. *Then* $f(x) = g(y(x))$ *is locally* d.c. *on* D.

This theorem is false (even for $n = m = 1$) if the assumption that E is open is omitted. In order to see this, let x and y be scalars, $g(y) = 1 - y^{1/2}$ ($\leqq 1$) on E: $0 \leqq y < 1$ and let $y = y(x) = |x - \frac{1}{2}|$ on D: $0 < x < 1$. Since $f(x) = g(y(x)) = 1 - |x - \frac{1}{2}|^{1/2}$ does not have finite left and right derivates at the interior point $x = \frac{1}{2}$, the function $f(x)$ is not d.c. at the point $x = \frac{1}{2}$.

It will be clear from the proof that (II) remains correct if the assumption that E is open is replaced by the following assumption on E and $g(y)$: if x_0 is any point of D and $y_0 = y(x_0)$, let there exists a convex y-neighborhood V of y_0 such that $g(y)$ satisfies a uniform Lipschitz condition on $V \cap E$. (This condition is always satisfied if y_0 is an interior point of E; cf., e.g., Lemma 3 below).

COROLLARY. *Let* D *be either an open or a closed convex set in the* $(x^1, \cdots, x^m)$-*space. Let* $f_1(x), f_2(x)$ *be* d.c. *functions on* D. *Then the product* $f_1(x)f_2(x)$ *and, if* $f_1(x) \neq 0$, *the quotient* $f_2(x)/f_1(x)$ *are* d.c. *functions on* D.

The assertion concerning the product follows from (I) and (II) by choosing y to be a binary vector $y = (y^1, y^2)$, $g(y) = y^1 y^2$, E the (y^1, y^2)-plane and $y^1 = f_1(x)$, $y^2 = f_2(x)$. Thus $f(x) \equiv g(y(x)) = f_1(x)f_2(x)$. Note that $g(y) = \frac{1}{2}(y^1 + y^2)^2 - \frac{1}{2}((y^1)^2 + (y^2)^2)$ is a d.c. function on E.

In the assertion concerning the quotient, it can be supposed that $f_2(x) \equiv 1$ and that $f_1(x) > 0$. Let y be a scalar, $g(y) \equiv 1/y$ on E: $y > 0$ and $y = f_1(x)$ on D. Thus $g(y)$ is convex on E and $f(x) \equiv g(y(x)) = 1/f_1(x)$.

3. Preliminary lemmas. It will be convenient to state some simple lemmas before proceeding to the proofs of (I) and (II). The proofs of these lemmas will be indicated for the sake of completeness.

In what follows, $x = (x^1, \cdots, x^m)$ is an m-dimensional Euclidean vector and $|x|$ is its length. D is a convex set in the x-space.

LEMMA 1. *Let* D *be either an open or a closed convex set having interior points. Let* $x = x_0$ *be a point of* D *and* U *a convex neighborhood of* x_0. *Let* $F(x)$ *be a convex function on* $D \cap U$. *Then there exists a neighborhood* U_1 *of* x_0 *and a function* $F_1(x)$ *defined and convex on* D *such that* $F(x) \equiv F_1(x)$ *on* $D \cap U_1$.

In order to see this, let U_2 be a small sphere $|x - x_0| < r$ such that $F(x)$ is bounded on the closure of $D \cap U_2$. Let $G(x) = K|x - x_0| +$

$F(x_0) - 1$, where K is a positive constant, chosen so large that $G(x) > F(x) + 1 > F(x)$ on the portion of the boundary of U_2 interior to D. Clearly $G(x) < F(x)$ holds for $x = x_0$, hence, for x on $D \cap U_1$ if U_1 is a suitably chosen neighborhood of x_0. If $x \in D$, define $F_1(x)$ to be $\max(F(x), G(x))$ or $G(x)$ according as x is or is not in $U_2 \cap D$. Since $\max(F(x), G(x))$ is convex on $U_2 \cap D$ and $\max(F(x), G(x)) \equiv G(x)$ for x in a vicinity (relative to D) of the boundary of U_2 in D, it follows that $F_1(x)$ is convex on D. Finally, $F_1(x) \equiv \max(F(x), G(x)) = F(x)$ for $x \in U_1 \cap D$.

