Pacific Journal of Mathematics

A SPACE OF MULTIPLIERS OF TYPE $L^p(-\infty, \infty)$

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Vol. 9, No. 3 July 1959

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1. Introduction. Let $V(G)$ denote the set of all functions having finite variation on G. Set $G = (-\infty, \infty) = \hat{G}$, and let $V_\infty(G)$ be the Banach space of all functions in $V(G)$ which vanish at infinity. If $f \in V_\infty(G)$, then there exists a bounded linear operator $(t_p f)$ on $L^p(\hat{G})$ such that

(i_0) (Fourier transform of $(t_p f)x$) = (Fourier transform of x) $\cdot f$

for all x in $L^p(\hat{G})$. This will be shown in 7.2. In the terminology of Hille [3, p. 566], functions f having property (i_0) are called "factor functions for Fourier transforms of type (L_p, L_p)".

Suppose $1 < p < \infty$. When $f \in L^1(G) \cap V(G) \subset V_\infty(G)$, then $(t_p f)$ is a singular integral operator: for all x in $L^p(\hat{G})$ it is found that $(t_p f)x$ has the form

$$[(t_p f)x]_\lambda = \frac{1}{2\pi i}\int_{-\infty}^{\infty} x(\theta)\frac{F(\theta - \lambda)}{\theta - \lambda}d\theta \qquad (\lambda \in \hat{G}) ,$$

where the integral is taken in the Cauchy principal value sense.

In 6.2 will be defined a set $\blacktriangle(L^p(\hat{G}))$ which contains all factor functions for Fourier transforms of type (L_p, L_p); the set $\blacktriangle(L^p(\hat{G}))$ is a slight extension of what Mihlin [6] calls "multipliers of Fourier integrals". We will find a number N_p such that

(i) *if* $f \in V_\infty(G)$ *then* $f \in \blacktriangle(L^p(\hat{G}))$ *and* $\|(t_p f)\| \leqq N_p \cdot \|f\|_v$,

where $\|f\|_v$ denotes the total variation on G of the function f. Let F_* be the mapping $\{x \to x * F\}$, where $x * F$ is the convolution of the functions x and F;

$$[x * F]_\lambda = \int_{-\infty}^{\infty} x(\theta) \cdot F(\theta - \lambda)d\theta \qquad (\lambda \in \hat{G}).$$

Let (Yf) denote the Fourier transform of the function f:

(ii) *if* $f \in L^1(G) \cap V(G)$, *then the transformation* $(Yf)_*$ *is a densely defined bounded operator, and* $(t_p f)$ *is its continuous linear extension to the whole space* $L^p(\hat{G})$.

Let us for a moment call $G = \{0, \pm 1, \pm 2, \cdots\}$ and $\hat{G} = [0, 1]$. In

Received December 8, 1958, and in revised form February 11, 1959. This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command under contract No. AF 49(638)-505.

a sense, the following relations are duals of (i) and (ii), respectively:

(i′) *if* $F \in V(\hat{G})$ *then* $(\hat{Y}F) \in \blacktriangle(L^p(\hat{G}))$ *and* $\|t_p(YF)\| \leqq k_p \cdot \|F\|_v$

(ii′) *if* $F \in V(\hat{G})$ *then* $F_* = t_p(YF)$ *is a bounded operator on* $L^p(\hat{G})$.

When $\hat{G} = [0, 1]$ these properties are easily verified (see 8.1). We will not[1] prove (i′)-(ii′) for other choices of G.

When $G = [0, 1]$, then (ii) is seen to be a theorem due to Stečkin [10]; by means of appropriate definitions, it could be shown that (i) also holds for this particular choice of G.

2. Applications. If f belongs to the class S of members of $L^1(G) \cap V(G)$ such that $(Yf) \in L^1(\hat{G})$, then $(Yf)_* = (t_p f)$ is a bounded operator defined on all of $L^p(\hat{G})$; it is interesting to compare this result with the conclusion $F_* = t_p(YF)$ of (ii′). All the classical convolution operators (Poisson, Picard, Weierstrass, Stieltjes, Dirichlet, Fejér,..etc. [7]) are of the form $(t_p f)$, where $f \in S$. See § 8.

