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MULTIPLICATION FORMULAE FOR THE *E*-FUNCTIONS REGARDED AS FUNCTIONS OF THEIR PARAMETERS

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1. Introduction. The formulae to be proved are

$$
\begin{aligned}
(1)\quad & \sum_{i,-i} \frac{1}{i} E(p; m\alpha_r : q; m\rho_s : ze^{i\pi}) \\
&= (2\pi)^{-\frac{1}{2}(m-1)(p-q-1)} m^{m(\Sigma\alpha_r - \Sigma\rho_s) - \frac{1}{2}(p-q-1)} \\
&\times \sum_{i,-i} \frac{1}{i} E\left\{ \begin{matrix} \alpha_1, \alpha_1 + \frac{1}{m}, \cdots, \alpha_1 + \frac{m-1}{m}, \cdots, \alpha_p + \frac{m-1}{m} : \\ \frac{1}{m}, \frac{2}{m}, \cdots, \frac{m-1}{m}, \rho_1, \cdots, \rho_q + \frac{m-1}{m} : \end{matrix} \left(\frac{z}{m^{p-q-1}}\right)^m e^{i\pi} \right\},
\end{aligned}
$$

where m is a positive integer, $p > q+1$, and $|\text{amp } z| < 1/2(p-q-1)\pi$. If $p \leq q+1$, both sides vanish identically.

For all values of p and q

$$
\begin{aligned}
& E(p; m\alpha_r : q; m\rho_s : ze^{\pm i\pi}) \\
&= (2\pi)^{-\frac{1}{2}(m-1)(p-q-1)} m^{m(\Sigma\alpha_r - \Sigma\rho_s) - \frac{1}{2}(p-q+1)} \\
(2)\quad & \times \sum_{n=0}^{m-1} \left(\frac{m^{p-q-1}}{z}\right)^n E\left\{ \begin{matrix} \alpha_1 + \frac{n}{m}, \cdots, \alpha_1 + \frac{n+m-1}{m}, \cdots, \alpha_p + \frac{n+m-1}{m} : \\ \frac{n+1}{m}, \frac{n+2}{m}, \cdots * \cdots, \frac{n+m}{m}, \rho_1 + \frac{n}{m}, \cdots, \end{matrix} \right. \\
& \left. \rho_q + \frac{n+m-1}{m} : \left(\frac{z}{m^{p-q-1}}\right)^m e^{\pm i\pi} \right\},
\end{aligned}
$$

the asterisk indicating that the parameter m/m is omitted.

The proof of (1) is based on the formula ([1], p. 374)

$$
(3)\qquad E(p; \alpha_r : q; \rho_s : z) = \frac{1}{2\pi i} \int \frac{\Gamma(\zeta) \Pi \Gamma(\alpha_r - \zeta)}{\Pi \Gamma(\rho_s - \zeta)} z^\zeta d\zeta ,
$$

where the integral is taken up the η -axis, with loops, if necessary, to ensure that the pole at the origin lies to the left and the poles at

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$\alpha_1, \alpha_2, \cdots, \alpha_p$ to the right of the contour. Zero and negative integral values of the α's and ρ's are excluded, and the α's must not differ by integral values. The contour must be modified if $p < q+1$; and if $p = q+1, |z| < 1$; but we are here concerned only with the case $p > q+1$. Then z must satisfy the condition $|\text{amp } z| < 1/2(p-q+1)\pi$.

From (3) it follows that, if $p > q+1$, $|\text{amp } z| < 1/2(p-q-1)\pi$,

$$(4) \qquad \sum_{i,-1} \frac{1}{i} E(p; \alpha_r : q; \rho_s : ze^{i\pi}) = \frac{1}{i} \int \frac{\Pi\Gamma(\alpha_r - \zeta)}{\Gamma(1-\zeta)\Pi\Gamma(\rho_s - \zeta)} z^{\zeta} d\zeta .$$

For, on substituting on the left from (3), a factor $(e^{i\pi\zeta} - e^{-i\pi\zeta})$ appears in the integral, and

$$\Gamma(\zeta) \sin \pi\zeta = \pi/\Gamma(1-\zeta) .$$

The three following formulae ([1], pp. 154, 406, 407) are also required.