LEMMA 2. *Let D be a closed, bounded convex set having $x = 0$ as an interior point. There exists a function $h(x)$ defined and convex for all x such that $h(x) \leq 1$ or $h(x) > 1$ according as $x \in D$ or $x \notin D$.*

In fact, $h(x)$ can also be chosen so as to satisfy $h(x) > 0$ for $x \neq 0$ and $h(cx) = ch(x)$ for $c > 0$. This function is then the supporting function of the polar convex set of D; Minkowski, cf. [2], § 4. The function $h(x)$ is given by 0 or $|x| \rho^{-1}(x/|x|)$ according as $x = 0$ or $x \neq 0$, where, if u is a unit vector, $\rho(u)$ is the distance from $x = 0$ to the point where the ray $x = tu, t > 0$ meets the boundary of D.

LEMMA 3. *Let D be a closed, bounded convex set having interior points and D_1 a closed convex set interior to D. Let $F(x)$ be a convex function on D. Then $F(x)$ satisfies a uniform Lipschitz condition on D_1.*

In fact, if $d > 0$ is the distance between the boundaries of D and D_1 and if $|F(x)| \leq M$ on D, then $|F(x_1) - F(x_2)| \leq 2M|x_1 - x_2|/d$ for $x_1, x_2 \in D_1$. This inequality follows from the fact that $F(x)$ is convex on the intersection of D and the line through x_1 and x_2.

4. *Proof of* (I). The proof will be given for the case of an open convex set D. It will be clear from the proof and from Lemma 1 how the proof should be modified for the case of a closed D.

To every point x_0 of D, there is a neighborhood $U = U(x_0)$, say U: $|x - x_0| < r(x_0)$, contained in D such that $f(x)$ is d.c. on U; that is, there exists a convex function $F(x) = F(x, x_0)$ such that $f(x) + F(x, x_0)$ is a convex function of x on $U(x_0)$. In view of Lemma 1, it can be supposed (by decreasing $r(x_0)$, if necessary) that $F(x, x_0)$ is defined and convex on D (although, of course, $f + F$ is convex only on U).

Let D_1 be a compact, convex subset of D. Then D_1 can be covered by a finite number of the neighborhoods $U(x_1), \cdots, U(x_k)$. Put $F(x) = F(x, x_1) + \cdots + F(x, x_k)$, so that $F(x)$ is defined and convex on D. Since $f(x) + F(x, x_j)$ is convex on $U(x_j)$, so is $f(x) + F(x) = f(x) + F(x, x_j) +$

$\sum_{i \neq j} F(x, x_i)$.   Hence $f + F$ is convex on D_1.

Thus there exists a sequence of open, bounded convex sets $D_1, D_2, \cdots$ with the properties that the closure of D_j is contained in D_{j+i}, $D = \bigcup D_j$, and to each D_j there corresponds a function $F_j(x)$ defined and convex on D such that $f(x) + F_j(x)$ is convex on D_j.

Introduce a sequence of closed convex sets $C^1, C^2, \cdots$ such that $C^1 \subset D_1 \subset C^2 \subset D_2 \subset \cdots$.   In particular, $D = \bigcup C^j$.

It will be shown that there is a function $G_1(x)$ with the properties that

(i) $G_1(x)$ is defined and convex on D,

(ii) $f(x) + G_1(x)$ is convex on D_2, and

(iii) $G_1(x) \equiv F_1(x)$ on C^1.

If this is granted for the moment, the proof of (1) can be completed as follows: If $G_1, \cdots, G_{k-1}$ have been constructed, let G_k be a function defined and convex on D such that $f + G_k$ is convex on D_{k+1} and $G_k \equiv G_{k-1}$ on C^k.   Then $F(x) = \lim G_k(x)$ exists uniformly on compact subsets of D; in fact, $F(x) \equiv G_j(x)$ on C^k for all $j \geq k$.   Hence, $F(x)$ is defined and convex on D.   Since $f(x) + F(x)$ is convex on C^k, $k = 1, 2, \cdots$, it is convex on D; that is, f is a d.c. function on D.