3. Preliminaries. We assume $1 < p < \infty$ throughout, and write $G = (-\infty, \infty)$. Denote by L^0 the set of step functions with compact support. Let V be the set of all functions a defined on G and such that $\|a\|_v \neq \infty$, where $\|a\|_v$ denotes the total variation on G.

3.1 Definitions. Let V_∞ be the set of all functions a in V such that $\lim a(\theta) = 0$ whenever $|\theta| \to \infty$. We will write L^p instead of $L^p(G)$. If $\iota = 0, 1$ and $f \in L^1$, then the Fourier transforms $[{}_\iota Yf]$ are the functions g_ι defined by

$$(1) \qquad [{}_\iota Yf]_\lambda = g_\iota(\lambda) = \int_{-\infty}^{\infty} \exp(2\pi i\lambda(-1)^\iota\theta) \cdot f(\theta)d\theta \qquad (\lambda \in G) .$$

To lighten the notation, we will write Yf for $[{}_1Yf]$ and Ψf for $[{}_0Yf]$.

3.2 Lemma. *If* $a \in L^1 \cap V$, *then* $a \in V_\infty$ *and*

$$(2) \qquad \int_{-\infty}^{\infty} e^{-2\pi i\theta t} da(t) = 2\pi i\theta \cdot [Ya]_\theta \qquad (\theta \in G) .$$

Proof. Since $a \in V$, the limits $a(\pm\infty) = \lim a(\theta)$ (when $\theta \to \pm\infty$) exist. Since $\|a\|_1 < \infty$ we have

$$(3) \qquad \lim_{\theta \to \pm\infty} \int_\theta^{\theta+1} |a| = 0 .$$

The eventuality $a(\pm\infty) \neq 0$ implies a contradiction of (3). Therefore

[1] It would be of interest to determine the validity of (i)-(ii) and (i′)-(ii′) in the general case where G is a connected locally compact abelian group with dual group $\hat{G}$. It is mainly in order to suggest such an investigation that (i′)-(ii′) are mentioned here.

$a(\pm\infty) = 0$, which permits the integration of (1) by parts to obtain (2).

3.3 DEFINITIONS. Let $\delta_* = (-\infty, -\delta] \cup [\delta, \infty)$ and let $(T_\delta a)x$ be the function defined by

$$[(T_\delta a)x]_\lambda = \int_{\delta_*} d\theta \frac{x(\lambda - \theta)}{2\pi i\theta} \int_{-\infty}^{\infty} e^{-2\pi i\theta t}\, da(t) \tag{4}$$

for all λ in G. We denote by V_1 the set of all members a of V such that, for all x in L^0, the limit

$$[(Ta)x]_\lambda = \lim_{\delta\to 0+} [(T_\delta a)x]_\lambda$$

exists almost-everywhere on G. Let Ta be the operator $\{x \to (Ta)x\}$ defined on L^0.

3.4 LEMMA. *If $h(\theta) = i\theta/|\theta|$, then $h \in V_1$ and Th is the restriction to L^0 of the Hilbert transformation. Moreover $\|(T_\delta h)x\|_p \leqq c_p \cdot \|x\|_p$, where c_p is the norm of Th.*

Proof. This follows from the statement in [8, p. 241] that $\|(T_\varepsilon h)x\|_p \leqq \|(Th)x\|_p$. Theorem G in [1, p. 251] yields a less precise result.

3.5. LEMMA. *If $a \in L^1 \cap V$ then $a \in V_1$ and $x * [Ya] = (Ta)x$ whenever $x \in L^0$.*

Proof. Suppose $\delta > 0$. By definition

$$(x * [Ya])_\lambda = \int_{-\infty}^{\infty} d\theta \cdot x(\lambda - \theta) \cdot [Ya]_\theta = E^\delta(\lambda) + G^\delta(\lambda)\ ,$$

where

$$G^\delta(\lambda) = \int_{\delta_*} d\theta \cdot x(\lambda - \theta) \cdot [Ya]_\theta \qquad (\lambda \in G)\ ,$$

while $E^\delta(\lambda)$ is the same integral over the interval $(-\delta, \delta)$. It is clear that $\lim E^\delta(\lambda) = 0$ when $\delta \to 0+$. On the other hand, $G^\delta = (T_\delta a)x$ follows immediately from (2) and (4). This concludes the proof.