If m is a positive integer,

$$(5) \qquad \Gamma(mz) = (2\pi)^{\frac{1}{2}-\frac{1}{2}m} m^{mz-\frac{1}{2}} \Gamma(z)\Gamma\left(z + \frac{1}{m}\right) \cdots \Gamma\left(z + \frac{m-1}{m}\right) ;$$

$$(6) \qquad \begin{aligned} &\int_0^\infty e^{-\lambda}\lambda^{k-1} E(p; \alpha_r : q; \rho_s : z/\lambda^m) d\lambda \\ &= (2\pi)^{\frac{1}{2}-\frac{1}{2}m} m^{k-\frac{1}{2}} E(p+m; \alpha_r : q; \rho_s : z/m^m) , \end{aligned}$$

where $R(k) > 0$, $\alpha_{p+1+\nu} = (k+\nu)/m$, $\nu = 0, 1, 2, \cdots, m-1$;

$$(7) \qquad \begin{aligned} &\frac{1}{2\pi i} \int e^{\zeta} \zeta^{-\rho} E(p; \alpha_r : q; \rho_s : \zeta^m z) d\zeta \\ &= (2\pi)^{\frac{1}{2}m-\frac{1}{2}} m^{\frac{1}{2}-\rho} E(p; \alpha_r : q+m; \rho_s : zm^m) , \end{aligned}$$

where the contour of integration starts from $-\infty$ on the ξ-axis, passes round the origin in the positive direction, and ends at $-\infty$ on the ξ-axis, amp ζ being $-\pi$ initially, and $\rho_{q+1+\nu} = (\rho+\nu)/m$, $\nu = 0, 1, 2, \cdots, m-1$.

2. Proofs of the formulae. On applying (4) on the left of (1) and replacing ζ by $m\zeta$ the left hand side becomes

$$\frac{m}{i} \int \frac{\pi\Gamma(m\alpha_r - m\zeta)}{\Gamma(1-m\zeta)\pi\Gamma(m\rho_s - m\zeta)} z^{m\zeta} d\zeta .$$

Here apply (5) and get

$$(2\pi)^{-\frac{1}{2}(m-1)(p-q-1)} m^{m(\Sigma\alpha_r-\Sigma\rho_s)-\frac{1}{2}(p-q-1)}$$

$$\times\frac{1}{i}\int\frac{\Pi\left\{\Gamma(\alpha_r-\zeta)\Gamma\left(\alpha_r+\frac{1}{m}-\zeta\right)\cdots\Gamma\left(\alpha_r+\frac{m-1}{m}-\zeta\right)\right\}}{\Gamma(1-\zeta)\Gamma\left(\frac{1}{m}-\zeta\right)\cdots\Gamma\left(\frac{m-1}{m}-\zeta\right)\Pi\left\{\Gamma(\rho_s-\zeta)\cdots\Gamma\left(\rho_s+\frac{m-1}{m}-\zeta\right)\right\}}$$

$$\times\left\{\frac{z}{m^{p-q-1}}\right\}^{m\zeta}d\zeta\ ,$$

and from (4), this is equal to the right hand side of (1).

Formula (2) can be obtained by showing that

$$\begin{aligned}E(:\ :e^{\pm i\pi}z) &= e^{1/z}\\ &= \sum_{n=0}^{m-1}\frac{(1/z)^n}{n!}F\left\{;\frac{n+1}{m},\cdots*\cdots,\frac{n+m}{m};(mz)^{-m}\right\}\\ &= (2\pi)^{\frac{1}{2}m-\frac{1}{2}}m^{-\frac{1}{2}}\sum_{n=0}^{m-1}\left(\frac{1}{mz}\right)^nE\left\{:\frac{n+1}{m},\cdots*\cdots,\frac{n+m}{m}:e^{\pm i\pi}(mz)^m\right\},\end{aligned}$$

and then generalizing by employing (6) and (7).

Note 1. Ragab's formula [2]

$$\sum_{i,-i}\frac{1}{i}\int_0^\infty e^{-pt}E\left(\alpha,\alpha+\frac{1}{m},\cdots,\alpha+\frac{m-1}{m}:\ :e^{i\pi}zm^{-m}/t\right)dt$$
$$= (2\pi)^{\frac{1}{2}+\frac{1}{2}m}m^{-m\alpha-\frac{1}{2}}p^{\alpha-1}z^{\alpha}\exp(-\,p^{1/m}z^{1/m})\ , \qquad (8)$$

where m is a positive integer greater than 1, p is positive, $|\text{amp } z| < 1/2(m-1)\pi$, can be derived by substituting on the left from (4), changing the order of integration, evaluating the inner integral, applying (5), replacing ζ by $\alpha-\zeta/m$, and applying (3).

Note 2. It has been pointed out by a referee that there seems to be some connection between the formulae of this paper and certain formulae of Meijer's for the G-function which are reproduced on pages 209, 210 of the first volume of Higher Transcendental Functions [McGraw Hill Book Co., 1953].

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2. F. M. Ragab, *The inverse Laplace transform of an exponential function*, New York University, Institute of Mathematical Sciences, Astia Document No. AD 133670.

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