Thus, in order to complete the proof of (I), it remains to construct a $G_1(x)$ with the properties (i) – (iii).   Let $k > 0$ be a constant so large that $F_2(x) - k \leq F_1(x)$ for $x \in C^1$.   Without loss of generality, it can be supposed that $x = 0$ is an interior point of C^1.   Let $h(x)$ be the function given by Lemma 2 when D there is replaced by C^1.   Put $H(x) = 0$ or $H(x) = K[h(x) - 1]$ according as $x \in C^1$ or $x \notin C^1$, where $K > 0$ is a constant.   Thus $H(x)$ is defined and convex for all x and $H(x) \equiv 0$ on C^1. In particular,

$$(2) \qquad\qquad F_2(x) - k + H(x) \leq F_1(x) \text{ for } x \in C^1 .$$

Choose K so large that

$$(3) \qquad\qquad F_2(x) - k + H(x) > F_1(x) \text{ on } D_1' ,$$

the boundary of D_1.   This is possible since $h(x) - 1 > 0$ for $x \notin C^1$.

Define $G_1(x)$ as follows:

$$(4) \qquad \begin{aligned} &G_1(x) = \max \left(F_1(x), F_2(x) - k + H(x) \right) \text{ for } x \in D_1 , \\ &G_1(x) = F_2(x) - k + H(x) \text{ for } x \in D - D_1 , \end{aligned}$$

where $D - D_1$ is the set of points in D, not in D_1.

Clearly, (2) and the first part of (4) imply property (iii),

$$(5) \qquad\qquad G_1(x) = F_1(x) \text{ if } x \in C^1 ,$$

and (3) implies that

(6) $G_1(x) = F_2(x) - k + H(x)$ for x on and near D_1',

the boundary of D_1.

By the first part of (4), $G_1(x)$ is convex on D_1. By the last part of (4) and by (6), $G_1(x)$ is convex in a vicinity of every point of $D - D_1$. Hence, $G_1(x)$ has property (i), that is $G_1(x)$ is convex on D.

Since $f(x) + F_1(x)$ is convex on D_1 and $f(x) + F_2(x)$, hence $f(x) + F_2(x) - k + H(x)$ is convex on $D_2 \supset D_1$, it follows that, on D_1, the function

$$f(x) + G_1(x) = \max(f + F_1, f + F_2 - k + H)$$

is convex. It also follows from the last part of (4) and from (6) that $f + G_1$ is convex in a vicinity of every point of $D_2 - D_1$. Hence G_1 has property (ii), that is, $f + G_1$ is convex on D_2. This completes the proof of (I).

5. *Proof of* (II). Without loss of generality, it can be supposed that $g(y)$ is convex on E.

Since $y^j(x)$ is a d.c. function on D, there exists a convex function $F(x)$ on D such that

(7) $\pm y^j(x) + F(x)$ are convex on D.

The function $F(x) = F(x, j, \pm)$ can be assumed to be independent of j, where $j = 1, \cdots, n$, and of $\pm$; for otherwise it can be replaced by $\sum_j F(x, j, +) + \sum_j F(x, j, -)$.

Let $x = x_0$ be a point of D and $y_0 = y(x_0)$. Let V be a convex neighborhood of y_0 such that g satisfies a uniform Lipschitz condition

(8) $|g(y_1) - g(y_2)| \leq M |y_1 - y_2|$

on V; cf. Lemma 3. Let U be a neighborhood of x_0 such that $y(x) \in V$ for $x \in U \cap D$. It will be shown that

(9) $f(x) + 3nMF(x)$ is convex on $D \cap U$,

so that f is d.c. at $x = x_0$.

It is clear that there is no loss of generality in assuming that $g(y)$ has continuous partial derivatives satisfying

(10) $|\partial g(y)/\partial y^j| \leq M$ for $j = 1, \cdots, n$ and $y \in V$.

For otherwise, g can be approximated by such functions.

In what follows, only x in $D \cap U$ and $y \in V$ occur. Let $x = x(s)$, where s is a real variable on some interval, be an arc-length parametrization of a line segment in $D \cap U$. The assertion (9) follows if it is shown that $e(s) + 3nMF(x(s))$, where $e(s) = f(x(s))$ is a convex function

of s. It is clear that $e(s)$ has left and right derivatives (whenever these are meaningful). Let $e'(s)$ denote a left or a right derivate of $e(s)$ and $F'(x(s))$, $y^{j'}(x(s))$ the corresponding derivates of $F(x(s))$, $y^j(x(s))$. Let $\Delta s > 0$, then $e(s) + 3nMF(x(s))$ is convex if and only if $\Delta e' + 3nM\Delta F' \geqq 0$, where $\Delta e' = e'(s + \Delta s) - e'(s)$ and $\Delta F' = F'(x(s + \Delta s)) - F'(x(s))$.