3.6 LEMMA. *Suppose $a \in V_1$ and $x \in L^0$. If there exists a number k_p such that $\|(T_\delta a)x\|_p \leqq k_p$ for all $\delta > 0$, then $\|(Ta)x\|_p \leqq k_p$.*

Proof. Set $q = p/(p-1)$. Observe first that

$$\|g\|_p = \sup\left\{\left|\int g \cdot \varphi\right| : \varphi \in L^q \text{ and } \|\varphi\|_q \leqq 1\right\}. \tag{5}$$

Next, we infer from a theorem of F. Riesz ([8], p. 227 footnote 10) that the uniform boundedness of $\|(T_\delta a)x\|_p$ implies that, for all φ in L^q with $\|\varphi\|_q \leqq 1$:

$$\int [(Ta)x]\cdot\varphi = \lim_{\delta\to 0+} \int [T_\delta a)x]\cdot\varphi \ . \tag{6}$$

By (5) we have $\left|\int [(T_\delta a)x]\cdot\varphi\right| \leqq k_p$; this enables us to use (6) to deduce $\left|\int [(Ta)x]\cdot\varphi\right| \leqq k_p$. The conclusion is reached by another application of (5).

3.7 LEMMA. *If $a \in L^1 \cap V$ and $x \in L^0$, then*

$$\|(Ta)x\|_p \leqq 2^{-1}c_p\|a\|_v\|x\|_p \ .$$

Proof. Suppose $\delta > 0$. Apply Fubini's theorem to (4):

$$[(T_\delta a)x]_\lambda = \int_{-\infty}^{\infty} da(t)e^{-2\pi i\lambda t}\int_{\delta_*} d\theta \frac{x(\lambda-\theta)}{2\pi i\theta} e^{2\pi i t(\lambda-\theta)} \ .$$

Set $x^t(\beta) = x(\beta)\exp(2\pi i t\beta)$. Keeping both (4) and 3.4 in mind, we can therefore write

$$[(T_\delta a)x]_\lambda = (2i)^{-1}\int_{-\infty}^{\infty} da(t)\{e^{-2\pi i\lambda t}[(T_\delta h)x^t]_\lambda\} \ . \tag{7}$$

This implies

$$\|(T_\delta a)x\|_p \leqq 2^{-1}\|a\|_v \sup_{t\in G}\|(T_\delta h)x^t\|_p \ . \tag{8}$$

The derivation of (8) from (7) is obtained by a standard procedure (e.g. as in [3, Lemma 21.2.1]); it rests upon (5) and requires a single application of the Fubini theorem. On the other hand, 3.4 implies that

$$\|(T_\delta h)x^t\|_p \leqq c_p\cdot\|x^t\|_p \leqq c_p\cdot\|x\|_p \ .$$

In view of (8) therefore: $\|(T_\delta a)x\|_p \leqq 2^{-1}c_p\|a\|_v\|x\|_p$. Use now 3.6 to reach the conclusion.

4. The Banach space V_∞. Let V_s denote the set of all functions in V which have compact support. The norm $\{a \to \|a\|_v\}$ makes the set $\{a \in V\colon a(-\infty) = 0\}$ into a Banach space V_0. Note that $V_s \subset V_\infty \subset V_0$. Henceforth V_∞ will be given the topology of V_0. We will write $\|a\|_\infty = \sup\{|a(\theta)|\colon \theta \in G\}$; it is easily checked that

$$\|a\|_\infty \leqq \|a\|_v \qquad \text{(when } a \in V_0) \ . \tag{9}$$

Let χ_n denote the characteristic function of the interval $(-n, n)$, and set $a_n = \chi_n \cdot a$.

4.1 LEMMA. *If* $a \in V_\infty$, *then* $\lim_{n\to\infty} \|a - a_n\|_v = 0$.

Proof. Suppose $f \in V$. Using the notation δ_* of 3.3, we have

$$\|f\|_v = v(f; [-\delta, \delta]) + v(f; \delta_*) , \tag{iii}$$

where $v(f; I)$ denotes the total variation over I. Set $\delta = n$ and $h_n = a - a_n$; therefore $v(h_n; [-\delta, \delta]) = |a(-\delta)| + |a(\delta)|$ and $v(h_n; \delta_*) = v(a; \delta_*)$. From (iii) therefore $\|h_n\|_v = |a(-\delta)| + |a(\delta)| + v(a; \delta_*)$, and the conclusion follows by letting $\delta \to \infty$.

4.2 REMARK. The set V_s is dense in V_∞ (since 4.1 and the fact that $a_n \in V_s$).

4.3 THEOREM. *The set* V_∞ *is a Banach space.*

Proof. Since V_∞ is a metric subspace of the Banach space V_0, it will suffice to show that V_∞ is complete. To that effect, consider a Cauchy sequence $\{g_k\}$ in V_∞; since $\{g_k\}$ is also in V_0, it will converge to some function f in V_0; therefore $f(-\infty) = 0$ and we need only establish that $f(\infty) = 0$. From (9) we see that

$$|f(\theta) - g_k(\theta)| \leqq \|f - g_k\|_v \qquad (\theta \in G) .$$

In view of $g_k(\infty) = 0$, the conclusion is obtained by letting $\theta \to \infty$ and $k \to \infty$.

5. The bilinear operator B_p. From 3.2 results that $V_s \subset L^1 \cap V \subset V_\infty$; it follows from 4.2 that $L^1 \cap V$ is dense in V_∞. Consider the bilinear operator $B = \{(x, a) \to (Ta)x\}$ which maps $L^0 \times (L^1 \cap V)$ into L^p. From 3.7 we see that B is a continuous bilinear mapping of $L^0 \times (L^1 \cap V)$ into L^p. Since L^0 and $L^1 \cap V$ are dense in L^p and V_∞, respectively, it follows that B has a (unique) continuous extension B_p to $L^p \times V_\infty$. Accordingly, if $a \in V_\infty$, then

$$\|B_p(x, a)\|_p \leqq 2^{-1}c_p\|a\|_v\|x\|_p \qquad (\text{if } x \in L^p) \tag{10}$$

If $a \in L^1 \cap V$, then (from 3.5)

$$B_p(x, a) = x * Ya \qquad (\text{if } x \in L^0) . \tag{11}$$

5.1 NOTATION. We henceforth identify functions equal almost-everywhere on G. If the sequence $\{f_n\}$ converges in the mean of order p (i.e., in the topology of L^p), then its limit will be denoted $(L^p)\lim f_n$.

5.2 LEMMA. *Let* $\bar{\chi}_n$ *be the function defined by*

$$\bar{\chi}_n(\theta) = (\sin 2\pi n\theta)/\pi\theta \qquad (\theta \in G) .$$

If $f \in L^p$, *then* $f = (L^p)\lim f * \bar{\chi}_n$ *as* $n \to \infty$.

Proof. Observe that Dunford's proof [2, p. 51, Lemma 3] for the case $p = 2$ holds without alteration whenever $1 < p < \infty$.

6. **The main result.** Suppose $\iota = 0, 1$. When f is a locally integrable function, we set

$$[({}_\iota Y_p)f] = (L^p)\lim_{n\to\infty} [{}_\iota Y(\chi_n \cdot f)] . \tag{12}$$

As in 3.1, we lighten the notation by writing $Y_p f = [({}_1 Y_p)f]$ and $\Psi_p f = [({}_0 Y_p)f]$.

6.1 REMARK. If $f \in L^1$ then $[({}_\iota Y_p)f] = [{}_\iota Yf]$. The following definition is an extension of the one used by Mihlin ("Multipliers of Fourier integrals"[2]).