By the definition of e,

(11)
$$e' = \sum (\partial g / \partial y^j) y^{j'} .$$

Hence,

(12)
$$\Delta e'(s) = \sum \Delta (\partial g / \partial y^j) y_0^{j'} + \sum (\partial g / \partial y^j)_1 \Delta_y^{j'} ,$$

where $y_0^{j'} = y^{j'}(x(s))$ and $(\partial g / \partial y^j)_1$ is the value of $\partial g / \partial y^j$ at $y = y(x(s + \Delta s))$.

The usual proofs of the mean value theorem of differential calculus (via Rolle's theorem) imply the existence of a $\theta = \theta_j$, $0 < \theta_j < 1$, such that

(13)
$$\Delta_y^j / \Delta s = y_\theta^{j'} ,$$

where $y_\theta^{j'}$ is a number between the left and right derivates of $y^j(x(s))$ at the s-point $s + \theta_j \Delta s$. By (13), the equation (12) can be written as

(14)
$$\Delta e' = \sum (\Delta \partial g / \partial y^j)(\Delta y^j / \Delta s) + \sum (\Delta \partial g / \partial y^j)(y_0^{j'} - y_\theta^{j'})$$
$$+ \sum (\partial g / \partial y^j)_1 \Delta y^{j'}) .$$

By (7),

$$| \Delta y^{j'} | \leqq \Delta F' \text{ and } | y_0^{j'} - y_\theta^{j'} | \leqq F_\theta' - F_0' \leqq \Delta F' ,$$

where F_θ' is the right derivate of $F(x(s))$ at the s-point $s + \theta_j \Delta s$ $(< s + \Delta s)$. Since $g(y)$ is convex, the first term on the right of (14) in non-negative. Hence (10) and (11) give

(15)
$$\Delta e' \geqq 0 - 2nM\Delta F' - Mn\Delta F' ,$$

so that $e(s) + 3nMF(x(s))$ is convex. This proves (II).

6. "Minimal" convex functions. Let $f(x)$ be d.c. on the unit sphere $|x| < 1$, so that there exist functions $F(x)$ on $|x| < 1$ such that

(16)
$$F(x) \text{ and } f(x) + F(x) \text{ are convex on } |x| < 1 .$$

The function $F(x)$ can be chosen so as to satisfy the normalization

(17)
$$F(0) = 0 \text{ and } F(x) \geqq 0 .$$

If x is a real variable, there exists a "least" $F(x)$, say $F_m(x)$, satisfying (16), (17) in the sense that (16), (17) hold for $F = F_m$ and (16), (17) imply

(18) $$F_m(x) \leqq F(x) \text{ on } |x| < 1 .$$

In fact, $F_m(x)$ can be obtained as follows: A (left or right) derivative $f'(x)$ of $f(x)$ is of bounded variation on every interval $|x| \leqq a < 1$ and so $f'(x)$ can be written as $f'(x) = P(x) - N(x)$, where $P(x)$, $N(x)$ are the positive, negative variation of f' on the interval between 0 and x, say, with the normalization $N(0) = 0$. In particular, P and N are non-decreasing on $|x| < 1$. In this case, $F_m(x)$ is given by

$$F_m(x) = \int_0^x N(x) dx .$$

On the other hand, if x is a vector, there need not exist a least $F = F_m(x)$. In order to see this, let x be a binary vector and write (x, y) instead of x. Let $f(x, y) = xy$. If $\varepsilon > 0$, $F(x, y) = \frac{1}{2}(\varepsilon x^2 + y^2/\varepsilon)$ satisfies (16), (17). If a least $F = F_m$ exists, then $0 \leqq F_m(x, y) \leqq \frac{1}{2}(\varepsilon x^2 + y^2/\varepsilon)$. In particular, $0 \leqq F_m(x, 0) \leqq \varepsilon x^2$, and, therefore, $F_m(x, 0) \equiv 0$. Similarly, $F_m(0, y) \equiv 0$. But since F_m is convex, it follows that $F_m \equiv 0$. This contradicts the case $F = F_m$ of (16) and so, a least $F = F_m$ does no exist.

Although a "least" F need not exist, it follows from Zorn's lemma that "minimal" F's do exist.

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