6.2 DEFINITION. A locally integrable function a is called a *"multiplier of type* L^p" if both the following conditions hold:

$$\begin{cases}\text{the transform } Y_p(a\cdot[\Psi x]) \text{ exists and belongs to } L^p \text{ whenever } x \in L^0\\ \qquad \infty \neq \sup\{\|Y_p(a\cdot[\Psi x])\|_p \colon x \in L^0 \text{ and } \|x\|_p \leqq 1\} .\end{cases}$$

Let $\blacktriangle(L^p)$ denote the set of all multipliers of type L^p. When $a \in \blacktriangle(L^p)$, then $(t_p a)$ is defined as the continuous extension to all of L^p of the transformation $\{x \to Y_p(a\cdot[\Psi x])\}$ defined on L^0.

6.3 THEOREM. *If* $a \in V_\infty$, *then* $a \in \blacktriangle(L^p)$ *and* $(t_p a)x = B_p(x, a)$ *for all* x *in* L^p.

Proof. Note first that $a_n = (\chi_n\cdot a) \in L^1 \cap V$. Suppose $x \in L^0$. From (11) we see that

$$[B_p(x, a_n)]_\lambda = \int d\theta\cdot x(\theta)\int dt\cdot e^{-2\pi i(\lambda-\theta)t}a_n(t) \qquad (\text{when } \lambda \in G) .$$

By Fubini's theorem

$$[B_p(x, a_n)]_\lambda = \int dt\cdot a_n(t)e^{-2\pi i\lambda t}[\Psi x]_t \qquad (\text{for all } \lambda \text{ in } G) .$$

Or, equivalently

$$B_p(x, a_n) = Y(\chi_n\cdot a\cdot[\Psi x]) .$$

[2] See [6]; in that article, Mihlin gives a condition which ensures that a differentiable function be in $\blacktriangle(L^p)$.

From (10) and 4.1 we can now infer that

$$B_p(x, a) = (L^p) \lim_{n\to\infty} Y(\chi_n \cdot \{a \cdot [\Psi x]\}) .$$

From the definition (12) now results that $B_p(x, a) = Y_p(a \cdot [\Psi x])$ for all x in L^0. This completes the proof, in view of (10) and 6.2.

7. Hille's definition. Set $q = p/(p-1)$. The following definition is found in [3, p. 566]: a function a is said to be a *factor function for Fourier transforms of type* (L_p, L_p) if and only if

$$a \cdot [\Psi_q x] \in \{\Psi_q z : z \in L^p\}$$

wherever $x \in L^p$. This definition seems to require the restriction $p \leqq 2$, since $[\Psi_q x]$ need not exist otherwise.

7.1 THEOREM. *Suppose $1 < p \leqq 2$. If a is a factor function for Fourier transforms of type (L_p, L_p), then $a \in \blacktriangle(L^p)$.*

Proof. If a is such a factor function, there exists a bounded linear mapping $(t'_p a)$ of $L^p(G)$ into itself (see [3, Theorem 21.2.1]); this operator is defined by

$$a \cdot [\Psi_q x] = \Psi_q((t'_p a)x) \qquad \text{for all } x \text{ in } L^p .$$

In view of [11, 5.17], this implies

$$Y_p(a \cdot [\Psi_q x]) = (t'_p a)x \qquad \text{for all } x \text{ in } L^p . \tag{13}$$

The conclusion follows from 6.1 and 6.2.

7.2 THEOREM. *Suppose $1 < p \leqq 2$ and $a \in V_\infty$. Then a is a factor function for Fourier transforms of type (L_p, L_p); moreover,*

$$\Psi_q(B_p(x, a)) = a \cdot [\Psi_q x] \qquad (\text{when } x \in L^p) . \tag{14}$$

Proof. Since $B_p(x, a) \in L^p$ when $x \in L^p$ (see §4), it will suffice to prove (14). Consider first the case $(x, a) \in L^0 \times V_s$. From (12) we see that

$$\Psi_q(B_p(x, a)) = (L^q) \lim_{n\to\infty} g_n , \tag{15}$$

where $g_n = \Psi[\chi_n \cdot B_p(x, a)]$. From (11):

$$g_n(\lambda) = \int_{-n}^{n} d\theta \cdot e^{2\pi i\lambda\theta} \int d\alpha \cdot x(\alpha)[Ya]_{\theta-\alpha} \qquad (\text{when } \lambda \in G) .$$

A repeated application of the Fubini theorem yields

$$g_n(\lambda) = \int dt\cdot a(t)[\Psi x]_t \int_{-n}^{n} d\theta \cdot e^{2\pi i(\lambda - t)\theta} \qquad \text{(when } \lambda \in G) .$$

In the notation of 5.2 we accordingly have

$$g_n = \{a\cdot[\Psi x]\} * \bar{\chi}_n .$$

Since $a \cdot [\Psi x]$ is in L^q, it can be inferred from 5.2 and (15) that

$$\Psi_q(B_p(x, a)) = (L^q) \lim_{n\to\infty} (\{a\cdot[\Psi x]\} * \bar{\chi}_n) = a\cdot[\Psi x] .$$

Keeping $\Psi x = \Psi_q x$ in mind (see 6.1), it is clear that (14) is now proved in the case $(x, a) \in L^0 \times V_s$. Consider the bilinear operator $R = \{(x, a) \to a \cdot \Psi_q x\}$ defined on $L^p \times V_\infty$; since $\|\Psi_q z\|_q \leqq \|z\|_p$, it follows that $\|R(x, a)\|_q \leqq \|x\|_p \|a\|_\infty$, and from (9) results that R is a bounded bilinear mapping of $L^p \times V_\infty$ into L^q. In view of (10), this remark also shows that the bilinear operator $J = \{(x, a) \to \Psi_q(B_p(x, a))\}$ is a bounded bilinear mapping of $L^p \times V_\infty$ into L^q.

Having shown that $R(x, a) = J(x, a)$ whenever $(x, a) \in L^0 \times V_s$, the desired conclusion $R = J$ can now be inferred from the denseness of L^0 and V_s in L^p and V_∞, respectively (see 4.2).

8. **Concluding remarks.** From 6.3, 3.2 and 3.5 follows that, if $f \in L^1 \cap V$ and $x \in L^p$, then $(t_p f)x = B_p(x, f) = Tf$; hence, if F is the Fourier-Stieltjes transform of f, we have (from 3.3) the relation

$$[(t_p f)x]_\lambda = \frac{1}{2\pi i}\int_{-\infty}^{\infty} x(\theta)\frac{F(\theta - \lambda)}{\theta - \lambda} d\theta \qquad (\lambda \in G)$$

which was announced in the introduction. Property (ii) of the introduction follows from (11) and 6.3. If $A \in L^1$ we denote by A_{*p} the bounded operator $\{x \to x * A\}$ defined on L^p. Let S be the set of all a in $L^1 \cap V$ such that $Ya \in L^1$, and observe that $(Ya)_{*p} = (t_p a)$ when $a \in S$. Again if $a \in S$, then $A = Ya \in L^1$ and $a = \Psi A$; from [4] it is seen that the spectrum of $(t_p a)$ is the closure of the range of a.

8.1 Remark. Set $\hat{G} = [0, 1]$ and $G = \{0, \pm 1, \pm 2, \cdots\}$. We will now sketch a proof of the properties (i′)-(ii′) described in §1. Denote by $\|A\|_v$ the total variation of A on $\hat{G}$, and suppose $\|A\|_v \neq \infty$. Observe that, since $A \in L^1(\hat{G})$, we may borrow from [5, p. 10] the following conclusion: $a = YA \in \blacktriangle(L^p(\hat{G}))$ and $t_p(YA) = A_*$ is a bounded linear operator on $L^p(\hat{G})$.

This is all of (i′)-(ii′) except for the inequality. The main result of [5] can be stated as follows[3]:

[3] The definition of $V_\sigma(a)$ is given in [5, p. 8].

$$\|t_p(a)\| \leqq 2k_p \cdot V_\sigma(a) . \tag{16}$$

Note also that $|[YA]_n| \leqq |2\pi n|^{-1}\|A\|_v$ when $n \in G$ (this is obtained by integrating by parts, as in 3.2); consequently $V_\sigma(a) = V_\sigma(YA) \leqq m_p\|A\|_v$. In view of (16), the proof of the inequality in (i′) is completed.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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Pacific Journal of Mathematics

Vol. 9, No. 3 July, 1959